ALMOST CONTINUITY IMPLIES CLOSURE CONTINUITY[†]

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Abstract. The purpose of this note is to answer in the affirmative a long standing open question raised by Singal and Singal—whether every almost continuous function is closure continuous (θ-continuous).

Introduction. Among other generalizations of continuity, the concepts of weak and closure continuity have been studied by many mathematicians: D. R. Andrew, J. Chew, L. Herrington, N. Levine, P. E. Long, T. Noire, J. Porter, M. Saleh, J. Tong, E. K. Whittlesy and others. In 1961, Levine introduced the concept of weak continuity as a generalization of continuity: later in 1966, Andrew and Whittlesy **[2]** introduced the concept of closure continuity which is stronger than weak continuity. Indeed closure continuity was introduced many years earlier by S. Fomin **[3]** precisely in 1941 as θ-continuity, but it seems that Andrew and Whittlesy were not aware of the paper by Fomin. In 1968, Singal and Singal **[9]** introduced almost continuity as another generalization and raised the following question in Remark 3.3: is every almost continuous function θ-continuous? In this short note we answer this question positively.

Definitions and notation. Let A be a subset of a topological space X. The closure and the interior of A in X are denoted, respectively, by $\overline{A}$, A°. A function $f : X \to Y$ is *closure continuous* (*θ-continuous*) at $x \in X$ if, given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq \overline{V}$. If this condition is satisfied at each $x \in X$, then f is said to be *closure continuous* (*θ-continuous*). A function $f: X \to Y$ is said to be *almost continuous in the sense of Singal and Singal* if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq \overline{V}^\circ$.

THEOREM 1. *Let $f: X \to Y$ be almost continuous. Then f is closure continuous.*

Proof. Let $x \in X$ and let V be an open set containing $f(x)$. By almost continuity of f, there exists an open set U containing x such that $f(U) \subseteq \overline{V}^\circ$. Let $y \in \overline{U}$. For any open set W containing $f(y)$ there exists, by almost continuity of f, an open set A containing y such that $f(A) \subseteq \overline{W}^\circ$. Since $y \in \overline{U}$, we have $A \cap U \neq \emptyset$. Therefore, $\emptyset \neq f(A \cap U) \subseteq \overline{V}^\circ \cap \overline{W}^\circ \subseteq \overline{W}$. Since $\overline{V}^\circ \cap \overline{W}^\circ$ is open we have $\overline{V}^\circ \cap \overline{W}^\circ \cap W \neq \emptyset$: that is, $\overline{V}^\circ \cap W \neq \emptyset$. Since this is true for every open set containing $f(y)$ we have $f(y) \in \overline{V}$. Also since this is true for every $y \in \overline{U}$ we obtain $f(\overline{U}) \subseteq \overline{V}$: that is, f is closure continuous.

Recall that a subset A of a space X is called *closure (almost) compact* if every open cover of A has a finite subcollection whose closures cover A. Closure compactness was introduced as $H(i)$—spaces in **[8]** as a generalization of absolutely closed (H-closed) spaces in **[1]**.

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Clearly every compact set is closure compact but not conversely as is shown in the next example.

EXAMPLE 1. Let X be any uncountable space with the cocountable topology. Then every subset of X is closure compact, but the only compact subsets of X are the finite ones.

The next theorem shows that closure compactness is preserved under closure continuous functions.

THEOREM 2. *Let $f: X \to Y$ be closure continuous and let K be a closure compact subset of X. Then $f(K)$ is a closure compact subset of Y.*

Proof. Let $\mathcal{V}$ be an open cover of $f(K)$. For each $k \in K, f(k) \in V_k$ for some $V_k \in \mathcal{V}$. By closure continuity of f, there exists an open set U_k containing x such that $f(\overline{U}_k) \subseteq \overline{V}_k$. The collection $\{U_k : k \in K\}$ is an open cover of K and so, since K is closure compact, there is a finite subcollection $\{U_k : k \in K_0\}$, where K_0 is a finite subset of K, and $\{\overline{U}_k : k \in K_0\}$ covers K. Clearly $\{\overline{V}_k : k \in K_0\}$ covers $f(K)$ and thus $f(K)$ is a closure compact subset of Y.

COROLLARY 1. *Let $f: X \to Y$ be almost continuous and let K be a closure compact subset of X. Then $f(K)$ is a closure compact subset of Y.*

As a consequence of the corollary, we get Theorem 3.3 and Lemmas 3.2, 3.3 in **[10]** and Theorem 3.4 in **[4]**.

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