COMPOSITIO MATHEMATICA

# Quaternionic Satake equivalence

Tsao-Hsien Chen, Mark Macerato, David Nadler and John O'Brien

Compositio Math. **161** (2025), 1615–1663.

doi: 10.1112/S0010437X25007146

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 161 (2025) 1615–1663
doi:10.1112/S0010437X25007146

# Quaternionic Satake equivalence

Tsao-Hsien Chen, Mark Macerato, David Nadler and John O'Brien

Abstract

We establish a derived geometric Satake equivalence for the quaternionic general linear group $\mathrm{GL}_n(\mathbb{H})$. By applying the real–symmetric correspondence for affine Grassmannians, we obtain a derived geometric Satake equivalence for the symmetric variety $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$. We explain how these equivalences fit into the general framework of a geometric Langlands correspondence for real groups and the relative Langlands duality conjecture. As an application, we compute the stalks of the IC-complexes for spherical orbit closures in the quaternionic affine Grassmannian and the loop space of $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$. We show that the stalks are given by the Kostka–Foulkes polynomials for $\mathrm{GL}_n$ but with all degrees doubled.

## Contents

**1 Introduction** **1616**
- 1.1 Real–symmetric correspondence . . . 1616
- 1.2 Reminder on derived Satake for $\mathrm{GL}_{2n}$ . . . 1617
- 1.3 Derived Satake for the quaternionic group $\mathrm{GL}_n(\mathbb{H})$ . . . 1617
- 1.4 Derived Satake for the symmetric variety $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ . . . 1619
- 1.5 Geometric Langlands correspondence for real groups . . . 1619
- 1.6 Relative Langlands duality conjectures . . . 1621
- 1.7 IC-stalks and Kostka–Foulkes polynomials . . . 1622
- 1.8 Outline of the proof . . . 1623
- 1.9 Organization . . . 1624

**2 Constructible sheaves on a semi-analytic stack** **1624**

**3 Spectral side** **1625**
- 3.1 Quaternionic group . . . 1625
- 3.2 Symplectic group . . . 1626
- 3.3 Notation related to root structure . . . 1626
- 3.4 Regular centralizers . . . 1627
- 3.5 Dual group . . . 1631
- 3.6 The partial Whittaker reduction . . . 1632

Received 6 June 2023, accepted in final form 7 January 2025.
*2020 Mathematics Subject Classification* 22E57 (primary), 14P15, 22E45 (secondary).
*Keywords:* Langlands duality; real reductive groups; perverse sheaves.

**4 Constructible side** **1633**
4.1 Twistor fibration . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1633
4.2 Equivariant cohomology of quaternionic projective spaces . . . . . . . . 1633
4.3 Two bases . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1635
4.4 Complex and quaternionic affine Grassmannians . . . . . . . . . . . . . 1636
4.5 Real nearby cycles functor . . . . . . . . . . . . . . . . . . . . . . . . . 1637
4.6 Equivariant homology and cohomology of affine Grassmannians . . . . . 1639
4.7 Fully-faithfulness . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1642
4.8 Ext algebras . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1649
4.9 IC-stalks, $q$-analogue of weight multiplicity, and Kostka–Foulkes polynomials . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1650
**5 Main results** **1652**
5.1 Formality . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1652
5.2 Derived geometric Satake equivalence for the quaternionic groups . . . . 1656
5.3 Spectral description of nearby cycles functors . . . . . . . . . . . . . . . 1658
5.4 Monoidal structures . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1660
**Acknowledgments** **1661**
**References** **1662**

## 1. Introduction

### 1.1 Real–symmetric correspondence

Let $G_{\mathbb{R}}$ be a real form of a connected complex reductive group $G$. Let $X = G/K$ be the associated symmetric variety under Cartan's bijection, where $K$ is the complexification of a maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$.

A fundamental feature of the representation theory of the real group $G_{\mathbb{R}}$ is that many results of an analytic nature have equivalent purely algebraic formulations in terms of the corresponding symmetric variety $X$. We will call this broad phenomenon the *real–symmetric correspondence*. It allows one to use algebraic tools on the symmetric side to study questions on the real side and, conversely, to use analytic tools on the real side to study questions on the symmetric side. Famous examples include Harish-Chandra's reformulation of admissible representations of real groups in terms of $(\mathfrak{g}, K)$-modules, the Kostant–Sekiguchi correspondence between real and symmetric nilpotent orbits, and the Matsuki correspondence between $G_{\mathbb{R}}$- and $K$-orbits on the flag manifold of $G$.

In [CN18], the first and third authors established a real–symmetric correspondence relating the dg derived category of spherical constructible sheaves on the real affine Grassmannian $\mathrm{Gr}_{G_{\mathbb{R}}}$ of $G_{\mathbb{R}}$ and the dg derived category of spherical constructible sheaves on the loop space $\mathfrak{L}X$ of $X$. We are interested in applying this real–symmetric correspondence to study questions in the real and relative geometric Langlands programs.

In the present paper, we consider the question of a geometric Satake equivalence for real groups and symmetric varieties. We focus on the case where the real group is the quaternionic group $G_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{H})$ with associated symmetric variety $X = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$. We prove the derived geometric Satake equivalence for $\mathrm{GL}_n(\mathbb{H})$ relating the dg constructible derived category of the quaternionic affine Grassmannian with the dg coherent derived category of a quotient stack associated to the Gaitsgory–Nadler dual group $\check{G}_X$ of $X$ (which is $\check{G}_X = \mathrm{GL}_n$ in this case). Via the real–symmetric correspondence, we obtain a derived geometric Satake equivalence for the symmetric variety $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$. As an application, we compute the stalks of the IC-complexes for



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

spherical orbit closures in the quaternionic affine Grassmannian and the loop space of $GL_{2n}/Sp_{2n}$. We show that the stalks are given by the Kostka–Foulkes polynomials for $GL_n$ but with all degrees doubled.

We explain how these equivalences fit into the general framework of a geometric Langlands correspondence for real groups, due to Ben-Zvi and the third author, and of the relative Langlands duality conjecture, due to Ben-Zvi, Sakellaridis, and Venkatesh.

From the point of view of real groups, the quaternionic group $GL_n(\mathbb{H})$ offers in some sense the simplest possible geometry: just as complex Grassmannians are simpler than real Grassmannians (Schubert cells are $2d$ versus $d$ real-dimensional), quaternionic Grassmannians are simpler still than complex Grassmannians (Schubert cells are $4d$ versus $2d$ real-dimensional). On the other hand, the geometry of the symmetric variety $GL_{2n}/Sp_{2n}$ is more complicated than that of $GL_{2n}$. The real–symmetric correspondence allows us to use the simpler quaternionic geometry of $GL_n(\mathbb{H})$ to answer questions about the more complicated geometry of $GL_{2n}/Sp_{2n}$.

We now describe the paper in more detail. We work throughout over the complex numbers, except where we specifically consider real forms. All topological sheaves are with respect to the classical topology with complex coefficients.

### 1.2 Reminder on derived Satake for $GL_{2n}$

Let $\mathfrak{L}GL_{2n}$ and $\mathfrak{L}^+GL_{2n}$ be the Laurent loop group and Taylor arc group of $GL_{2n}$. The affine Grassmannian $Gr_{2n} = \mathfrak{L}GL_{2n}/\mathfrak{L}^+GL_{2n}$ for $GL_{2n}$ is the ind-variety classifying $\mathbb{C}[[t]]$-lattices in $\mathbb{C}((t))^n$. The arc group $\mathfrak{L}^+GL_{2n}$ acts naturally on $Gr_{2n}$, and we denote by $D^b(\mathfrak{L}^+GL_{2n}\backslash Gr_{2n})$ the monoidal dg-category of $\mathfrak{L}^+GL_{2n}$-equivariant constructible complexes on $Gr_{2n}$ with monoidal structure given by convolution.

Let $\mathfrak{gl}_{2n}$ be the Lie algebra of $GL_{2n}$. Write $\mathrm{Sym}(\mathfrak{gl}_{2n}[-2])$ for the symmetric algebra of $\mathfrak{gl}_{2n}[-2]$ viewed as a dg-algebra with trivial differential. The group $GL_{2n}$ acts on $\mathrm{Sym}(\mathfrak{gl}_{2n}[-2])$ via the adjoint action, and we denote by $D^{GL_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2]))$ the monoidal dg-category of perfect $GL_{2n}$-equivariant dg-modules over $\mathrm{Sym}(\mathfrak{gl}_{2n}[-2]))$ with monoidal structure given by the (derived) tensor product of dg-modules.

One of the versions of the derived Satake equivalence in [BF08] says that there is an equivalence of monoidal dg-categories

$$\Psi : D^b(\mathfrak{L}^+GL_{2n}\backslash Gr_{2n}) \simeq D^{GL_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2]))$$

extending the geometric Satake equivalence $\mathrm{Perv}(Gr_{2n}) \simeq \mathrm{Rep}(GL_{2n})$, where $\mathrm{Perv}(Gr_{2n}) \subset D^b(\mathfrak{L}^+GL_{2n}\backslash Gr_{2n})$ is the subcategory of $\mathfrak{L}^+GL_{2n}$-equivariant perverse sheaves on $Gr_{2n}$ and $\mathrm{Rep}(GL_{2n}) \subset D^{GL_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2]))$ is the subcategory of representations of $GL_{2n}$.[1]

### 1.3 Derived Satake for the quaternionic group $GL_n(\mathbb{H})$

Let $GL_n(\mathbb{H}) \subset GL_{2n}$ be the real form given by the rank $n$ quaternionic group. Let $\mathfrak{L}GL_n(\mathbb{H})$ and $\mathfrak{L}^+GL_n(\mathbb{H})$ be the real Laurent loop group and Taylor arc group for $GL_n(\mathbb{H})$. By the real affine Grassmannian for the quaternionic group $GL_n(\mathbb{H})$, we will mean the ind semi-analytic variety $Gr_{n,\mathbb{H}} = \mathfrak{L}GL_n(\mathbb{H})/\mathfrak{L}^+GL_n(\mathbb{H})$ classifying $\mathbb{H}[[t]]$-lattices in $\mathbb{H}((t))^n$.[2]

[1] The embedding $\mathrm{Rep}(GL_{2n}) \subset D^{GL_{2n}}_{\mathrm{perf}}\mathrm{Sym}(\mathfrak{gl}_{2n}[-2])$ is given by $V \mapsto \mathrm{Sym}(\mathfrak{gl}_{2n}[-2]) \otimes_{\mathbb{C}} V$.

[2] By definition, a $\mathbb{H}[[t]]$-lattice $\Lambda$ in $\mathbb{H}((t))^n$ is a finitely generated right $\mathbb{H}[[t]]$-submodule of $\mathbb{H}((t))^n$ such that $\Lambda \otimes_{\mathbb{H}[[t]]} \mathbb{H}((t)) = \mathbb{H}((t))^n$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

The real arc group $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$ acts naturally on $\mathrm{Gr}_{n,\mathbb{H}}$, and we denote by $D^b(\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})\backslash\mathrm{Gr}_{n,\mathbb{H}})$ the monoidal dg-category of $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$-equivariant constructible complexes on $\mathrm{Gr}_{n,\mathbb{H}}$ with monoidal structure given by convolution. The $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$-orbits on $\mathrm{Gr}_{n,\mathbb{H}}$ are all even real-dimensional (in fact, $4d$ real-dimensional; see § 4.4), and hence middle perversity makes sense. We denote by $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ the category of $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$-equivariant perverse sheaves on $\mathrm{Gr}_{n,\mathbb{H}}$. In [Nad05], the third author established a real geometric Satake equivalence, giving an equivalence of monoidal abelian categories $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})\simeq\mathrm{Rep}(\mathrm{GL}_n)$ in the case at hand.

The first main result of this paper is the following equivalence of monoidal dg derived categories, to be called derived Satake for $\mathrm{GL}_n(\mathbb{H})$.

THEOREM 1.1 (See Theorem 5.5). *There is an equivalence of monoidal dg-categories*

$$\Psi_{\mathbb{H}}: D^b(\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})\backslash\mathrm{Gr}_{n,\mathbb{H}})\simeq D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4]))$$

*extending the real geometric Satake equivalence* $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})\simeq\mathrm{Rep}(\mathrm{GL}_n)$.

A key ingredient in the proof of Theorem 1.1 (as in the proof of the abelian quaternionic geometric Satake) is a nearby cycles functor

$$\mathrm{R}: D^b(\mathfrak{L}^+\mathrm{GL}_{2n}\backslash\mathrm{Gr}_{2n})\to D^b(\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})\backslash\mathrm{Gr}_{n,\mathbb{H}}) \tag{1.1}$$

associated to a real form of the Beilinson–Drinfeld Grassmannian with generic fibers isomorphic to the complex affine Grassimannian $\mathrm{Gr}_{2n}$ and special fiber isomorphic to the quaternionic affine Grassimannian $\mathrm{Gr}_{n,\mathbb{H}}$ (see § 4.5). Note that, unlike in the complex algebraic setting, the nearby cycles functor R is not $t$-exact: it maps perverse sheaves to direct sums of shifts of perverse sheaves (see Proposition 4.5). As a corollary of the proof, we obtain the following spectral description of the nearby cycles functor.

Consider the graded scheme

$$\widetilde{\mathfrak{gl}}_{2n}=\left\{\begin{pmatrix} A[0] & B[-2]\\ C[2] & D[0]\end{pmatrix}\middle| A,B,C,D\in\mathfrak{gl}_n\right\}.$$

We have the natural embedding of (even graded) schemes

$$\tau:\mathfrak{gl}_n[4]\to\widetilde{\mathfrak{gl}}_{2n}[2],\quad \tau(C[4])=\begin{pmatrix} 0 & \mathrm{Id}_n\\ C[4] & 0\end{pmatrix}, \tag{1.2}$$

where $\mathrm{Id}_n$ is the rank $n$ identity matrix. Note that the map $\tau$ is $\mathrm{GL}_n$ adjoint-equivariant, where $\mathrm{GL}_n$ acts on $\widetilde{\mathfrak{gl}}_{2n}[2]$ via the diagonal embedding $\mathrm{GL}_n\to\mathrm{GL}_{2n}$. Hence pullback along $\tau$ provides a functor

$$\tau^*: D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\widetilde{\mathfrak{gl}}_{2n}[-2]))\to D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4])).$$

Here we view the rings of functions on $\mathfrak{gl}_n[4]$ and $\widetilde{\mathfrak{gl}}_{2n}[2]$ as the dg symmetric algebras $\mathrm{Sym}(\mathfrak{gl}_{2n}[-4])$ and $\mathrm{Sym}(\widetilde{\mathfrak{gl}}_{2n}[-2])$ with trivial differential. Introduce the functor

$$\Phi: D^{\mathrm{GL}_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2]))\longrightarrow D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\widetilde{\mathfrak{gl}}_{2n}[-2]))\xrightarrow{\tau^*} D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4]))$$

where the first functor is the sheared forgetful functor associated to the $\mathbb{G}_m$-action on $\mathfrak{gl}_{2n}[-2]$ via the co-character $2\rho_L:\mathbb{G}_m\to\mathrm{GL}_{2n}$ (see (5.17)). Here $L$ is the complexification of the Levi subgroup of the minimal parabolic subgroup of $\mathrm{GL}_n(\mathbb{H})$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Theorem 1.2 (See Theorem 5.7). *The following square is naturally commutative*

$$\begin{array}{ccc} D^b(\mathfrak{L}^+\mathrm{GL}_{2n}\backslash \mathrm{Gr}_{2n}) & \xrightarrow{\mathrm{R}} & D^b(\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})\backslash \mathrm{Gr}_{n,\mathbb{H}}) \\ {\scriptstyle\Psi}\downarrow\simeq & & {\scriptstyle\Psi_{\mathbb{H}}}\downarrow\simeq \\ D^{\mathrm{GL}_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2])) & \xrightarrow{\Phi} & D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4])) \end{array}$$

*where $\Psi$ and $\Psi_{\mathbb{H}}$ are the complex and quaternionic derived Satake equivalences, respectively.*

Later, in § 1.6, we will discuss how Theorem 1.2 fits into the general framework of duality for Hamiltonian spaces.

## 1.4 Derived Satake for the symmetric variety $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$

Let $\mathfrak{L}\,\mathrm{Sp}_{2n}$ be the Laurent loop group of the symmetric subgroup $\mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$. There is a natural action of $\mathfrak{L}\,\mathrm{Sp}_{2n}$ on $\mathrm{Gr}_{2n}$, and we denote by $D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash \mathrm{Gr}_{2n})$ the dg-category of $\mathfrak{L}\,\mathrm{Sp}_{2n}$-equivariant constructible complexes on $\mathrm{Gr}_{2n}$.

In [CN18, Theorem 8.1] it is shown that there is an equivalence of dg-categories

$$D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash \mathrm{Gr}_{2n}) \simeq D^b(\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})\backslash \mathrm{Gr}_{n,\mathbb{H}}). \tag{1.3}$$

In [CN18, Theorem 9.1] it is shown that the equivalence is compatible with the natural monoidal actions of $D^b(\mathfrak{L}^+\mathrm{GL}_{2n}\backslash \mathrm{Gr}_{2n})$, where the action on the right-hand side is through the nearby cycles functor (1.1). One can view the above equivalence as an example of the real–symmetric correspondence for the affine Grassmannian $\mathrm{Gr}_{2n}$. Combining this with Theorem 1.1, we obtain a derived Satake equivalence for $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$.

Theorem 1.3. *There is an equivalence of dg-categories*

$$\Psi_X : D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash \mathrm{Gr}_{2n}) \simeq D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4]))$$

*compatible with the monoidal actions of* $D^b(\mathfrak{L}^+\mathrm{GL}_{2n}\backslash \mathrm{Gr}_{2n}) \simeq D^{\mathrm{GL}_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2]))$.

*Remark* 1.4. In general, the $\mathfrak{L}\,\mathrm{Sp}_{2n}$-orbits on $\mathrm{Gr}_{2n}$ are neither finite-dimensional nor finite-codimensional. Thus there is not a naive approach to sheaves on $\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash \mathrm{Gr}_{2n}$ with traditional methods. To overcome this, we use the observation in [CN18] that the based loop group $\Omega\mathrm{Sp}(n)$ of the compact real form $\mathrm{Sp}(n)$ of $\mathrm{Sp}_{2n}$ acts freely on $\mathrm{Gr}_{2n}$ and the quotient $\Omega\mathrm{Sp}(n)\backslash \mathrm{Gr}_{2n}$ is a semi-analytic space of ind-finite type, i.e. an inductive limit of real analytic schemes of finite type. We define $D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash \mathrm{Gr}_{2n})$ to be the category of sheaves on $\Omega\mathrm{Sp}(n)\backslash \mathrm{Gr}_{2n}$ constructible with respect to the stratification coming from the descent of the $\mathfrak{L}\,\mathrm{Sp}_{2n}$-orbits stratification on $\mathrm{Gr}_{2n}$; see [CN18] Definition 1.3] and also Remark 1.10.

## 1.5 Geometric Langlands correspondence for real groups

We discuss here how one might interpret our results in terms of geometric Langlands for real groups [BZN15]. Our results specifically relate to the curve $\mathbb{P}^1$ with its standard real structure with real points $\mathbb{RP}^1$ (whereas connections to Langlands parameters have been explored [BZN13] for $\mathbb{P}^1$ with its antipodal real structure with no real points).

For complex reductive groups, it is known that the derived Satake equivalence implies the geometric Langlands correspondence over the projective line $\mathbb{P}^1$ via a Radon transform. To state a version of this in the setting at hand, let $\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1)$ be the moduli stack of $\mathrm{GL}_{2n}$-bundles over $\mathbb{P}^1$, and let $\mathrm{Loc}_{\mathrm{GL}_{2n}}(\mathrm{S}^2)$ be the moduli stack of Betti $\mathrm{GL}_{2n}$-local systems on the 2-sphere $\mathrm{S}^2$. Let $D_!(\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1))$ be the dg-category of constructible complexes on $\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1)$ that are extensions by zero off of a finite-type substack, and let $\mathrm{Coh}(\mathrm{Loc}_{\mathrm{GL}_{2n}}(\mathrm{S}^2))$ be the dg-category of coherent complexes on $\mathrm{Loc}_{\mathrm{GL}_{2n}}(\mathrm{S}^2)$ with bounded cohomology.

In this setting, the geometric Langlands correspondence for $\mathbb{P}^1$ constructed in [Laf09] takes the form of an equivalence

$$D_!(\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1)) \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Loc}_{\mathrm{GL}_{2n}}(\mathrm{S}^2)). \tag{1.4}$$

Moreover, it fits into a commutative diagram of equivalences

$$\begin{array}{ccc} D_!(\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1)) & \xrightarrow{\simeq} & \mathrm{Coh}(\mathrm{Loc}_{\mathrm{GL}_{2n}}(\mathrm{S}^2)) \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ D^b(\mathfrak{L}^+\mathrm{GL}_{2n}\backslash\mathrm{Gr}_{2n}) & \xrightarrow{\simeq} & D^{\mathrm{GL}_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2])) \end{array} \tag{1.5}$$

where the left vertical equivalence

$$D_!(\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1)) \simeq D^b(\mathfrak{L}^+\mathrm{GL}_{2n}\backslash\mathrm{Gr}_{2n}) \tag{1.6}$$

is given by the Radon transform (see [Laf09, Proposition 2.1]) and the right vertical equivalence is given by the the Koszul duality equivalence

$$\mathrm{Coh}(\mathfrak{gl}_{2n}[-1]/\mathrm{GL}_{2n}) \simeq D^{\mathrm{GL}_{2n}}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_{2n}[-2])) \tag{1.7}$$

under the isomorphisms $\mathrm{Loc}_{\mathrm{GL}_{2n}}(\mathrm{S}^2) \simeq \mathrm{pt}/\mathrm{GL}_{2n} \times_{\mathfrak{gl}_{2n}/\mathrm{GL}_{2n}} \mathrm{pt}/\mathrm{GL}_{2n} \simeq \mathfrak{gl}_{2n}[-1]/\mathrm{GL}_{2n}$.

As a special case of the affine Matsuki correspondence established in [CN18], we have a real group version of the equivalence (1.6) taking the form

$$D_!(\mathrm{Bun}_{\mathrm{GL}_n(\mathbb{H})}(\mathbb{RP}^1)) \simeq D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathrm{Gr}_{2n}). \tag{1.8}$$

Here $\mathrm{Bun}_{\mathrm{GL}_n(\mathbb{H})}(\mathbb{RP}^1)$ is the real form of $\mathrm{Bun}_{\mathrm{GL}_{2n}}(\mathbb{P}^1)$ classifying $\mathrm{GL}_n(\mathbb{H})$-bundles on the real projective line $\mathbb{RP}^1$. Combining this with the derived Satake equivalence for $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ in Theorem 1.3, we obtain the following geometric Langlands correspondence for $\mathrm{GL}_n(\mathbb{H})$.

Let $\mathrm{Loc}_{\mathrm{GL}_n}(\mathrm{S}^4)$ be the moduli stack of Betti $\mathrm{GL}_n$-local systems on the 4-sphere $\mathrm{S}^4$. Note that the presentation $\mathrm{S}^4 = \mathrm{D}^4 \cup_{\mathrm{S}^3} \mathrm{D}^4$ (where $\mathrm{D}^4$ is the four-dimensional disk in $\mathbb{R}^4$) gives an isomorphism of dg-stacks:

$$\mathrm{Loc}_{\mathrm{GL}_n}(\mathrm{S}^4) \simeq \mathrm{pt}/\mathrm{GL}_n \times_{\mathfrak{gl}_n[-2]/\mathrm{GL}_n} \mathrm{pt}/\mathrm{GL}_n \simeq \mathfrak{gl}_n[-3]/\mathrm{GL}_n.$$

From the Koszul duality $\mathrm{Coh}(\mathfrak{gl}_n[-3]/\mathrm{GL}_n) \simeq D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4]))$, we obtain

$$\mathrm{Coh}(\mathrm{Loc}_{\mathrm{GL}_n}(\mathrm{S}^4)) \simeq \mathrm{Coh}(\mathfrak{gl}_n[-3]/\mathrm{GL}_n) \simeq D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4])). \tag{1.9}$$

THEOREM 1.5. *There is an equivalence*

$$D_!(\mathrm{Bun}_{\mathrm{GL}_n(\mathbb{H})}(\mathbb{RP}^1)) \simeq \mathrm{Coh}(\mathrm{Loc}_{\mathrm{GL}_n}(\mathrm{S}^4))$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

*that fits into a commutative diagram of equivalences as follows.*

$$\begin{array}{ccc} D_!(\mathrm{Bun}_{\mathrm{GL}_n(\mathbb{H})}(\mathbb{RP}^1)) & \xrightarrow{\simeq} & \mathrm{Coh}(\mathrm{Loc}_{\mathrm{GL}_n}(\mathrm{S}^4)) \\ \downarrow\simeq & & \downarrow\simeq \\ D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathrm{Gr}_{2n}) & \xrightarrow{\simeq} & D^{\mathrm{GL}_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{gl}_n[-4])) \end{array} \tag{1.10}$$

*Here the left and right vertical equivalence are the affine Matsuki correspondence (1.8) and Koszul duality (1.9), respectively, and the bottom equivalence is the derived Satake equivalence for* $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$.

*Remark* 1.6. On the one hand, the appearance of the 4-sphere $S^4$ in the above version of geometric Langlands for $\mathrm{GL}_n(\mathbb{H})$ is perhaps not so surprising owing to the identification $\mathbb{HP}^1 \simeq S^4$. Moreover, the twistor fibration $\mathbb{CP}^3 \to \mathbb{HP}^1 \simeq S^4$ arises naturally in the proof of Theorem 1.1 (see § 4.1). On the other hand, the appearance of connections on $S^4$ is quite mysterious (at least to the authors). From the perspective of geometric Langlands for real groups, we expect the spectral side to be expressible in terms of $\mathrm{GL}_{2n}$-connections on a disk with a partial oper structure along the boundary. This should reflect the usual Satake $\mathrm{GL}_{2n}$-Hecke operators in the bulk and the real Satake $\mathrm{GL}_n$-Hecke operators along the boundary.

*Remark* 1.7. More generally, the real–symmetric correspondence (1.3) and affine Matsuki correspondence (1.8) hold for any real group $G_{\mathbb{R}}$. It follows that a derived Satake equivalence for real groups or symmetric varieties will imply a version of geometric Langlands correspondence over $\mathbb{RP}^1$ for real groups and vice versa.

### 1.6 Relative Langlands duality conjectures

A far-reaching program of Ben-Zvi, Sakellaridis, and Venkatesh proposes relative Langlands duality conjectures between periods and L-functions (see e.g. [BZSV24]). A fundamental conjecture in the program predicts that given a complex reductive group $G$, with dual group $\check{G}$, and a homogeneous spherical $G$-variety $X$, there exists a (graded) Hamiltonian $\check{G}$-variety $\check{M}$ together with a moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$ equipped with a commuting $\mathbb{G}_m$-action of weight 2, and an equivalence

$$D^b(\mathfrak{L}X/\mathfrak{L}^+G) \simeq \mathrm{Coh}(\check{M}/\check{G}) \tag{1.11}$$

where $\mathrm{Coh}(\check{M}/\check{G})$ is the dg-category of $\check{G}$-equivariant perfect dg-modules over the ring of functions on $\check{M}$ viewed as a dg-algebra with trivial differential and grading given by the above $\mathbb{G}_m$-action. Moreover, this equivalence should be compatible with the derived Satake equivalence $D^b(\mathfrak{L}^+G\backslash\mathrm{Gr}_G) \simeq D^{\check{G}}_{\mathrm{perf}}(\mathrm{Sym}(\check{\mathfrak{g}}[-2])) \simeq \mathrm{Coh}(\check{\mathfrak{g}}^*[2]/\check{G})$, in the sense that the right convolution action of $D^b(\mathfrak{L}^+G\backslash\mathrm{Gr}_G)$ on $D^b(\mathfrak{L}X/\mathfrak{L}^+G)$ should correspond to the tensor product action of $\mathrm{Coh}(\check{\mathfrak{g}}^*[2]/\check{G})$ on $\mathrm{Coh}(\check{M}/\check{G})$ through the moment map $\mu$.

We now explain how the derived Satake equivalence for the symmetric variety $X = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ fits into the general setting of relative Langlands duality. On the one hand, by [CN24, Proposition 8.1], there are an isomorphism of stacks

$$\mathfrak{L}X/\mathfrak{L}^+\mathrm{GL}_{2n} \cong \mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathfrak{L}\mathrm{GL}_{2n}/\mathfrak{L}^+\mathrm{GL}_{2n} \cong \mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathrm{Gr}_{2n}$$

and hence equivalences of categories

$$D^b(\mathfrak{L}X/\mathfrak{L}^+\mathrm{GL}_{2n}) \simeq D^b(\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathrm{Gr}_{2n}) \tag{1.12}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

where $D^b(\mathfrak{L}X/\mathfrak{L}^+\mathrm{GL}_{2n})$ is the dg-category of $\mathfrak{L}^+\mathrm{GL}_{2n}$-equivariant constuctible complexes on the loop space $\mathfrak{L}X$ of $X$.

On the other hand, it is expected that the Hamiltonian $\check{G}$-space $\check{M}$ associated to the symmetric variety $X = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ (note that symmetric varieties are spherical) is given by $\check{M} = T^*(\mathrm{GL}_{2n}/\mathrm{GL}_n \ltimes U, \psi)$, the partial Whittaker reduction of $T^*\mathrm{GL}_{2n}$ with respect to the generic homomorphism $\psi$ of the Shalika subgroup $\mathrm{GL}_n \ltimes U$ of $\mathrm{GL}_{2n}$:

$$\mathrm{GL}_n \ltimes U = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \mathrm{Id}_n & 0 \\ C & \mathrm{Id}_n \end{pmatrix} \middle| A \in \mathrm{GL}_n,\, C \in \mathfrak{gl}_n \right\}, \quad \psi\left( \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \mathrm{Id}_n & 0 \\ C & \mathrm{Id}_n \end{pmatrix} \right) = -\mathrm{tr}(C) \tag{1.13}$$

(see the list of examples of relative duality in [Wan]).

By Lemma 3.2, there is an isomorphism $\check{M} \simeq \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n$ such that the induced isomorphism $\check{M}/\mathrm{GL}_{2n} \simeq \mathfrak{gl}_n/\mathrm{GL}_n$ fits into a commutative diagram as follows.

$$\begin{array}{ccc} \check{M}/\mathrm{GL}_{2n} & \xrightarrow{\simeq} & \mathfrak{gl}_n/\mathrm{GL}_n \\ \downarrow \mu & & \downarrow \tau \\ \mathfrak{gl}_{2n}^*/\check{\mathrm{GL}}_{2n} & \xrightarrow{\simeq} & \mathfrak{gl}_{2n}/\mathrm{GL}_{2n} \end{array}$$

Here $\mu$ is the moment map, $\tau$ is the embedding in (1.2) (disregarding the cohomological grading), and we identify $\mathfrak{gl}_{2n}^*/\mathrm{GL}_{2n} \simeq \mathfrak{gl}_{2n}/\mathrm{GL}_{2n}$ via the trace form. Thus, in view of (1.12), the equivalence of Theorem 1.3 gives an instance of (1.11) of the form[3]

$$D^b(\mathfrak{L}X/\mathfrak{L}^+\mathrm{GL}_{2n}) \simeq \mathrm{Coh}(\check{M}/\mathrm{GL}_{2n}).$$

*Remark* 1.8. Our work suggests an interesting relationship between real groups and periods of automorphic forms associated to the corresponding symmetric varieties, and we plan to investigate this relationship in more detail. The case of the quaternionic group $\mathrm{GL}_n(\mathbb{H})$ is related to the so-called symplectic periods and Jacquet–Shalika periods [JR92, JS90].

### 1.7 IC-stalks and Kostka–Foulkes polynomials

As an application of the proof of Theorem 1.1, we determine the stalk cohomology of the IC-complexes for the $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$-orbit closures in the quaternionic affine Grassmannian $\mathrm{Gr}_{n,\mathbb{H}}$ and the $\mathfrak{L}\,\mathrm{Sp}_{2n}$-orbit closures in the complex affine Grassmannian $\mathrm{Gr}_{2n}$.

The $\mathfrak{L}^+\mathrm{GL}_n$-orbits (respectively, $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$ and $\mathfrak{L}\,\mathrm{Sp}_{2n}$-orbits) on $\mathrm{Gr}_n$ (respectively, $\mathrm{Gr}_{n,\mathbb{H}}$ and $\mathrm{Gr}_{2n}$) are in bijection with the set of dominant coweights $\Lambda_n^+$ of $\mathrm{GL}_n$; see § 4.4. For any $\lambda \in \Lambda_n^+$ we denote by $\mathrm{Gr}_n^\lambda$ (respectively, $\mathrm{Gr}_{n,\mathbb{H}}^\lambda$ and $\mathrm{Gr}_{2n,X}^\lambda$) the corresponding orbit and by $\mathrm{IC}(\overline{\mathrm{Gr}_n^\lambda})$ (respectively, $\mathrm{IC}(\overline{\mathrm{Gr}_{n,\mathbb{H}}^\lambda})$ and $\mathrm{IC}(\overline{\mathrm{Gr}_{2n,X}^\lambda})$) the intersection cohomology complex on the orbit closure. We will write $\mathscr{H}^i(\mathcal{F})$ for the $i$th cohomology sheaf of a complex $\mathcal{F}$ and $\mathscr{H}_x^i(\mathcal{F})$ for its stalk at a point $x$.

For any pair of dominant coweights $\lambda, \mu \in \Lambda_n^+$, we denote by $K_{\lambda,\mu}(q)$ the associated Kostka–Foulkes polynomial with variable $q$. Denote by $\rho_n$ the half-sum of positive roots of $\mathrm{GL}_n$. A well-known result of Lusztig [Lus81] says that we have $\mathscr{H}^{i-2\langle\lambda,\rho_n\rangle}(\mathrm{IC}(\overline{\mathrm{Gr}_n^\lambda})) = 0$ for $2 \nmid i$ and

$$\sum_i \dim \mathscr{H}_x^{2i-2\langle\lambda,\rho_n\rangle}(\mathrm{IC}(\overline{\mathrm{Gr}_n^\lambda}))q^i = q^{\langle\lambda-\mu,\rho_n\rangle} K_{\lambda,\mu}(q^{-1}) \quad \text{for } x \in \mathrm{Gr}_n^\mu.$$

[3]Here we have not been precise about cohomological degrees on the right-hand side.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

We have the following real and symmetric analogue.

THEOREM 1.9 (See Corollary 4.14 and Theorem 4.22). *Let* $\lambda,\mu\in\Lambda_n^+$. *For any* $x\in\mathrm{Gr}_{n,\mathbb{H}}^{\mu}$ *and* $y\in\mathrm{Gr}_{2n,X}^{\mu}$, *the following hold:*

(1) $\mathscr{H}^{i-4\langle\lambda,\rho_n\rangle}(\mathrm{IC}(\overline{\mathrm{Gr}_{n,\mathbb{H}}^{\lambda}}))=\mathscr{H}^{i-4\langle\lambda,\rho_n\rangle}(\mathrm{IC}(\overline{\mathrm{Gr}_{2n,X}^{\lambda}}))=0$ *for* $4\nmid i$;

(2) $\sum_i \dim\mathscr{H}_x^{4i-4\langle\lambda,\rho_n\rangle}(\mathrm{IC}(\overline{\mathrm{Gr}_{n,\mathbb{H}}^{\lambda}}))q^i=\sum_i \dim\mathscr{H}_y^{4i-4\langle\lambda,\rho_n\rangle}(\mathrm{IC}(\overline{\mathrm{Gr}_{2n,X}^{\lambda}}))q^i=q^{\langle\lambda-\mu,\rho_n\rangle}K_{\lambda,\mu}(q^{-1})$.

In other words, the theorem says that the IC-complex for the $\mathfrak{L}^+\mathrm{GL}_n(\mathbb{H})$ and $\mathfrak{L}K$-orbit closures $\overline{\mathrm{Gr}_{n,\mathbb{H}}^{\lambda}}$ and $\overline{\mathrm{Gr}_{2n,X}^{\lambda}}$ have the same stalk cohomology as the $\overline{\mathrm{Gr}_{n}^{\lambda}}$ ones for $\mathrm{GL}_n$, but with all degrees doubled.

*Remark* 1.10. To define the IC-stalk $\mathscr{H}_y^i\mathrm{IC}(\overline{\mathrm{Gr}_{2n,X}^{\lambda}})$ at $y\in\mathrm{Gr}_{2n,X}^{\mu}$, we use the observation that $\mathrm{Gr}_{2n,X}^{\mu}$ has finite codimension in $\overline{\mathrm{Gr}_{2n,X}^{\lambda}}$ and hence the IC-stalk makes sense. This can be made precise using the observation in [CN18] that the image $\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathrm{Gr}_{2n,X}^{\lambda}$ of the $\mathfrak{L}\,\mathrm{Sp}_{2n}$-orbits $\mathrm{Gr}_{2n,X}^{\lambda}$ in the quotient $\Omega\mathrm{Sp}(n)\backslash\mathrm{Gr}_{2n}$ is finite-dimensional with even real dimension and the collection $\{\mathfrak{L}\,\mathrm{Sp}_{2n}\backslash\mathrm{Gr}_{2n,X}^{\lambda}\}_{\lambda\in\Lambda_n^+}$ forms a nice stratification of $\Omega\mathrm{Sp}(n)\backslash\mathrm{Gr}_{2n}$. This allows us to define the IC-complex $\mathrm{IC}(\overline{\mathrm{Gr}_{2n,X}^{\lambda}})$ of $\overline{\mathrm{Gr}_{2n,X}^{\lambda}}$ (and hence the IC-stalks) as the IC-complex for the orbit closure $\Omega\mathrm{Sp}(n)\backslash\overline{\mathrm{Gr}_{2n,X}^{\lambda}}$ of $\Omega\mathrm{Sp}(n)\backslash\mathrm{Gr}_{2n,X}^{\lambda}$ inside $\Omega\mathrm{Sp}(n)\backslash\mathrm{Gr}_{2n}$.

*Remark* 1.11. We first prove Theorem 1.9 in the real case using the nice geometry of the quaternionic Grassmannian $\mathrm{Gr}_{n,\mathbb{H}}$ and then deduce the symmetric case via the real–symmetric correspondence. At the moment, we do not know a direct argument in the symmetric case.

### 1.8 Outline of the proof

We briefly explain the proof of Theorem 1.1. Similar to the proof of the derived Satake for complex reductive groups [BF08], the desired equivalence follows from the following two statements: (1) the de-equivariantized extension algebra $\mathrm{Ext}^*_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n))$ is isomorphic to the dg symmetric algebra $\mathrm{Sym}(\mathfrak{gl}_n[-4])$ (see Proposition 4.21) and (2) the dg algebra $\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n))$ is formal (see Proposition 5.3).

We deduce (1) from a fully-faithfulness property of the equivariant cohomology functor. In [BF08], this was proved using a general result of Ginzburg [Gin91], whose proof uses Hodge theory and hence does not apply directly to the real analytic setting. We observe that in the situation of the quaternionic affine Grassmannian, the stalks of the IC-complexes satisfy a parity vanishing property and, as observed in [AR15], one can use parity considerations in place of Hodge theory. To prove the parity vanishing result of the IC-stalks, we show that fibers of certain convolution Grassmannians (which are basically quaternionic Springer fibers) admit pavings by quaternionic affine spaces.

The proof of (2) in [BF08] also relies on Hodge theory (or theory of weights) and hence must be modified in the real setting. We observe that the nearby cycles functor (1.1) induces a surjective homomorphism from the dg-algebra $\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_{2n}))$ associated to the complex affine Grassimannian $\mathrm{Gr}_{2n}$ to the dg-algebra $\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n))$. Since the former dg-algebra is formal, thanks to [BF08], the desired claim follows from a general criterion of formality; see Lemma 5.4.

*Remark* 1.12. We expect that the proof strategy outlined above is applicable to general real groups: the parity vanishing, fully-faithfulness, and formality results should hold in general.

### 1.9 Organization

We briefly summarize the main goals of each section. In § 2, we recall some notation and results on constructible sheaves on a semi-analytic stack. In § 3, we study the spectral side of the quaternionic Satake equivalence, including results on quaternionic groups, symplectic groups, regular centralizers group schemes, and Whittaker reduction. In § 4, we study the constructible side of the equivalence, including the study of nearby cycles functors, parity vanishing results, fully-faithfulness of the equivariant cohomology functor, and the computation of the IC-stalks and the de-equivariantized extension algebra. Finally, in § 5, we prove the formality result and deduce the derived Satake equivalence for quaternionic groups, including a version involving nilpotent singular supports (Theorem 5.5), and also the spectral description of the nearby cycles functor (Theorem 5.7).

## 2. Constructible sheaves on a semi-analytic stack

We will work with $\mathbb{C}$-linear dg-categories (see e.g. [GR17, Chapter 1, § 10] for a concise summary of the theory of dg-categories). Unless specified otherwise, all dg-categories will be assumed to be cocomplete, i.e. containing all small colimits, and all functors between dg-categories will be assumed to be continuous, i.e. preserving all small colimits.

We collect some facts about constructible sheaves on a semi-analytic stack. Recall that a subset $Y$ of a real analytic manifold $M$ is said to be semi-analytic if any point $y \in Y$ has a open neighbourhood $U$ such that the intersection $Y \cap U$ is a finite union of sets of the form

$$\{y \in U \mid f_1(y) = \cdots = f_r(y) = 0, g_1(y) > 0, \ldots, g_l(y) > 0\},$$

where the $f_i$ and $g_j$ are real analytic functions on $U$. A map $f: Y \to Y'$ between two semi-analytic sets is said to be semi-analytic if it is continuous and its graph is a semi-analytic set.

Let Grpd be the $\infty$-category of spaces or, equivalently, $\infty$-groupoids. Let RSp be the site of semi-analytic sets where the coverings are étale (i.e. locally bi-analytic) maps $\{S_i \to S\}_{i \in I}$ such that the map $\bigsqcup S_i \to S$ is surjective. A semi-analytic pre-stack is a functor $\mathcal{Y}: \mathrm{RSp} \to \mathrm{Grpd}$, and a semi-analytic stack is a pre-stack that is a sheaf. We will view any semi-analytic set as a semi-analytic stack via the Yoneda embedding.

For any semi-analytic set $Y$, we define $D(Y) = \mathrm{Ind}(D^b(Y))$ to be the ind-completion of the bounded dg-category $D^b(Y)$ of $\mathbb{C}$-constructible sheaves on $Y$. For any semi-analytic stack $\mathcal{Y}$ we define $D(\mathcal{Y}) := \lim_I D(Y)$, where the index category is that of semi-analytic sets equipped with a semi-analytic map to $\mathcal{Y}$ and the transition functors are given by !-pullback. Since we are in the constructible context, !-pullback admits a left adjoint, given by !-pushforward, and it follows that $D(\mathcal{Y}) = \mathrm{colim}_I D(Y)$. In particular, $D(\mathcal{Y})$ is compactly generated. We let $D(\mathcal{Y})^c$ be the full subcategory of compact objects and $D^b(\mathcal{Y}) \subset D(\mathcal{Y})$ the full subcategory of objects that pull back to an object of $D^b(Y)$ for any $Y$ mapping to $\mathcal{Y}$. Note that we have the natural inclusion $D(\mathcal{Y})^c \subset D^b(\mathcal{Y})$, but the inclusion is in general not an equality. For example, the constant sheaf $\mathbb{C}_{\mathcal{Y}} \in D^b(\mathcal{Y})$ for the classifying stack $\mathcal{Y} = B(\mathrm{GL}_1(\mathbb{C}))$ is not compact.

Let $f: \mathcal{Y} \to \mathcal{Y}'$ be a map between semi-analytic stacks. We have the usual six-functor formalism $f_*$, $f^!$, $f_*$, $f_!$, $\otimes$, $\underline{\mathrm{Hom}}$.

For a semi-analytic stack $\mathcal{Y}$, with projection map $p: \mathcal{Y} \to \mathrm{pt}$, and $\mathcal{F} \in D(Y)$, we will write $H^*(Y, \mathcal{F}) := p_*(\mathcal{F})$ for the cohomology of $\mathcal{F}$. If $\mathcal{Y}$ is isomorphic to the quotient stack $\mathcal{Y} \simeq G\backslash Y$, where $Y$ is a semi-analytic set acted on real analytically by a Lie group $G$, we will write $H^*_G(Y, \mathcal{F}) := (p_{BG})_*(\mathcal{F})$ for the equivariant cohomology of $\mathcal{F}$, where $p_{BG}: \mathcal{Y} = G\backslash Y \to BG$ is the projection map. When it is clear from the context we will abbreviate $H^*(Y, \mathcal{F})$ and $H^*_G(Y, \mathcal{F})$ by $H^*(\mathcal{F})$ and $H^*_G(\mathcal{F})$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

For an ind semi-analytic stack $\mathcal{Y}=\operatorname{colim}_I\mathcal{Y}_i$ we define $D(\mathcal{Y})=\lim_I D(\mathcal{Y}_i)$, where the limit is taken with respect to the !-pull-back along the closed embedding $\iota_{i,i'}:\mathcal{Y}_i\to\mathcal{Y}_{i'}$, $i,i'\in I$.

## 3. Spectral side

### 3.1 Quaternionic group

For any positive integer $n$, we denote by $G_n=\mathrm{GL}_n(\mathbb{C})$ the complex Lie group of $n\times n$ invertible matrices and $\mathfrak{g}_n=\mathfrak{gl}_n(\mathbb{C})$ its Lie algebra of $n\times n$ matrices. We write $B_n$, $N_n$, and $T_n$ for the subgroups of $G_n$ consisting of upper triangular matrices, upper triangular unipotent matrices, and diagonal matrices, respectively, and write $\mathfrak{b}_n$, $\mathfrak{n}_n$, and $\mathfrak{t}_n$ for their Lie algebras. We denote by $\mathrm{W}_n$ the Weyl group of $G_n$ acting on $\mathfrak{t}_n$ by the permutation action. We let $\mathfrak{c}_n=\mathfrak{t}_n//\mathrm{W}_n$. We will identify $\mathfrak{c}_n$ with the space of degree $n$ monic polynomials in such a way that under the above identification, the Chevalley map $\chi_n:\mathfrak{g}_n\to\mathfrak{g}_n//G_n\simeq\mathfrak{c}_n$ becomes the map sending a matrix to its characteristic polynomial. We will fix the coordinates $(c_1,\dots,c_n)$ on $\mathfrak{c}_n$ given by the coefficients of a degree $n$ monic polynomial listed in increasing degree. We will identify $\mathfrak{g}_n\simeq\mathfrak{g}_n^*$ using the trace paring $\mathfrak{g}_n\times\mathfrak{g}_n\to\mathbb{C}$, $(A,B)\mapsto\operatorname{tr}(AB)$.

Let $\mathbb{H}=\{a+ib+jc+kd\}$ denote the quaternions. Consider the quaternionic vector space $\mathbb{H}^n$ where $\mathbb{H}$ acts via right multiplication. Let $G_{n,\mathbb{H}}$ be the Lie group of $\mathbb{H}$-linear invertible endomorphisms of $\mathbb{H}^n$, which can be identified with the space $\mathrm{GL}_n(\mathbb{H})$ of $n\times n$ invertible quaternionic matrices, and let $\mathfrak{g}_{n,\mathbb{H}}$ be the Lie algebra of $\mathbb{H}$-linear endomorphisms of $\mathbb{H}^n$, which can be identified with the space $\mathfrak{gl}_n(\mathbb{H})$ of $n\times n$ quaternionic matrices.

Using the identification $\mathbb{C}^{2n}\simeq\mathbb{H}^n$ sending $(z,w)\to q=z+jw$ for $z,w\in\mathbb{C}^n$, one can realize $G_{n,\mathbb{H}}$ as a real form of $G_n$. Specifically, the endomorphism of $\mathbb{H}^n$ sending $q\to qj$ corresponds to the endomorphism of $\mathbb{C}^{2n}$ sending

$$(z,w)\to(-\bar{w},\bar{z}),\tag{3.1}$$

and we can identify $\mathfrak{g}_{n,\mathbb{H}}$ and $G_{n,\mathbb{H}}$ as the subsets of $\mathfrak{g}_n$ and $G_n$ consisting of $\mathbb{C}$-linear endomorphisms of $\mathbb{C}^{2n}$ that commute with the map (3.1). Equivalently, consider the $2n\times 2n$ matrix

$$S_n=\begin{pmatrix}0 & -\mathrm{Id}_n\\ \mathrm{Id}_n & 0\end{pmatrix}$$

where $\mathrm{Id}_n$ is the $n\times n$ identity matrix. Then the endomorphism $\eta$ of $\mathfrak{g}_{2n}$ (respectively, $G_{2n}$) sending $X\in\mathfrak{g}_{2n}$ (respectively, $X\in G_{2n}$) to

$$\eta(X)=S_n\overline{X}S_n^{-1}$$

defines a real form of $\mathfrak{g}_{2n}$ (respectively, $G_{2n}$), that is, an anti-holomorphic conjugation on $\mathfrak{g}_{2n}$ (respectively, $G_{2n}$), and $\mathfrak{g}_{n,\mathbb{H}}$ and $G_{n,\mathbb{H}}$ are the $\eta$-fixed points in $\mathfrak{g}_{2n}$ and $G_{2n}$. Concretely, $\mathfrak{g}_{n,\mathbb{H}}$ (respectively, $G_{n,\mathbb{H}}$) consists of $2n\times 2n$ matrices (respectively, invertible matrices) of the form

$$\begin{pmatrix}A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$$

where $A,B\in\mathfrak{g}_n$.

We denote by $\mathfrak{t}_{n,\mathbb{H}}\subset\mathfrak{g}_{n,\mathbb{H}}$ (respectively, $T_{n,\mathbb{H}}\subset G_{n,\mathbb{H}}$) the Cartan subalgebra (respectively, Cartan subgroup) consisting of matrices (respectively, invertible matrices)

$$\begin{pmatrix}A & 0\\ 0 & \overline{A}\end{pmatrix}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

where $A \in \mathfrak{t}_n$.

We denote by $P_{n,\mathbb{H}} = M_{n,\mathbb{H}} A_{n,\mathbb{H}} N_{n,\mathbb{H}}$ the standard minimal parabolic subgroup of $G_{n,\mathbb{H}}$ consisting of invertible upper triangular quaternionic matrices and $\mathfrak{p}_{n,\mathbb{H}} = \mathfrak{m}_{n,\mathbb{H}} \oplus \mathfrak{a}_{n,\mathbb{H}} \oplus \mathfrak{n}_{n,\mathbb{H}}$ its Lie algebra.

### 3.2 Symplectic group

According to the Cartan classification of real forms, the conjugation $\eta$ corresponds to a holomorphic involution $\theta$ on $G_{2n}$ (respectively, $\mathbb{C}$-linear involution of $\mathfrak{g}_{2n}$) characterized by the property that $\eta \circ \theta = \theta \circ \eta$ is a compact real form, that is, the fixed-point subgroup (respectively, subalgebra) of $\eta_c := \eta \circ \theta$,

$$G_c = (G_{2n})^{\eta_c} \quad \text{(respectively, } \mathfrak{g}_c = (\mathfrak{g}_{2n})^{\eta_c}),$$

is compact. In our case, we will take $\theta$ to be

$$\theta(X) = S_n (X^t)^{-1} S_n^{-1} \quad \text{(respectively, } \theta(X) = -S_n (X^t) S_n^{-1}),$$

where $X \in G_{2n}$ (respectively, $X \in \mathfrak{g}_{2n}$), and we have

$$\eta_c(X) = (\overline{X}^t)^{-1} \quad \text{(respectively, } \theta(X) = -\overline{X}^t),$$

and the corresponding compact subgroup $G_c = (G_{2n})^{\eta_c}$ is the group of $2n \times 2n$ unitary matrices.

The $\theta$-fixed-point subgroup $K = (G_{2n})^\theta = \mathrm{Sp}_{2n}$ is the symplectic group of rank $n$, and the intersection

$$K_c := \mathrm{Sp}_{2n} \cap G_c = \mathrm{Sp}(n)$$

is the compact symplectic group. The Lie algebras $\mathfrak{k}$ and $\mathfrak{k}_c$ consist of matrices

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

where $A, B, C \in \mathfrak{g}_n$ satisfy $B = B^t$ and $C = C^t$ for $\mathfrak{k}$ and the additional conditions $A = -\bar{A}^t$ and $C = -\overline{B}$ for $\mathfrak{k}_c$.

Recall the Cartan decompostion of the Lie algebra $\mathfrak{g}_{2n} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the $(-1)$-eigenspace of $\theta$. The Cartan decompostion induces a decomposition $\mathfrak{t}_{2n} = \mathfrak{t} \oplus \mathfrak{s}$ where $\mathfrak{t} = \mathfrak{t}_{2n} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ consisting of diagonal matrices of the form

$$\mathrm{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n)$$

and $\mathfrak{s} = \mathfrak{t}_{2n} \cap \mathfrak{p} \subset \mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}$ consisting of diagonal matrices of the form

$$\mathrm{diag}(s_1, \ldots, s_n, s_1, \ldots, s_n).$$

We denote by W the Weyl group of $K$ and $\mathrm{W}_\mathfrak{s} = N_K(\mathfrak{s})/Z_K(\mathfrak{s})$ the little Weyl group. We have $\mathrm{W} \simeq \mathrm{W}_n \ltimes \{\pm 1\}^n$ and $\mathrm{W}_\mathfrak{s} \simeq \mathrm{W}_n$. We let $\mathfrak{c} = \mathfrak{t}//\mathrm{W}$. Then the natural inclusion $\mathfrak{t} \to \mathfrak{t}_{2n}$ gives rise to an embedding

$$\mathfrak{c} = \mathfrak{t}//\mathrm{W} \longrightarrow \mathfrak{t}_{2n}//\mathrm{W}_{2n} = \mathfrak{c}_{2n} \tag{3.2}$$

whose image consists of monic polynomials of degree $2n$ with vanishing odd terms.

Finally, we denote by $X = G_{2n}/K$ and $X_c = G_c/K_c$ the symmetric space and compact symmetric space associated to $K$ and $K_c$.

### 3.3 Notation related to root structure

Let $\Lambda_n = \mathrm{Hom}(\mathbb{C}^\times, T_n)$ be the coweight lattice of $T_n$ and let $\Lambda_n^+$ be the set of dominant coweights with respect to $B_n$. Let $2\rho_n$ be the sum of the positive roots of $G_n$ and let $\langle \lambda, 2\rho_n \rangle \in \mathbb{Z}$ be the natural paring for a coweight $\lambda \in \Lambda_n$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Let $S\subset T$ be the maximal split torus corresponding to the maximal abelian subspace $\mathfrak{s}\subset\mathfrak{p}$, let $\Lambda_S=\mathrm{Hom}(\mathbb{C}^\times,S)$ be the set of real coweights, and let $\Lambda_S^+=\Lambda_S\cap\Lambda_{2n}^+$ be the set of dominant real coweights. There is a natural identification $S\simeq T_n$ sending $\mathrm{diag}(s_1,\dots s_n,s_1,\dots,s_n)$ to $\mathrm{diag}(s_1,\dots,s_n)$ and hence natural identifications $\Lambda_S\simeq\Lambda_n$ and $\Lambda_S^+\simeq\Lambda_n^+$.

### 3.4 Regular centralizers

3.4.1. Recall the group scheme of centralizers $I_n\to\mathfrak{g}_n$ whose fiber over $x\in\mathfrak{g}_n$ is the centralizer $(G_n)^x=\{g\in G_n\mid \mathrm{Ad}_g(x)=x\}$ of $x$ in $G_n$. Let $\mathfrak{g}_n^{\mathrm{reg}}\subset\mathfrak{g}_n$ be the open subset of regular elements in $\mathfrak{g}_n$. It is shown in [Ngô06, § 3] that the base-change $I_n|_{\mathfrak{g}_n^{\mathrm{reg}}}\to\mathfrak{g}_n^{\mathrm{reg}}$ is a smooth group scheme over $\mathfrak{g}_n^{\mathrm{reg}}$ which descends to a smooth group scheme $J_n\to\mathfrak{c}_n$ over $\mathfrak{c}_n$, known as the group scheme of regular centralizers.

3.4.2. Consider the embedding

$$\tau:\mathfrak{g}_n\to\mathfrak{g}_{2n},\quad \tau(C)=\begin{pmatrix}0 & \mathrm{Id}_n\\ C & 0\end{pmatrix}. \tag{3.3}$$

Note that the map $\tau$ is $G_n$-equivariant, where $G_n$ acts on $\mathfrak{g}_{2n}$ via diagonal embedding $\delta:G_n\to G_{2n}$. Thus it induces an embedding on the invariant quotients (denoted again by $\tau$),

$$\tau:\mathfrak{c}_n=\mathfrak{g}_n//G_n\to\mathfrak{g}_{2n}//G_{2n}\simeq\mathfrak{c}_{2n},\quad \tau(c_1,\dots,c_n)=(0,c_1,0,c_2,\dots,0,c_n), \tag{3.4}$$

whose image consists of monic polynomials of degree $2n$ with vanishing odd terms. Note that the image of $\tau$ is equal to the image of the map $\mathfrak{c}=\mathfrak{t}//\mathrm{W}\to\mathfrak{c}_{2n}=\mathfrak{t}_{2n}//\mathrm{W}_{2n}$ in (3.2), and hence there is a natural identification

$$\mathfrak{c}\simeq\mathfrak{c}_n \tag{3.5}$$

such that $\tau:\mathfrak{c}_n\simeq\mathfrak{c}\to\mathfrak{c}_{2n}$.

Recall the group scheme of centralizers $I_n$ (respectively, $I_{2n}$) over $\mathfrak{g}_n$ (respectively, $\mathfrak{g}_{2n}$) and the group scheme of regular centralizers $J_n$ (respectively, $J_{2n}$) over $\mathfrak{c}_n$ (respectively, $\mathfrak{c}_{2n}$).

Lemma 3.1. *There is a natural closed embedding of group schemes, $J_n\to J_{2n}$,* that *fits into a commutative diagram*

$$\begin{array}{ccc} J_n & \longrightarrow & J_{2n}\\ \downarrow & & \downarrow\\ \mathfrak{c}_n & \longrightarrow & \mathfrak{c}_{2n}\end{array} \tag{3.6}$$

*where the bottom arrow is the map in (3.4).*

*Proof.* We first claim that $\tau(\mathfrak{g}_n^{\mathrm{reg}})=\mathfrak{g}_{2n}^{\mathrm{reg}}\cap\tau(\mathfrak{g}_n)$. Let $x=\tau(C)=\begin{pmatrix}0 & \mathrm{Id}_n\\ C & 0\end{pmatrix}$. If $x$ is in $\mathfrak{g}_{2n}^{\mathrm{reg}}$, then the centralizer $(G_{2n})^x$ of $x$ in $G_{2n}$ is abelian. By direct calculation, the inclusion $G_n\to G_{2n}$ takes the centralizer $(G_n)^C$ of $C$ in $G_n$ into the centralizer $(G_{2n})^x$, and hence $(G_n)^C$ is also abelian. Hence the characterization of regular elements for $\mathfrak{g}_n$ implies that $C\in\mathfrak{g}_n^{\mathrm{reg}}$. On the other hand, if $C\in\mathfrak{g}_n^{\mathrm{reg}}$, then without loss of generality we can assume that $C$ is a companion matrix, and an easy computation shows that $x$ is in $\mathfrak{g}_{2n}^{\mathrm{reg}}$ (see (3.8)). The claim follows.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Let $I_n^{\mathrm{reg}} = I_n|_{\mathfrak{g}_n^{\mathrm{reg}}}$ and $I_{2n}^{\mathrm{reg}} = I_{2n}|_{\mathfrak{g}_{2n}^{\mathrm{reg}}}$. Then the claim implies that we have a commutative diagram as follows.

$$\begin{array}{ccc} I_n^{\mathrm{reg}} & \longrightarrow & I_{2n}^{\mathrm{reg}} \\ \downarrow & & \downarrow \\ \mathfrak{g}_n^{\mathrm{reg}} & \xrightarrow{\tau} & \mathfrak{g}_{2n}^{\mathrm{reg}} \end{array} \tag{3.7}$$

Since $J_{2n} \simeq I_{2n}^{\mathrm{reg}}//G_{2n}$ is the descent of $I_{2n}^{\mathrm{reg}}$ along the map $\mathfrak{g}_{2n}^{\mathrm{reg}} \to \mathfrak{c}_{2n}$, the restriction $J_{2n}|_{\mathfrak{c}_n}$ is the descent of $I_{2n}^{\mathrm{reg}}|_{\mathfrak{g}_n^{\mathrm{reg}}}$ along the map $\mathfrak{g}_n^{\mathrm{reg}} \to \mathfrak{c}_n$:

$$J_{2n}|_{\mathfrak{c}_n} \simeq (I_{2n}^{\mathrm{reg}}//G_{2n})|_{\mathfrak{c}_n} \simeq I_{2n}^{\mathrm{reg}}|_{\mathfrak{g}_n^{\mathrm{reg}}}//G_n.$$

Since the maps in (3.7) are compatible with the natural $G_n$-action and the desired map is the map on the GIT quotients,

$$J_n \simeq I_n^{\mathrm{reg}}//G_n \longrightarrow I_{2n}^{\mathrm{reg}}|_{\mathfrak{g}_n^{\mathrm{reg}}}//G_n \simeq J_{2n}|_{\mathfrak{c}_n}.$$

□

3.4.3 *Kostant sections.* We give an alternative construction of the map $J_n \to J_{2n}$ in (3.6) using Kostant sections.

Consider the following two ordered bases of $\mathbb{C}^{2n}$: the standard basis $\{e_1 = (1, 0, \ldots, 0), \ldots, e_{2n} = (0, \ldots, 0, 1)\}$ and the basis $\{w_1 = e_1, w_2 = e_3, \ldots, w_n = e_{2n-1}, w_{n+1} = e_2, w_{n+2} = e_4, \ldots, w_{2n} = e_{2n}\}$. Let $P \in G_{2n}$ be the matrix associated to the linear map $w_i \mapsto e_i$ in the basis $w_1, \ldots, w_{2n}$.

For any positive integer $s$, consider the Kostant section $\kappa_s : \mathfrak{c}_s \to \mathfrak{g}_s$ for $G_s$ given by

$$\kappa_s(c) = \begin{pmatrix} 0 & 1 & & \\ \vdots & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -c_s & -c_{s-1} & \ldots & -c_1 \end{pmatrix}, \quad c = (c_1, \ldots, c_s) \in \mathfrak{c}_s.$$

A direct computation shows that

$$\begin{pmatrix} 0 & 1 & & \\ \vdots & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -c_{2n} & -c_{2n-1} & \ldots & -c_1 \end{pmatrix} = P \begin{pmatrix} 0 & \mathrm{Id}_n \\ C & D \end{pmatrix} P^{-1}$$

where

$$C = \begin{pmatrix} 0 & 1 & & \\ \vdots & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -c_{2n} & -c_{2n-2} & \ldots & -c_2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ -c_{2n-1} & -c_{2n-3} & \ldots & -c_1 \end{pmatrix}.$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

It follows that for any $\mathfrak{c} = (c_1, \dots, c_n) \in \mathfrak{c}_n$ with $\tau(c) = (0, c_1, 0, c_2, \dots, 0, c_n) \in \mathfrak{c}_{2n}$, we have

$$\begin{pmatrix} 0 & 1 & \dots & \dots & \dots & \dots & 0 \\ \vdots & 0 & \ddots & & & & \\ \vdots & & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & \ddots & & 1 \\ -c_n & 0 & -c_{n-1} & 0 & \dots & 0 & -c_1 \end{pmatrix} = P \begin{pmatrix} 0 & \mathrm{Id}_n \\ \kappa_n(c) & 0 \end{pmatrix} P^{-1}. \tag{3.8}$$

Thus there is the following commutative diagram.

$$\begin{array}{ccc} \mathfrak{g}_n^{\mathrm{reg}} & \xrightarrow{\tau} & \mathfrak{g}_{2n}^{\mathrm{reg}} \\ {\scriptstyle \kappa_n}\uparrow & & \uparrow{\scriptstyle \mathrm{Ad}_P^{-1} \circ \kappa_{2n}} \\ \mathfrak{c}_n & \xrightarrow{\tau} & \mathfrak{c}_{2n} \end{array} \tag{3.9}$$

In particular, we have

$$\tau \circ \kappa_n : \mathfrak{c}_n \to \mathfrak{g}_{2n}^{\mathrm{reg}}.$$

The pullback of the group scheme $I_{2n}^{\mathrm{reg}}$ along the map above $\tau \circ \kappa_n$ is isomorphic to

$$(\tau \circ \kappa_n)^*(I_{2n}^{\mathrm{reg}}) \simeq ((\kappa_{2n})^* \mathrm{Ad}^*_{P^{-1}}(I_{2n}^{\mathrm{reg}}))|_{\mathfrak{c}_n} \simeq (\kappa_{2n})^*(I_{2n}^{\mathrm{reg}})|_{\mathfrak{c}_n} \simeq J_{2n}|_{\mathfrak{c}_n},$$

and the desired map is given by pullback of (3.7) along the map $\tau \circ \kappa_n$:

$$J_n \simeq \kappa_n^*(I_n^{\mathrm{reg}}) \longrightarrow \kappa_n^*(I_{2n}^{\mathrm{reg}}|_{\mathfrak{g}_n^{\mathrm{reg}}}) \simeq (\tau \circ \kappa_n)^*(I_{2n}^{\mathrm{reg}}) \simeq J_{2n}|_{\mathfrak{c}_n}. \tag{3.10}$$

3.4.4. The identification $\mathfrak{c} \simeq \mathfrak{c}_n$ in (3.5) gives rise to a map

$$\mathfrak{t} \to \mathfrak{c} \simeq \mathfrak{c}_n$$

sending $(t_1, \dots, t_n, -t_1, \dots, -t_n) \in \mathfrak{t}$ to the coefficients $(c_1, \dots, c_n) \in \mathfrak{c}_n$ of the monic polynomial $f(x) = \prod_{i=1}^n (x - t_i^2)$ of degree $n$. We shall give a description of the pullback

$$J_n \times_{\mathfrak{c}_n} \mathfrak{t} \to J_{2n} \times_{\mathfrak{c}_{2n}} \mathfrak{t} \tag{3.11}$$

of (3.6) along $\mathfrak{t} \to \mathfrak{c}_n$. Consider the map

$$e^{T_{2n}} : \mathfrak{t}_{2n} \to \mathfrak{g}_{2n}, \quad e^{T_{2n}}(t) = \begin{pmatrix} t_1 & 1 & & \\ 0 & t_2 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \dots & 0 & t_{2n} \end{pmatrix}, \quad t = \mathrm{diag}(t_1, \dots, t_{2n}). \tag{3.12}$$

Note that the image of $e^{T_{2n}}$ consists of regular elements. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{t}_{2n} & \xrightarrow{e^{T_{2n}}} & \mathfrak{g}_{2n} \\ \downarrow & & \downarrow \\ \mathfrak{c}_{2n} & \longrightarrow & \mathfrak{c}_{2n} \end{array}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

where the vertical arrows are the natural adjoint quotient maps. If follows that there is a canonical isomorphism

$$J_{2n} \times_{\mathfrak{c}_{2n}} \mathfrak{t}_{2n} \simeq (e^T)^* I_{2n} = (G_{2n} \times \mathfrak{t}_{2n})^{e^{T_{2n}}}$$

where $(G_{2n} \times \mathfrak{t}_{2n})^{e^{T_{2n}}} = \{(g,t) \in G_{2n} \times \mathfrak{t}_{2n} \mid \mathrm{Ad}_g(e^{T_{2n}}(t)) = e^{T_{2n}}(t)\}$ denotes the subgroup scheme of the constant group scheme $G_{2n} \times \mathfrak{t}_{2n}$ over $\mathfrak{t}_{2n}$ of centralizers of the section $e^{T_{2n}}$.

Consider the restriction $e^T = e^{T_{2n}}|_{\mathfrak{t}} : \mathfrak{t} \to \mathfrak{g}_{2n}$. Concretely, we have

$$e^T : \mathfrak{t} \to \mathfrak{g}_{2n}, \quad e^T(t) = \begin{pmatrix} t_1 & 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & t_n & \ddots & & \vdots \\ \vdots & & \ddots & -t_1 & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & -t_n \end{pmatrix}, \quad t = \mathrm{diag}(t_1, \dots, t_n, -t_1, \dots, -t_{2n}). \tag{3.13}$$

It is clear that

$$J_{2n} \times_{\mathfrak{c}_{2n}} \mathfrak{t} \simeq (e^T)^* I_{2n} = (G_{2n} \times \mathfrak{t})^{e^T}.$$

Consider the map

$$e^T_X : \mathfrak{t} \to \mathfrak{g}_n, \quad e^T_X(t) = \begin{pmatrix} t_1^2 & 1 & & \\ \vdots & t_2^2 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & 0 & \dots & t_n^2 \end{pmatrix}. \tag{3.14}$$

The image of $e^T_X$ consists of regular elements, and we have the following commutative diagram.

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{e^T_X} & \mathfrak{g}_n \\ \downarrow & & \downarrow \\ \mathfrak{c}_n & \xrightarrow{id} & \mathfrak{c}_n \end{array}$$

It follows that we have a canonical isomorphism

$$J_n \times_{\mathfrak{c}_n} \mathfrak{t} \simeq (e^T_X)^* I_n = (G_n \times \mathfrak{t})^{e^T_X}$$

of group schemes over $\mathfrak{t}$. For any $t \in \mathfrak{t}$, we have

$$\tau \circ e^T_X(t) = \begin{pmatrix} 0 & \mathrm{Id}_n \\ C & 0 \end{pmatrix}, \quad C = \begin{pmatrix} t_1^2 & 1 & & \\ \vdots & t_2^2 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & 0 & \dots & t_n^2 \end{pmatrix}. \tag{3.15}$$

Note that the elements $\tau \circ e^T_X(t)$ and $e^T(t)$ are regular and have the same characteristic polynomial and hence lie in the same $G_{2n}$-orbit. Pick an element $g_t \in G_{2n}$ such that

$$e^T(t) = g_t(\tau \circ e^T_X(t))g_t^{-1}.$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Then the conjugation map $\mathrm{Ad}_{g_t}: G_{2n} \to G_{2n}, g \to g_t g g_t^{-1}$ restricts to a map between the centralizers,

$$(G_n)^{e_X^T(t)} \xrightarrow{\delta} (G_{2n})^{e^T(t)} \xrightarrow{\mathrm{Ad}_{g_t}} (G_{2n})^{e^T(t)}.$$

Since centralizers of a regular element form a commutative group, the map above is independent of the choice of the element $g_t$ and hence is canonical. Then, as $t$ varies over $\mathfrak{t}$, we obtain a map between the corresponding centralizer group schemes,

$$J_n \times_{\mathfrak{c}_n} \mathfrak{t} \simeq (G_n \times \mathfrak{t})^{e_X^T} \to (G_{2n} \times \mathfrak{t})^{e^T} \simeq J_{2n} \times_{\mathfrak{c}_X} \mathfrak{t}, \tag{3.16}$$

which is the map in (3.11).

Alternatively, the assignment $t \to g_t$ gives rise to an element

$$\Phi \in G_{2n} \otimes R_T, \quad \Phi(t) = g_t,$$

where we set $R_T = \mathcal{O}(\mathfrak{t})$. If we regard the maps $e^T$ and $\tau \circ e_X^T$ as elements in $\mathfrak{g}_{2n} \otimes R_T$, we have

$$e^T = \Phi(\tau \circ e_X^T)\Phi^{-1} \in \mathfrak{g}_{2n} \otimes R_T \tag{3.17}$$

(we will give a canonical construction of the element $\Phi$; see Remark 5.2). Then the composition

$$\mathrm{Ad}_\Phi \circ \delta : G_n \times \mathfrak{t} \to G_{2n} \times \mathfrak{t}, \quad (g,t) \to \Phi(t)(\delta(g))\Phi^{-1}(t)$$

restricts to the map (3.16) between the corresponding centralizer group schemes.

### 3.5 Dual group

In [Nad05], the author associated to each real form $G_\mathbb{R}$ of a complex reductive group $G$, or equivalently a symmetric space $X$ of $G$, a complex reductive group $\check{G}_X$ together with a homomorphism $\delta : \check{G}_X \to \check{G}$. In the case of $G = G_{2n}$ and $G_\mathbb{R} = G_{n,\mathbb{H}}$, or equivalently $X = G_{2n}/\mathrm{Sp}_{2n}$, we have $\check{G} = G_{2n}$ and $\check{G}_X = G_n$, and the homomorphism is the diagonal embedding

$$\delta : G_n \to G_{2n}, \quad \delta(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}. \tag{3.18}$$

Let $P = LN$ be the complexification of the minimal parabolic $P_{n,\mathbb{H}}$. The Levi subgroup $L$ consists of matrices of the form

$$L = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{2n} \;\middle|\; A, B, C, \text{and } D \text{ are diagonal matrices} \right\}.$$

Consider the principal $\mathrm{SL}_2$ of $L$ given by

$$\psi : \mathrm{SL}_2 \to L, \quad \psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = a\mathrm{Id}_n$, $B = b\mathrm{Id}_n$, etc. The restriction of $\psi$ to the torus $\mathbb{G}_m \subset \mathrm{SL}_2$ is the co-character

$$2\rho_L : \mathbb{G}_m \to L, \quad 2\rho_L(h) = \mathrm{diag}(h, \ldots, h, h^{-1}, \ldots, h^{-1})$$

corresponding to the sum of the positive roots of the Levi factor $L$. A direct computation shows that the image $\psi(\mathrm{SL}_2) \subset G_{2n}$ centralizes the subgroup $\delta(G_n) \subset G_{2n}$, and hence we obtain a homomorphism

$$\psi_X : \check{G}_X \times \mathrm{SL}_2 \to G_{2n}, \quad \psi_X(g,y) \to \delta(g)\psi(y). \tag{3.19}$$

### 3.6 The partial Whittaker reduction

Consider the identification $T^*G_{2n} \simeq G_{2n} \times \mathfrak{g}_{2n}^*$ by considering $\mathfrak{g}_{2n}^*$ as left-invariant differential forms on $G_{2n}$. The group $G_{2n} \times G_{2n}$ acts on $G_{2n}$ via left and right multiplication, and the induced action on $T^*G_{2n} \simeq G_{2n} \times \mathfrak{g}_{2n}^*$ is given by $(g,h)(x,v) = (gxh^{-1}, \mathrm{Ad}_h v)$. The moment map $(\mu_l, \mu_r) : T^*G_{2n} \to \mathfrak{g}_{2n}^* \times \mathfrak{g}_{2n}^*$ with respect to the $G_{2n} \times G_{2n}$-action is given by $(\mu_l, \mu_r)(x,v) = (\mathrm{Ad}_x v, -v)$.

Consider the Shalika subgroup $G_n \ltimes U$ and the generic morphism $\psi$ in (1.13). Let $\mathfrak{g}_n \times \mathfrak{u}$ be the Lie algebra of $G_n \ltimes U$. Then one can view $\psi$ as an element $\psi = (0, -\mathrm{tr})$ in $\mathfrak{g}_n^* \times \mathfrak{u}^*$:

$$\psi\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\right) = -\mathrm{tr}(C).$$

The moment map for the right $G_n \ltimes U$-action on $T^*G_{2n}$ is given by

$$\mu : T^*G_{2n} \xrightarrow{\mu_r} \mathfrak{g}_{2n}^* \to \mathfrak{g}_n^* \times \mathfrak{u}^*$$

where $\mu_r$ is the right moment map above and the second map is the natural restriction map. The partial Whittaker reduction $\check{M}$ of $T^*G_{2n}$ with respect to the right $G_n \ltimes U$-action is given by

$$\check{M} = T^*(G_{2n}/G_n \ltimes U, \psi) := \mu^{-1}(\psi)/G_n \ltimes U.$$

LEMMA 3.2. *There is an isomorphism $\check{M} \simeq G_{2n} \times^{G_n} \mathfrak{g}_n$ fitting into a commutative diagram*

$$\begin{array}{ccc} \check{M} & \xrightarrow{\simeq} & G_{2n} \times^{G_n} \mathfrak{g}_n \\ \downarrow & & \downarrow \\ \mathfrak{g}_{2n}^* & \xrightarrow{\simeq} & \mathfrak{g}_{2n} \end{array}$$

*where the left vertical arrow is the left moment map $\mu_l$, the bottom arrow is induced by the trace pairing $(A,B) \to \mathrm{tr}(AB)$, and the right vertical map is given by*

$$(x, C) \mapsto \mathrm{Ad}_x \begin{pmatrix} 0 & \mathrm{Id}_n \\ C & 0 \end{pmatrix}.$$

*Proof.* We will identify $\mathfrak{g}_{2n}^*$ with $\mathfrak{g}_{2n}$ via the trace pairing. The pre-image of $\psi = (0, -\mathrm{tr}) \in \mathfrak{g}_n^* \times \mathfrak{u}^*$ in $\mathfrak{g}_{2n}^* \simeq \mathfrak{g}_{2n}$ is given by

$$\mathfrak{g}_{2n,\psi}^* := \left\{ \begin{pmatrix} A & -\mathrm{Id}_n \\ C & -A \end{pmatrix} \middle| A, C \in \mathfrak{g}_n \right\},$$

and it follows that

$$\check{M} \simeq \mu^{-1}(\psi)/G_n \ltimes U \simeq \mu_r^{-1}(\mathfrak{g}_{2n,\psi}^*)/G_n \ltimes U \simeq G_{2n} \times^{G_n \ltimes U} (-\mathfrak{g}_{2n,\psi}^*)$$

(recall that $\mu_r(x,v) = -v$). On the other hand, a direct computation shows that the action of $U$ on $-\mathfrak{g}_{2n,\psi}^*$ is free and any $U$-orbit on $-\mathfrak{g}_{2n,\psi}^*$ contains a unique element of the form $\begin{pmatrix} 0 & \mathrm{Id}_n \\ C & 0 \end{pmatrix}$ with $C \in \mathfrak{g}_n$.[4] Thus there is an isomorphism

$$\check{M} \simeq G_{2n} \times^{G_n \ltimes U} (-\mathfrak{g}_{2n,\psi}^*) \simeq G_{2n} \times^{G_n} \mathfrak{g}_n$$

such that the left moment map is given by $\mu_l(x, C) = \mathrm{Ad}_x \begin{pmatrix} 0 & \mathrm{Id}_n \\ C & 0 \end{pmatrix}$. The lemma follows. □

[4]Indeed, this follows from $\begin{pmatrix} \mathrm{Id}_n & 0 \\ X & \mathrm{Id}_n \end{pmatrix} \begin{pmatrix} A & \mathrm{Id}_n \\ C & -A \end{pmatrix} \begin{pmatrix} \mathrm{Id}_n & 0 \\ -X & \mathrm{Id}_n \end{pmatrix} = \begin{pmatrix} A - X & \mathrm{Id}_n \\ C + XA + AX - X^2 & X - A \end{pmatrix}$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

## 4. Constructible side

### 4.1 Twistor fibration

Consider the complex projective space $\mathbb{P}^{2n-1}$ and the quaternionic projective space $\mathbb{HP}^{n-1}$. Recall the identification $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ sending

$$(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_n) \to z + jw = (q_1 = z_1 + jw_1, \ldots, q_n = z_n + jw_n).$$

If to each complex line in $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ we associate the quaternionic line it generates, we get a map

$$f : \mathbb{P}^{2n-1} \to \mathbb{HP}^{n-1}, \quad [z, w] \to [q_1, \ldots, q_n] \tag{4.1}$$

between the corresponding complex and quaternionic projective spaces, called the *twistor fibration* for $\mathbb{HP}^{n-1}$. The fiber of $f$ over a quaternionic line (a copy of $\mathbb{H} \simeq \mathbb{C}^2$) consists of all complex lines generating that quaternionic line, which is a copy of $\mathbb{P}^1 \simeq S^2$. Thus the twistor fibration $f$ is a fiber bundle with fiber $\mathbb{P}^1$. In the $n = 2$ case, we have $\mathbb{HP}^{n-1} = \mathbb{HP}^1 \simeq S^4$ and the map (4.1) is the well-known twistor fibration

$$f : \mathbb{P}^3 \to S^4$$

for $S^4$.

Consider the standard action of the complex torus $T_{2n}$ (respectively, $T_n$) on $\mathbb{P}^{2n-1}$ (respectively, $\mathbb{HP}^{n-1}$):

$$x \cdot [z_1, \ldots, z_{2n}] = [x_1 z_1, \ldots, x_{2n} z_{2n}], \quad x = (x_1, \ldots, x_{2n}) \in T_{2n}$$

$$(\text{respectively,} \quad x \cdot [q_1, \ldots, q_n] = [x_1 q_1, \ldots, x_n q_n], \qquad x = (x_1, \ldots, x_n) \in T_n).$$

Then the twistor map $f : \mathbb{P}^{2n-1} \to \mathbb{HP}^{n-1}$ is $T_n$-equivariant, where $T_n$ acts on $\mathbb{P}^{2n-1}$ through the embedding

$$T_n \xrightarrow{\sim} T_{n,\mathbb{H}} \subset T_{2n}, \quad (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n)$$

(recall that $T_{n,\mathbb{H}}$ is the Cartan subgroup of $G_{n,\mathbb{H}}$). Indeed, for any $x = (x_1, \ldots, x_n) \in T_n$, we have

$$\begin{aligned} f(x \cdot [z, w]) &= f([x_1 z_1, \ldots, x_n z_n, \bar{x}_1 w_1, \ldots, \bar{x}_n w_n]) = [x_1 z_1 + j\bar{x}_1 w_1, \ldots, x_n z_n + j\bar{x}_n w_n] \\ &= [x_1 z_1 + x_1 j w_1, \ldots, x_n z_n + x_n j w_n] = x \cdot [q_1, \ldots, q_n]. \end{aligned}$$

### 4.2 Equivariant cohomology of quaternionic projective spaces

Consider the inverse action of $T_{2n}$ on $\mathbb{P}^{2n-1}$.[5] Recall the following well known description of the $T_{2n}$-equivariant cohomology of $\mathbb{P}^{2n-1}$:

$$H^*_{T_{2n}}(\mathbb{P}^{2n-1}) \simeq \mathbb{C}[t_1, \ldots, t_{2n}][\xi] / \prod_{i=1}^{2n} (\xi - t_i) \tag{4.2}$$

where

$$\xi = c_1^{T_{2n}}(\mathcal{O}(1)) \in H^*_{T_{2n}}(\mathbb{P}^{2n-1})$$

is the first equivariant Chern class of the line bundle $\mathcal{O}(1)$ over $\mathbb{P}^{2n-1}$ and $H^*_{T_{2n}}(\mathrm{pt}) \simeq \mathcal{O}(\mathfrak{t}_{2n}) \simeq \mathbb{C}[t_1, \ldots, t_{2n}]$.

[5] The reason to consider the inverse of the standard action will become clear later; see the proof of Lemma 4.9.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

The imbedding $T_n \simeq T_{n,\mathbb{H}} \subset T_{2n}$ gives rise to a map $H^*_{T_{2n}}(\mathrm{pt}) \to H^*_{T_n}(\mathrm{pt})$, and a direct computation show that under the isomorphism $\mathbb{C}[t_1,\dots,t_{2n}] \simeq H^*_{T_{2n}}(\mathrm{pt})$ and $\mathbb{C}[t_1,\dots,t_n] \simeq H^*_{T_n}(\mathrm{pt})$, the map is given by

$$\mathbb{C}[t_1,\dots,t_{2n}] \to \mathbb{C}[t_1,\dots,t_{2n}]/(t_1+t_{n+1}, t_2+t_{n+2},\dots,t_n+t_{2n}) \simeq \mathbb{C}[t_1,\dots,t_n].$$

It follows that

$$H^*_{T_n}(\mathbb{P}^{2n-1}) \simeq H^*_{T_{2n}}(\mathbb{P}^{2n-1}) \otimes_{H^*_{T_{2n}}(\mathrm{pt})} H^*_{T_n}(\mathrm{pt}) \simeq \mathbb{C}[t_1,\dots,t_n][\xi]/\prod_{i=1}^{n}(\xi^2-t_i^2). \tag{4.3}$$

Similarly, we consider the inverse $T_n$-action on $\mathbb{HP}^{n-1}$. Let $\mathcal{O}_{\mathbb{H}}(-1)$ be the tautological $\mathbb{H}$-line bundle $\mathcal{O}_{\mathbb{H}}(-1)$ over $\mathbb{HP}^{n-1}$. It is canonically $T_n$-equivariant, and we denote by

$$\eta = -e^{T_n}(\mathcal{O}_{\mathbb{H}}(-1)) \in H^4_{T_n}(\mathbb{HP}^{n-1})$$

the negative of the equivariant Euler class of $\mathcal{O}_{\mathbb{H}}(-1)$.

LEMMA 4.1. *There is an isomorphism*

$$H^*_{T_n}(\mathbb{HP}^{n-1}) \simeq \mathbb{C}[t_1,\dots t_n][\eta]/\prod_{i=1}^{n}(\eta-t_i^2)$$

*making a diagram commute*

$$\begin{array}{ccc} H^*_{T_n}(\mathbb{HP}^{n-1}) & \xrightarrow{\quad f^* \quad} & H^*_{T_n}(\mathbb{P}^{2n-1}) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathbb{C}[t_1,\dots t_n][\eta]/\prod_{i=1}^{n}(\eta-t_i^2) & \longrightarrow & \mathbb{C}[t_1,\dots t_n][\xi]/\prod_{i=1}^{n}(\xi^2-t_i^2) \end{array}$$

*where the bottom arrow is the natural $\mathbb{C}[t_1,\dots t_n]$-linear embedding sending $\eta$ to $\xi^2$, that is, we have $f^*(\eta)=\xi^2$.*

*Proof.* The $T_n$-fixed points on $\mathbb{HP}^{n-1}$ are $p_0=[1,0,\dots,0]_{\mathbb{H}}$, $p_2=[0,1,0,\dots,0]_{\mathbb{H}},\dots,p_n= [0,0,\dots,0,1]_{\mathbb{H}}$. Write $s_i:\{p_i\}\to\mathbb{HP}^{n-1}$ for the inclusion map. Then equivariant localization says that we have an injective map of rings

$$\mathrm{Loc} = \bigoplus s_i^* : H^*_{T_n}(\mathbb{HP}^{n-1}) \longrightarrow R_{T_n}^{\oplus n}.$$

The fiber of $\mathcal{O}_{\mathbb{H}}(-1)|_{p_i}$ over $p_i$ is the $\mathbb{H}$-line spanned by the $i$th coordinate vector of $\mathbb{H}^n$, and hence the action of $T_n$ factors though the $i$th projection $T_n \to \mathbb{G}_m$, $(x_1,\dots,x_n)\to x_i$. It follows that, in terms of the coordinate $\mathbb{C}^2 \simeq \mathcal{O}_{\mathbb{H}}(-1)|_{p_i}$, $(z_i,w_i)\to z_i+jw_i$ (and hence a chosen orientation), the (inverse) action is given by $x_i(z_i,w_i)=(x_i^{-1}z_i,\bar{x}_i^{-1}w_i))$ and hence

$$s_i^*(\eta) = s_i^*(-e^{T_n}(\mathcal{O}_{\mathbb{H}}(-1))) = -e^{T_n}(\mathcal{O}_{\mathbb{H}}(-1)|_{p_i}) = t_i^2.$$

Thus we have

$$\mathrm{Loc}(\eta) = \mathrm{Loc}(-e^{T_n}(\mathcal{O}_{\mathbb{H}}(-1))) = (t_1^2,\dots,t_n^2) \in R_{T_n}^{\oplus n},$$

and it follows that $\mathrm{Loc}(\prod_{i=1}^n(\eta-t_i^2))=0$; as Loc is injective, this implies $\prod_{i=1}^n(\eta-t_i^2)=0$.

To see that $f^*(\eta)=\xi^2$, we observe that the pre-image $f^{-1}(p_i)$ is isomorphic to the projection line $\mathbb{P}^1_i=[z_i,w_i]\subset\mathbb{P}^{2n-1}$. The $T_n$-action preserves $\mathbb{P}^1_i$ and is given by $(x_1,\dots,x_n)[z_i,w_i]= [x_i^{-1}z_i,\bar{x}_i^{-1}w_i]$. The localization map $\mathrm{Loc}' : H^*_{T_n}(\mathbb{P}^{2n-1}) \to \bigoplus H^*_{T_n}(\mathbb{P}^1_i) = \mathbb{C}[t_i][\xi_i]/(\xi_i^2-t_i^2)$ is



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

injective and we have $\mathrm{Loc}'(\xi^2)=(\xi_1^2,\dots,\xi_n^2)$. On the other hand, we have

$$\mathrm{Loc}'(f^*\eta)=f^*(\mathrm{Loc}(\eta))=f^*((t_1^2,\dots,t_n^2))=(t_1^2,\dots,t_n^2)=(\xi_1^2,\dots,\xi_n^2)\in\bigoplus H^*_{T_n}(\mathbb{P}^1_i)$$

as $\xi_i^2=t_i^2$ in $H^*_{T_n}(\mathbb{P}^1_i)$. We conclude that $\mathrm{Loc}'(\xi^2)=\mathrm{Loc}'(f^*\eta)$ and hence $\xi^2=f^*\eta$. □

*Remark* 4.2. Here is an alternative argument. One can show that there is an isomorphism of $T_n$-equivariant complex vector bundles

$$f^*\mathcal{O}_{\mathbb{H}}(-1)\simeq\mathcal{O}(-1)\oplus\overline{\mathcal{O}(-1)}$$

over $\mathbb{P}^{2n-1}$. Here $\overline{\mathcal{O}(-1)}$ is the complex conjugate of $\mathcal{O}(-1)$ (note that a choice of a hermitian metric on $\mathcal{O}(-1)$ induces an isomorphism $\overline{\mathcal{O}(-1)}\simeq\mathcal{O}(-1)^\vee\simeq\mathcal{O}(1)$). Since $e^T(\overline{\mathcal{O}(-1)})=-e^T(\mathcal{O}(-1))=-\xi$, it follows that

$$f^*(\eta)=-f^*(e^T(\mathcal{O}_{\mathbb{H}}(-1))=-e^T(\mathcal{O}(-1)\oplus\overline{\mathcal{O}(-1)})=e^T(\mathcal{O}(-1))^2=\xi^2.$$

Now the lemma follows from the fact that $f^*:H^*_{T_n}(\mathbb{HP}^{n-1})\to H^*_{T_n}(\mathbb{P}^{2n-1})$ is injective and $f^*(\prod_{i=1}^n(\eta-t_i^2))=\prod_{i=1}^n(f^*\eta-t_i^2)=\prod_{i=1}^n(\xi^2-t_i^2)=0$ in $H^*_{T_n}(\mathbb{P}^{2n-1})$.

Consider the pushforward functor $f_*:D^b_{T_n}(\mathbb{P}^{2n-1})\to D^b_{T_n}(\mathbb{HP}^{n-1})$.

Lemma 4.3. *We have that* $f_*(\mathbb{C}_{\mathbb{P}^{2n-1}})\simeq\mathbb{C}_{\mathbb{HP}^{n-1}}\oplus\mathbb{C}_{\mathbb{HP}^{n-1}}[-2]$.

*Proof.* Since $f$ is a $\mathbb{P}^1$-fibration, we have a distinguished triangle

$$\mathbb{C}_{\mathbb{HP}^{n-1}}\to f_*(\mathbb{C}_{\mathbb{P}^{2n-1}})\to\mathbb{C}_{\mathbb{HP}^{n-1}}[-2]\to\mathbb{C}_{\mathbb{HP}^{n-1}}[1]$$

and we need to show that it splits. But this follows from

$$\mathrm{Hom}(\mathbb{C}_{\mathbb{HP}^{n-1}}[-2],\mathbb{C}_{\mathbb{HP}^{n-1}}[1])\simeq\mathrm{Ext}^3(\mathbb{C}_{\mathbb{HP}^{n-1}},\mathbb{C}_{\mathbb{HP}^{n-1}})\simeq H^3_{T_n}(\mathbb{HP}^{n-1})=0.$$ □

### 4.3 Two bases

Consider the subvarieties $\mathbb{P}^{i-1}=\{[z_1,\dots,z_i,0,\dots,0]\}\subset\mathbb{P}^{2n-1}$ for $i=1,\dots,2n$. If we write $[\mathbb{P}^{i-1}]\in H^{T_{2n}}_{2i-2}(\mathbb{P}^{2n-1})\simeq H^{4n-2i}_{T_{2n}}(\mathbb{P}^{2n-1})$ for the corresponding fundamental class in the equivariant Borel–Moore homology, then the collection $\{[\mathbb{P}^{i-1}]\}_{i=1,\dots,2n}$ forms a basis of the free $R_{T_{2n}}$-module $H^*_{T_{2n}}(\mathbb{P}^{2n-1})$. Moreover, one can check that the image of the fundamental class $[\mathbb{P}^{i-1}]$ under the identification (4.3) is given by

$$\Upsilon:H^*_{T_{2n}}(\mathbb{P}^{2n-1})\simeq\mathbb{C}[t_1,\dots,t_{2n}][\xi]/\prod_{i=1}^{2n}(\xi-t_i),$$

$$\Upsilon([\mathbb{P}^{2n-1}])=1,\quad \Upsilon([\mathbb{P}^{i-1}])=\prod_{s=i+1}^{2n}(\xi-t_s)\quad\text{for } i=1,\dots,2n-1.$$

Consider the subvarieties $\mathbb{HP}^{i-1}=\{[q_1,\dots,q_i,0,\dots,0]\}\subset\mathbb{HP}^{n-1}$ for $i=1,\dots,n$. If we write $[\mathbb{HP}^{i-1}]\in H_{4i-4,T_n}(\mathbb{HP}^{n-1})\simeq H^{4n-4i}_{T_n}(\mathbb{P}^{2n-1})$ for the corresponding fundamental class in the equivariant Borel–Moore homology, then the collection $\{[\mathbb{HP}^{i-1}]\}_{i=1,\dots,n}$ forms a basis of the free $R_{T_n}$-module $H^*_{T_n}(\mathbb{HP}^{n-1})$. Moreover, one can check that the image of the fundamental class $[\mathbb{HP}^{i-1}]$ under the identification in (4.1) is given by

$$\Upsilon_{\mathbb{H}}:H^*_{T_n}(\mathbb{HP}^{n-1})=\mathbb{C}[t_1,\dots,t_n][\eta]/\prod_{i=1}^{n}(\eta-t_i^2),$$

$$\Upsilon_{\mathbb{H}}([\mathbb{HP}^{n-1}])=1,\quad \Upsilon_{\mathbb{H}}([\mathbb{HP}^{i-1}])=\prod_{s=i+1}^{n}(\eta-t_s^2)\quad\text{for } i=1,\dots,n-1.$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

The isomorphism $f_*\mathbb{C}_{\mathbb{P}^{2n-1}} \simeq \mathbb{C}_{\mathbb{HP}^{n-1}} \oplus \mathbb{C}_{\mathbb{HP}^{n-1}}[-2]$ gives rise to a decomposition

$$\Upsilon' : H^*_{T_n}(\mathbb{HP}^{n-1}) \oplus H^{*-2}_{T_n}(\mathbb{HP}^{n-1}) \simeq H^*_{T_n}(\mathbb{P}^{2n-1}) \simeq \mathbb{C}[t_1, \ldots, t_n][\xi]/\prod_{i=1}^{n}(\xi^2 - t_i^2),$$

and one can check that the image of the basis $\{[\mathbb{HP}^{i-1}]\} \cup \{[\mathbb{HP}^{i-1}][2]\}$ of $H^*_{T_n}(\mathbb{HP}^{n-1}) \oplus H^{*-2}_{T_n}(\mathbb{HP}^{n-1})$ under the map above is given by

$$\begin{aligned} \Upsilon'([\mathbb{HP}^{n-1}]) = 1, \quad \Upsilon'([\mathbb{HP}^{i-1}]) &= \prod_{s=i+1}^{n}(\xi^2 - t_s^2) \quad \text{for } i = 1, \ldots, n-1, \\ \Upsilon'([\mathbb{HP}^{n-1}[2]]) = \xi, \quad \Upsilon'([\mathbb{HP}^{i-1}[2]]) &= \xi \prod_{s=i+1}^{n}(\xi^2 - t_s^2) \quad \text{for } i = 1, \ldots, n-1. \end{aligned} \tag{4.4}$$

LEMMA 4.4.

(1) *In terms of the ordered basis* $\{[\mathbb{P}^0], [\mathbb{P}^1], \ldots, [\mathbb{P}^{2n}]\}$, *the cup product action* $c_1^{T_{2n}}(\mathcal{O}(1)) \cup (-) \in \mathrm{End}_{R_{T_n}}(H^*_{T_n}(\mathbb{P}^{2n-1}))$ *is given by the element* $e^{T_{2n}}$ *in* (3.12):

$$e^{T_{2n}} = \begin{pmatrix} t_1 & 1 & & \\ 0 & t_2 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \ldots & 0 & t_{2n} \end{pmatrix}.$$

(2) *In terms of the ordered basis* $\{[\mathbb{HP}^0][2], \ldots, [\mathbb{HP}^{n-1}][2], [\mathbb{HP}^0], \ldots, [\mathbb{HP}^{n-1}]\}$, *the cup product action* $c_1^{T_n}(\mathcal{O}(1)) \cup (-) \in \mathrm{End}_{R_{T_n}}(H^*_{T_n}(\mathbb{P}^{2n-1}))$ *is given by the element* $\tau \circ e_X^T$ *in* (3.14):

$$\tau \circ e_X^T = \begin{pmatrix} 0 & \mathrm{Id}_n \\ C & 0 \end{pmatrix}, \quad C = \begin{pmatrix} t_1^2 & 1 & & \\ \vdots & t_2^2 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & 0 & \ldots & t_n^2 \end{pmatrix}.$$

*Proof.* The cup product action is given by mutiplication by $\xi$ and the claim is a straightforward computation. □

## 4.4 Complex and quaternionic affine Grassmannians

We denote by $\mathrm{Gr}_{2n} = \mathfrak{L}G_{2n}/\mathfrak{L}^+G_{2n}$ the complex affine Grassmannian for $G_{2n}$, where $\mathfrak{L}G_{2n} = G_{2n}(\mathbb{C}((t)))$ and $\mathfrak{L}^+G_{2n} = G_{2n}(\mathbb{C}[[t]])$ are the Laurent loop group and Taylor arc group for $G_{2n}$, respectively. We denote by $D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})$ the dg-category of $\mathfrak{L}^+G_{2n}$-equivariant constructible complexes on $\mathrm{Gr}_{2n}$ and by $\mathrm{Perv}(\mathrm{Gr}_{2n})$ the abelian category of $\mathfrak{L}^+G_{2n}$-equivariant perverse sheaves on $\mathrm{Gr}_{2n}$.

We denote by $\mathrm{Gr}_{n,\mathbb{H}} = \mathfrak{L}G_{n,\mathbb{H}}/\mathfrak{L}^+G_{n,\mathbb{H}}$ the real affine Grassmannian for the quaternionic group $G_{n,\mathbb{H}}$, where $\mathfrak{L}G_{n,\mathbb{H}} = G_{n,\mathbb{H}}(\mathbb{R}((t)))$ and $\mathfrak{L}^+G_{n,\mathbb{H}} = G_{n,\mathbb{H}}(\mathbb{R}[[t]])$ are the real Laurent loop group and real Taylor arc group for $G_{n,\mathbb{H}}$. The $\mathfrak{L}^+G_{n,\mathbb{H}}$-orbits on $\mathrm{Gr}_{n,\mathbb{H}}$ are of the form $\mathrm{Gr}^\lambda_{n,\mathbb{H}} = \mathfrak{L}^+G_{n,\mathbb{H}} \cdot t^\lambda$ where $(\lambda : \mathbb{G}_m \to S) \in \Lambda_S^+$ is a dominant real coweight. By [Nad05, Proposition 3.6.1], each orbit $\mathrm{Gr}^\lambda_{n,\mathbb{H}}$ is a real vector bundle over the quaternionic flag manifold $G_{n,\mathbb{H}}/P^\lambda_{n,\mathbb{H}}$ of real dimension $2\langle\lambda, \rho_{2n}\rangle$. We denote by $D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})$ the dg-category of $\mathfrak{L}^+G_{n,\mathbb{H}}$-equivariant constructible complexes on $\mathrm{Gr}_{n,\mathbb{H}}$. Since $\langle\lambda, \rho_{2n}\rangle = 4\langle\lambda, \rho_n\rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda_S^+$ (in the second paring we regard $\lambda$ as an element in $\Lambda_n$), all the orbits $\mathrm{Gr}^\lambda_{n,\mathbb{H}}$ have real even dimension, and hence middle perversity



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

makes sense and we denote by $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ the category of $\mathfrak{L}^+G_{n,\mathbb{H}}$-equivariant perverse sheaves on $\mathrm{Gr}_{n,\mathbb{H}}$. Note also that, as $P^\lambda_{n,\mathbb{H}}$ is connected, all the $G_{n,\mathbb{H}}$-equivariant local systems on $\mathrm{Gr}^\lambda_{n,\mathbb{H}}$ are trivial and hence the irreducible objects in $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ are intersection cohomology complexes $\mathrm{IC}_\lambda = \mathrm{IC}(\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}})$, $\lambda \in \Lambda^+_S$, for the closure $\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}} \subset \mathrm{Gr}_{n,\mathbb{H}}$.

Like in the case of complex reductive groups, there is a natural monoidal structure on $D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})$ given by the convolution product: consider the convolution diagram

$$\mathrm{Gr}_{n,\mathbb{H}} \times \mathrm{Gr}_{n,\mathbb{H}} \overset{p}{\leftarrow} \mathfrak{L}G_{n,\mathbb{H}} \times \mathrm{Gr}_{n,\mathbb{H}} \overset{q}{\rightarrow} \mathrm{Gr}_{n,\mathbb{H}} \tilde{\times} \mathrm{Gr}_{n,\mathbb{H}} := \mathfrak{L}G_{n,\mathbb{H}} \times^{\mathfrak{L}^+G_{n,\mathbb{H}}} \mathrm{Gr}_{n,\mathbb{H}} \overset{m}{\rightarrow} \mathrm{Gr}_{n,\mathbb{H}}$$

where $p$ and $q$ are the natural quotient maps and $m(x, y \mod \mathfrak{L}^+G_{n,\mathbb{H}}) = xy \mod \mathfrak{L}^+G_{n,\mathbb{H}}$. For any $\mathcal{F}_1, \mathcal{F}_2 \in D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})$, the convolution is defined as

$$\mathcal{F}_1 \star \mathcal{F}_2 = m_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$$

where $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2 \in D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}} \tilde{\times} \mathrm{Gr}_{n,\mathbb{H}})$ is the unique complex such that $q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2) \simeq p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$.

### 4.5 Real nearby cycles functor

We shall recall the construction of the real nearby cycles functor in [Nad05]. Consider the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}^{(2)}_{2n} \to \mathbb{C}$ over the complex line $\mathbb{C}$ classifying a $G_{2n}$-bundle $\mathcal{E} \to \mathbb{C}$, a point $x \in \mathbb{C}$, and a section $\nu : \mathbb{C} \setminus \{\pm x\} \to \mathcal{E}|_{\mathbb{C}\setminus\{\pm x\}}$. It is well known that there are canonical isomorphisms

$$\mathrm{Gr}^{(2)}_{2n}|_{\{0\}} \simeq \mathrm{Gr}_{2n},$$

$$\mathrm{Gr}^{(2)}_{2n}|_{\mathbb{C}\setminus\{0\}} \simeq \mathrm{Gr}_{2n} \times \mathrm{Gr}_{2n} \times \mathbb{C} \setminus \{0\}.$$

It is shown in [Nad05, Proposition 4.3.1] that the real form $G_{n,\mathbb{H}}$ of $G_{2n}$ together with real form $i\mathbb{R}$ of $\mathbb{C}$ (corresponding to the complex conjugation $x \to -\bar{x}$ on $\mathbb{C}$) defines a real form $\mathrm{Gr}^{(2)}_{n,\mathbb{H}} \to i\mathbb{R}$ of $\mathrm{Gr}^{(2)}_{2n}$ such that there are canonical isomorphisms

$$\mathrm{Gr}^{(2)}_{n,\mathbb{H}}|_{\{0\}} \simeq \mathrm{Gr}_{n,\mathbb{H}}$$

$$\mathrm{Gr}^{(2)}_{n,\mathbb{H}}|_{i\mathbb{R}\setminus\{0\}} \simeq \mathrm{Gr}_{2n} \times (i\mathbb{R} \setminus \{0\})$$

Consider the following diagram.

$$\begin{array}{ccccccccc}
\mathrm{Gr}_{2n} \times i\mathbb{R} & \xrightarrow{\simeq} & \mathrm{Gr}^{(2)}_{n,\mathbb{H}}|_{i\mathbb{R}_{>0}} & \xrightarrow{j} & \mathrm{Gr}^{(2)}_{n,\mathbb{H}}|_{i\mathbb{R}_{\geq 0}} & \xleftarrow{i} & \mathrm{Gr}^{(2)}_{n,\mathbb{H}}|_{\{0\}} & \xleftarrow{\simeq} & \mathrm{Gr}_{n,\mathbb{H}} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & i\mathbb{R}_{>0} & \longrightarrow & i\mathbb{R}_{\geq 0} & \longleftarrow & \{0\} & &
\end{array}$$

Note that the maps in the above diagram are all $K_c$-equivariant, and we define the functor

$$\mathrm{R}' : D^b(K_c\backslash \mathrm{Gr}_{2n}) \to D^b(K_c\backslash \mathrm{Gr}_{n,\mathbb{H}})$$

by the formula

$$\mathrm{R}'(\mathcal{F}) = i^* j_*(\mathcal{F} \boxtimes \mathbb{C}_{i\mathbb{R}_{\geq 0}}). \tag{4.5}$$

By [Nad05, Proposition 4.5.1], the functor $\mathrm{R}'$ takes $\mathfrak{L}^+G_{2n}$-constructible complexes to $\mathfrak{L}^+G_{n,\mathbb{H}}$-constructible complexes. Introduce the subcategory $D^b_{\{\mathfrak{L}^+G_{n,\mathbb{H}}\}}(K_c\backslash \mathrm{Gr}_{n,\mathbb{H}})$ (respectively,



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

$D^b_{\{\mathfrak{L}^+G_{2n}\}}(G_c\backslash\mathrm{Gr}_{2n})$ and $D^b_{\{\mathfrak{L}^+G_{2n}\}}(K_c\backslash\mathrm{Gr}_{2n})$) of $D^b(K_c\backslash\mathrm{Gr}_{n,\mathbb{H}})$ (respectively, $D^b(G_c\backslash\mathrm{Gr}_{2n})$ and $D^b(K_c\backslash\mathrm{Gr}_{2n})$) consisting $\mathfrak{L}^+G_{n,\mathbb{H}}$-constructible complexes (respectively, $\mathfrak{L}^+G_{2n}$-constructible complexes). Since the quotients $\mathfrak{L}^+G_{2n}/G_c$ and $\mathfrak{L}^+G_{n,\mathbb{H}}/K_c$ are contractible, we have natural equivalences $D^b_{\{\mathfrak{L}^+G_{n,\mathbb{H}}\}}(K_c\backslash\mathrm{Gr}_{n,\mathbb{H}})\simeq D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})$ and $D^b_{\{\mathfrak{L}^+G_{2n}\}}(G_c\backslash\mathrm{Gr}_{2n})\simeq D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})$, and the nearby cycles functor $\mathrm{R}'$ above induces a functor

$$\mathrm{R}': D^b_{\{\mathfrak{L}^+G_{2n}\}}(K_c\backslash\mathrm{Gr}_{2n})\to D^b_{\{\mathfrak{L}^+G_{n,\mathbb{H}}\}}(K_c\backslash\mathrm{Gr}_{n,\mathbb{H}})\simeq D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}}). \tag{4.6}$$

Finally, the real nearby cycles functor is defined as

$$\mathrm{R}: D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})\simeq D^b_{\{\mathfrak{L}^+G_{2n}\}}(G_c\backslash\mathrm{Gr}_{2n})\to D^b_{\{\mathfrak{L}^+G_{2n}\}}(K_c\backslash\mathrm{Gr}_{2n})\overset{\mathrm{R}'}{\to} D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}}) \tag{4.7}$$

where the middle arrow is the natural forgetful functor.

The following properties of $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ and R can be deduced from [Nad05].

PROPOSITION 4.5.

(1) *There is a tensor equivalence* $\mathrm{Rep}(G_n)\simeq\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ *sending the irreducible representation* $V_\lambda$ *of* $G_n$ *with highest weight* $\lambda\in\Lambda_S^+$ *to* $\mathrm{IC}_\lambda$.

(2) *The real nearby cycle functor* R *preserves semisimplicity, that is, we have*

$$\mathrm{R}(\mathcal{F})\simeq\bigoplus_{n\in\mathbb{Z}}{}^pH^n\mathrm{R}(\mathcal{F})[-n]$$

*for any semisimple complex* $\mathcal{F}$ *in* $D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})$.

(3) *Consider the monoidal subcategory*

$$\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})_{\mathbb{Z}}:=\bigoplus_{n\in\mathbb{Z}}\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})[n]\subset D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}}).$$

*The real nearby cycle functor restricts to a monoidal functor*

$${}^p\mathrm{R}=\bigoplus_{n\in\mathbb{Z}}{}^pH^n\mathrm{R}(\mathcal{F}):\mathrm{Perv}(\mathrm{Gr}_{2n})\to\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})_{\mathbb{Z}}$$

*such that there is a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Perv}(\mathrm{Gr}_{2n}) & \xrightarrow{{}^p\mathrm{R}} & \mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})_{\mathbb{Z}} \\
\downarrow\simeq & & \downarrow\simeq \\
\mathrm{Rep}(G_{2n}) & \longrightarrow & \mathrm{Rep}(G_n\times\mathbb{G}_m)
\end{array}$$

*where the vertical tensor equivalences come from the complex and quaternionic Satake isomorphisms (part (1)) and the bottom arrow is the restriction map to the subgroup* $G_n\times\mathbb{G}_m\subset G_{2n}$ *as in* § 3.5.

*Proof.* Part (1) is proved in [Nad05, Theorem 1.2.2], and part (3) is proved in [Nad05, § 10.3]. To prove part (2), it suffices to show that $\mathrm{R}(\mathrm{IC}_\lambda)$ is semisimple for all dominant $\lambda$. It is shown in [Nad05, Corollary 1.2.1 and § 6.4] that R is monoidal and that given two semisimple objects $\mathcal{F}_1$ and $\mathcal{F}_2$ in the essential image of R, the convolution $\mathcal{F}_1\star\mathcal{F}_2$ is again semisimple. Let $\omega_1$ and $\epsilon$ (respectively, $\omega_1'$ and $\epsilon'$) be the highest weights of the standard representation and determinant character of $G_n$ (respectively, $G_{2n}$), respectively. Since $\mathrm{Rep}(G_{2n})$ is tensor-generated by the standard representation $V_{\omega_1'}$ and the determinant character $V_{\epsilon'}$, it suffices to show that $\mathrm{R}(\mathrm{IC}_{\omega_1'})$ and $\mathrm{R}(\mathrm{IC}_{\epsilon'})$ are semisimple. It follows from part (3) that ${}^p\mathrm{R}(\mathrm{IC}_{\epsilon'})\simeq\mathrm{IC}_{2\epsilon}$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

and ${}^p\mathrm{R}(\mathrm{IC}_{\omega_1'}) \simeq \mathrm{IC}_{\omega_1} \oplus \mathrm{IC}_{\omega_1}$ where $\mathrm{IC}_{\omega_1}$ is the IC-complex of $\mathrm{Gr}_{n,\mathbb{H}}^{\omega_1} \simeq \mathbb{HP}^{n-1}$. Since the $\mathbb{G}_m$-weights of $\det_{2n}$ and $V_{\omega_1'}$ with respect to the co-character $\mathbb{G}_m \simeq \{e\} \times \mathbb{G}_m \subset G_n \times \mathbb{G}_m \subset G_{2n}$ in part (3) are equal to 0 and $\{1,-1\}$, respectively, this implies that $\mathrm{R}(\mathrm{IC}_{\epsilon'}) \simeq {}^p\mathrm{R}(\mathrm{IC}_{\epsilon'}) \simeq \mathrm{IC}_{2\epsilon}$ is a simple perverse sheaf, and $\mathrm{R}(\mathrm{IC}_{\omega_1'})$ admits a filtration with associated graded given by $\mathrm{IC}_{\omega_1}[1] \oplus \mathrm{IC}_{\omega_1}[-1]$. Since $\mathrm{Ext}^1(\mathrm{IC}_{\omega_1}[-1], \mathrm{IC}_{\omega_1}[1]) = \mathrm{Ext}^3(\mathbb{C}_{\mathbb{HP}^{n-1}}, \mathbb{C}_{\mathbb{HP}^{n-1}}) \simeq H^3_{G_{n,\mathbb{H}}}(\mathbb{HP}^{n-1}) \subset H^3_{T_n}(\mathbb{HP}^{n-1}) = 0$, it follows that the filtration splits and hence that $\mathrm{R}(\mathrm{IC}_{\omega_1'}) \simeq \mathrm{IC}_{\omega_1}[1] \oplus \mathrm{IC}_{\omega_1}[-1]$ is semisimple. □

In the course of the proof, together with Lemma 4.3, we have shown the following.

Corollary 4.6. *There is an isomorphism* $\mathrm{R}(\mathrm{IC}_{\omega_1'}) \simeq f_*(\mathrm{IC}_{\mathbb{P}^{2n-1}}) \simeq \mathrm{IC}_{\mathbb{HP}^{n-1}}[1] \oplus \mathrm{IC}_{\mathbb{HP}^{n-1}}[-1]$.

*Remark* 4.7. In Theorem 5.7, we will give a spectral description of the nearby cycle functor R on the whole derived category (not just its restriction ${}^p\mathrm{R}$ to the subcategory of perverse sheaves).

Recall the nearby cycles functor $\mathrm{R}': D^b_{\{\mathfrak{L}^+G_{2n}\}}(K_c\backslash \mathrm{Gr}_{2n}) \to D^b_{\{\mathfrak{L}^+G_{n,\mathbb{H}}\}}(K_c\backslash \mathrm{Gr}_{n,\mathbb{H}})$ in (4.6). It extends to the ind-completion (denoted again by $\mathrm{R}'$)

$$\mathrm{R}': \mathrm{Ind}\, D^b_{\{\mathfrak{L}^+G_{2n}\}}(K_c\backslash \mathrm{Gr}_{2n}) \to \mathrm{Ind}\, D^b_{\{\mathfrak{L}^+G_{n,\mathbb{H}}\}}(K_c\backslash \mathrm{Gr}_{n,\mathbb{H}}).$$

Lemma 4.8. *The functor* $\mathrm{R}'$ *admits the left adjoint*

$${}^L\mathrm{R}': \mathrm{Ind}\, D^b_{\{\mathfrak{L}^+G_{n,\mathbb{H}}\}}(K_c\backslash \mathrm{Gr}_{n,\mathbb{H}}) \to \mathrm{Ind}\, D^b_{\{\mathfrak{L}^+G_{2n}\}}(K_c\backslash \mathrm{Gr}_{2n}).$$

*Moreover, we have* ${}^L\mathrm{R}'(\mathbb{C}_{\mathrm{Gr}_{n,\mathbb{H}}}) \simeq \mathbb{C}_{\mathrm{Gr}_{2n}}$.

*Proof.* By [Nad05, Proposition 4.5.1], the ind-proper family $\mathrm{Gr}^{(2)}_{n,\mathbb{H}} \to i\mathbb{R}_{\geq 0}$ is a Thom stratified map with respect to a Whitney stratification $\mathcal{T}$ on $\mathrm{Gr}^{(2)}_{n,\mathbb{H}}$ and the stratification $i\mathbb{R}_{>0} \cup \{0\}$ on $i\mathbb{R}_{\geq 0}$ such that $\mathcal{T}$ restricts to the $\mathfrak{L}^+G_{2n}$-orbits stratification on the generic fiber $\mathrm{Gr}_{2n}$ and to the $\mathfrak{L}^+G_{n,\mathbb{H}}$-orbits stratification on the special fiber $\mathrm{Gr}_{n,\mathbb{H}}$. The construction in [GM83, § 6], together with the results in [PW19, Theorem 1.1] (extending Mather's theory of control data to the equivariant setting), implies that the nearby cycles functor $\mathrm{R}'$ is isomorphic to the functor given by $*$-pushforward along a $K_c$-equivariant specialization map $\psi: \mathrm{Gr}_{2n} \to \mathrm{Gr}_{n,\mathbb{H}}$, and hence admits a left adjoint given by the $*$-pullback $\psi^*$. It is clear that $\psi^*$ sends constant sheaf to constant sheaf. The lemma follows. □

## 4.6 Equivariant homology and cohomology of affine Grassmannians

4.6.1 We reviewed the description of the equivariant homologies $H^{T_{2n}}_*(\mathrm{Gr}_{2n})$ and $H^{T_c}_*(\mathrm{Gr}_{n,\mathbb{H}})$ of $\mathrm{Gr}_{2n}$ and $\mathrm{Gr}_{n,\mathbb{H}}$ in [O'Br23, YZ11]. Recall that for an ind-proper semi-analytic set $Y = \mathrm{colim}_I Y_i$ acting real analytically by a Lie group $G$, the $G$-equivariant homology $H^G_*(Y)$ of $Y$ is defined as $H^G_*(Y) := \mathrm{colim}_I H^*_G(Y_i, \omega_i)$, where $\omega_i \in D(G\backslash Y_i)$ is the dualizing sheaf of $G\backslash Y_i$ and the colimit is induced by the natural adjunction map $(\iota_{i,i'})_*\omega_i \simeq (\iota_{i,i'})_!\omega_i \to \omega_{i'}$, and the $G$-equivariant cohomology $H^*_G(Y)$ of $Y$ is defined as $H^*_G(Y) := \lim_I H^*_G(Y_i, \mathbb{C})$, where the limit is induced by the natural restriction map $H^*_G(Y_{i'}, \mathbb{C}) \to H^*_G(Y_i, \mathbb{C})$.

Let $\mathcal{L}$ be the determinant line bundle on $\mathrm{Gr}_{2n}$ and let $c_1^{T_{2n}}(\mathcal{L}) \in H^2_{T_{2n}}(\mathrm{Gr}_{2n})$ be its equivariant first Chern class. It is shown in [YZ11, Lemma 2.2] that there is an isomorphism of functors

$$H^*_{T_{2n}}(\mathrm{Gr}_{2n}, -) \simeq H^*(\mathrm{Gr}_{2n}, -) \otimes R_{T_{2n}} : \mathrm{Perv}(\mathrm{Gr}_{2n}) \to R_{T_{2n}}\text{-mod} \tag{4.8}$$

induced by the canonical splitting of the MV-filtration associated to the semi-infinite orbits $S^\lambda_{2n}$, the $\mathfrak{L}N_{2n}$-orbits through $\lambda \in \Lambda_{2n}$. (Recall that $N_{2n} \subset G_{2n}$ denotes the subgroup of upper



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

triangular unipotent matrices.) Moreover, the isomorphism respects the natural monoidal structures on $H^*_{T_{2n}}(\mathrm{Gr}_{2n},-)$ coming from fusion and the one on $H^*(\mathrm{Gr}_{2n},-)\otimes R_{T_{2n}}$ induced from $H^*(\mathrm{Gr}_{2n},-)$. The cup product action $\wedge c_1^{T_{2n}}(\mathcal{L})$ on $H^*_{T_{2n}}(\mathrm{Gr}_{2n},\mathcal{F})$ for $\mathcal{F}\in\mathrm{Perv}(\mathrm{Gr}_{2n})$ gives rise to a tensor endomorphism of $H^*_{T_{2n}}(\mathrm{Gr}_{2n},-)$ and hence, by the Tannakian formalism, gives rise to an element $c^{T_{2n}}\in\mathfrak{g}_{2n}\otimes R_{T_{2n}}$. One can regard the element $c^{T_{2n}}$ as a map

$$c^{T_{2n}}:\mathfrak{t}_{2n}\to\mathfrak{g}_{2n}. \tag{4.9}$$

The equivariant homology $H_*^{T_{2n}}(\mathrm{Gr}_{2n})$ is a commutative and cocommutative Hopf algebra over $R_{T_{2n}}$, and there is an isomorphism of group schemes

$$\mathrm{Spec}(H_*^{T_{2n}}(\mathrm{Gr}_{2n}))\simeq(G_{2n}\times\mathfrak{t}_{2n})^{c^{T_{2n}}} \tag{4.10}$$

where $(G_{2n}\times\mathfrak{t}_{2n})^{c^{T_{2n}}}$ is the centralizer of $c^{T_{2n}}$ in $G_{2n}\times\mathfrak{t}_{2n}$.

We have a similar result for quaternionic Grassmannians. Let $\mathcal{L}_{\mathbb{H}}$ be the quaternionic determinant line bundle on $\mathrm{Gr}_{n,\mathbb{H}}$ and let $p^T(\mathcal{L}_{\mathbb{H}})\in H^4_{T_c}(\mathrm{Gr}_{n,\mathbb{H}})$ be its equivariant Pontryagin class. It is shown in [O'Br23, Theorem 3] that there is an isomorphism of functors

$$H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}},-)\simeq H^*(\mathrm{Gr}_{n,\mathbb{H}},-)\otimes_{\mathbb{C}}R_T:\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})\to R_T\text{-mod} \tag{4.11}$$

induced by the canonical splitting of the real MV-filtration associated to the real semi-infinite orbits $S^\lambda_{n,\mathbb{H}}$, the $\mathfrak{L}N_{n,\mathbb{H}}$-orbits through $\lambda\in\Lambda_S$. Moreover, the isomorphism above respects the natural monoidal structures on $H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}},-)$ coming from fusion and the one on $H^*(\mathrm{Gr}_{n,\mathbb{H}},-)\otimes R_T$ induced from $H^*(\mathrm{Gr}_{n,\mathbb{H}},-)$. The cup product action of $p^T(\mathcal{L}_{\mathbb{H}})$ on $H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}},\mathcal{F})$ for $\mathcal{F}\in\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})\simeq\mathrm{Rep}(G_n)$ gives rise to a tensor endomorphism of $H^*_T(\mathrm{Gr}_{n,\mathbb{H}},-)$ and hence an element $p^T_X\in\mathfrak{g}_n\otimes R_T$. Let

$$p^T_X:\mathfrak{t}\to\mathfrak{g}_n \tag{4.12}$$

be the corresponding map. The main result in [O'Br23, Theorem 9 and Corollary 1] says that there an isomorphism of group schemes

$$\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}}))\simeq(G_n\times\mathfrak{t})^{p^T_X} \tag{4.13}$$

where $(G_n\times\mathfrak{t})^{p^T_X}$ is the centralizer of $p^T_X$ in $G_n\times\mathfrak{t}$.

Recall the maps $e^{T_{2n}}$ and $e^T_X$ introduced in (3.13) and (3.14), respectively.

Lemma 4.9. *We have $c^{T_{2n}}=e^{T_{2n}}$ and $p^T_X=-e^T_X$. Thus there are isomorphisms of group schemes*

$$\mathrm{Spec}(H_*^{T_{2n}}(\mathrm{Gr}_{2n}))\simeq(G_{2n}\times\mathfrak{t}_{2n})^{e^{T_{2n}}}\simeq J_{2n}\times_{\mathfrak{c}_{2n}}\mathfrak{t}_{2n},$$

$$\mathrm{Spec}(H_*^{T}(\mathrm{Gr}_{n,\mathbb{H}}))\simeq(G_n\times\mathfrak{t})^{e^T_X}\simeq J_n\times_{\mathfrak{c}_n}\mathfrak{t}$$

*over $\mathfrak{t}_{2n}$ and $\mathfrak{t}$, respectively, and isomorphisms of group schemes*

$$H_*^{G_c}(\mathrm{Gr}_{2n})\simeq H_*^{T_{2n}}(\mathrm{Gr}_{2n})^{\mathrm{W}_{2n}}\simeq(J_{2n}\times_{\mathfrak{c}_{2n}}\mathfrak{t}_{2n})^{\mathrm{W}_{2n}}\simeq J_{2n},$$

$$H_*^{K_c}(\mathrm{Gr}_{n,\mathbb{H}})\simeq H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}})^{\mathrm{W}}\simeq(J_n\times_{\mathfrak{c}_n}\mathfrak{t})^{\mathrm{W}}\simeq J_n$$

*over $\mathfrak{c}_{2n}=\mathfrak{t}_{2n}//\mathrm{W}_{2n}$ and $\mathfrak{t}//\mathrm{W}\simeq\mathfrak{c}_n$, respectively.*

*Proof.* The result follows from the computations in [YZ11, § 5] and [O'Br23]. We give an alternative (and more direct) proof using the computation in § 4.3.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

It suffices to show that the element

$$c^{T_{2n}}=c_1^{T_{2n}}(\mathcal{L})\cup(-)\in \mathrm{End}(H^*_{T_{2n}}(\mathrm{Gr}_{2n},\mathrm{IC}_{\omega_1'}))\simeq \mathrm{End}(H^*(\mathrm{Gr}_{2n},\mathrm{IC}_{\omega_1'})\otimes R_{T_{2n}})$$
$$\simeq \mathrm{End}(V_{\omega_1'}\otimes R_{T_{2n}})\simeq \mathfrak{g}_{2n}\otimes R_T$$

(respectively, $p_X^T=p_1^T(\mathcal{L}_{\mathbb{H}})\cup(-)\in \mathrm{End}(H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_{\omega_1}))\simeq \mathrm{End}(H^*(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_{\omega_1})\otimes R_T)$

$$\simeq \mathrm{End}(V_{\omega_1}\otimes R_T)\simeq \mathfrak{g}_n\otimes R_T)$$

is given by $e^{T_{2n}}$ (respectively, $-e_X^T$). We have the following observations:

(1) there is a $T_{2n}$-equivariant (respectively, $T_c$-equivariant) isomorphism $\mathrm{Gr}_{2n}^{\omega_1'}\simeq\mathbb{P}^{2n-1}$ (respectively, $\mathrm{Gr}_{n,\mathbb{H}}^{\omega_1}\simeq\mathbb{HP}^{n-1}$), where $T_{2n}$ (respectively, $T_c$) acts on $\mathbb{P}^{2n-1}$ (respectively, $\mathbb{HP}^{n-1}$) via the inverse of the natural action;[6]

(2) the restriction $\mathcal{L}|_{\mathrm{Gr}_{2n}^{\omega_1'}}$ (respectively, $\mathcal{L}_{\mathbb{H}}|_{\mathrm{Gr}_{n,\mathbb{H}}^{\omega_1}}$) is isomorphic to $\mathcal{O}(1)$ (respectively, $\mathcal{O}_{\mathbb{H}}(1)$, the $\mathbb{H}$-dual of the $\mathcal{O}_{\mathbb{H}}(-1)$);

(3) the composed isomorphism

$$R_{T_{2n}}\otimes V_{\omega_1'}\simeq R_{T_{2n}}\otimes H^*(\mathrm{Gr}_{2n},\mathrm{IC}_{\omega_1'})\simeq H^*_{T_{2n}}(\mathrm{Gr}_{2n},\mathrm{IC}_{\omega_1'})\simeq H^*_{T_{2n}}(\mathbb{P}^{2n-1},\mathrm{IC}_{\mathbb{P}^{2n-1}})$$

(respectively, $R_T\otimes V_{\omega_1}\simeq R_T\otimes H^*(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_{\omega_1})\simeq H^*_T(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_{\omega_1})\simeq H^*_T(\mathbb{HP}^{n-1},\mathrm{IC}_{\mathbb{HP}^{n-1}})$)

sends the vectors $1\otimes e_i$, $i=1,\dots,2n$, to the fundamental class

$$[\mathbb{P}^{i-1}]\in H^{2n+1-2i}_{T_{2n}}(\mathbb{P}^{2n-1},\mathrm{IC}_{\mathbb{P}^{2n-1}})=H^{4n-2i}_{T_{2n}}(\mathbb{P}^{2n-1})$$

(respectively, $1\otimes e_i$, $i=1,\dots,n$, to the fundamental class

$$[\mathbb{HP}^{i-1}]\in H_T^{2n+2-4i}(\mathbb{HP}^{n-1},\mathrm{IC}_{\mathbb{HP}^{n-1}})=H_T^{4n-4i}(\mathbb{HP}^{n-1})).$$

From the above observations, we see that $c^{T_{2n}}\in\mathfrak{g}_{2n}\otimes R_{T_{2n}}$ (respectively, $p_X^T\in\mathfrak{g}_n\otimes R_T$) is the matrix presentation of the cup product action $c_1^{T_{2n}}(\mathcal{O}(1))\cup(-)$ (respectively, $e^T(\mathcal{O}_{\mathbb{H}}(1))\cup(-)$)[7] in the basis $\{[\mathbb{P}^{i-1}]\}_{i=1,\dots,2n}$ (respectively, $\{[\mathbb{HP}^{i-1}]\}_{i=1,\dots,n}$), and the desired claim follows from Lemma 4.4. □

4.6.2 Recall that for any Lie group $G$ and any ind-proper $G$-variety $Y$ we have a paring

$$H^*_G(Y)\times H^G_*(Y)\to H^*_G(\mathrm{pt})\simeq R_G$$

induced by the action of cohomology on homology and then the pushforward map in the Borel–Moore homology $H_*^G(Y)\to H^*_G(\mathrm{pt})$. On the other hand, for any commutative affine group scheme $H$ over $S$ there is a canonical paring

$$U(\mathrm{Lie}H)\times\mathcal{O}(H)\to\mathcal{O}(S),\quad (\xi,f)\to\xi(f)|_e$$

between the relative universal enveloping algebra $U(\mathrm{Lie}H)$ and the ring of functions on $H$. Here $e:S\to H$ is the unity map.

According to [BFM05, Remark 2.13], there are isomorphisms

$$H^*_{G_c}(\mathrm{Gr}_{2n})\simeq U(\mathrm{Lie}J_{2n})\quad\text{and}\quad H^*_{K_c}(\mathrm{Gr}_{n,\mathbb{H}})\simeq U(\mathrm{Lie}J_n)\tag{4.14}$$

[6]This is because the isomorphism $\mathrm{Gr}_{2n}^{\omega_1'}\simeq\mathbb{P}^{2n-1}$ is given by the composition of the canonical $T_{2n}$-equivariant isomorphism $\mathrm{Gr}_{2n}^{\omega_1'}\simeq\mathrm{Gr}(2n-1,\mathbb{C}^{2n})$, where $\mathrm{Gr}(2n-1,\mathbb{C}^{2n})$ is the Grassmannian variety of $(2n-1)$-dimensional subspaces of $\mathbb{C}^{2n}$, with the duality $\mathrm{Gr}(2n-1,\mathbb{C}^{2n})\simeq\mathrm{Gr}(1,(\mathbb{C}^{2n})^*)\simeq\mathbb{P}^{2n-1}$.

[7]Note that the underlying complex rank 2 bundles of $\mathcal{O}_{\mathbb{H}}(-1)$ and $\mathcal{O}_{\mathbb{H}}(1)$ are complex conjugate to each other and hence $e^T(\mathcal{O}_{\mathbb{H}}(1))\simeq(-1)^2e^T(\mathcal{O}_{\mathbb{H}}(-1))=e^T(\mathcal{O}_{\mathbb{H}}(-1))$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

such that the paring above between the cohomology and homology of $Y=\mathrm{Gr}_{2n}$ (respectively, $\mathrm{Gr}_{n,\mathbb{H}}$) becomes the paring between the universal enveloping algebra and ring of functions for the group scheme $H=J_{2n}$ (respectively, $J_n$).

### 4.7 Fully-faithfulness

A key ingredient in the proof of the (complex) derived Satake theorem is the fully-faithfulness of the equivariant cohomology functor $H^*_{\mathfrak{L}^+G_{2n}}(\mathrm{Gr}_{2n},-)$ into the category of modules over the global cohomology $H^*_{\mathfrak{L}^+G_{2n}}(\mathrm{Gr}_{2n},\mathbb{C})$. In [BF08], this was established using general results of Ginzburg [Gin91]. Ginzburg's arguments appeal to Hodge theory and therefore must be modified in the real setting. As in [AR15], we can use parity considerations in place of Hodge theory. More precisely, we will make use of the theory of parity sheaves [JMW14]. Our first step, therefore, is to establish that the complexes $\mathrm{IC}_\lambda$ (for $\lambda\in\Lambda_S^+$) are even.

*Remark* 4.10. In fact, our situation is simpler than the modular setting considered in [JMW14] owing to the fact that the tensor category $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ of spherical perverse sheaves on $\mathrm{Gr}_{n,\mathbb{H}}$ is semisimple (see Proposition 4.5).

4.7.1 Recall that if a coweight $\mu\in\Lambda_S^+$ is minuscule, the orbit $\mathrm{Gr}^{\mu}_{n,\mathbb{H}}$ is closed. Such an orbit is necessarily smooth.

Lemma 4.11. *Let $\mu_1,\dots,\mu_k\in\Lambda_S^+$ denote minuscule coweights. Consider the convolution morphism*

$$m:\mathrm{Gr}^{\mu_\bullet}_{n,\mathbb{H}}:=\mathrm{Gr}^{\mu_1}_{n,\mathbb{H}}\widetilde{\times}\cdots\widetilde{\times}\mathrm{Gr}^{\mu_k}_{n,\mathbb{H}}\to\mathrm{Gr}_{n,\mathbb{H}}.$$

*Then the non-empty fibers of $m$ are paved by quaternionic affine spaces.*

*Proof.* In the complex setting, this result is due to [Hai06]. We proceed by induction on $k$. When $k=1$, there is nothing to prove. In general, we factor $m$ as follows.

$$\begin{array}{ccc} \mathrm{Gr}^{\mu_\bullet}_{n,\mathbb{H}} & \xrightarrow{q} & \mathrm{Gr}_{n,\mathbb{H}}\,\widetilde{\times}\,\mathrm{Gr}^{\mu_k}_{n,\mathbb{H}} \\ & \searrow^{m} & \downarrow{\scriptstyle p} \\ & & \mathrm{Gr}_{n,\mathbb{H}} \end{array}$$

Here, $q$ is induced by multiplying the first $k-1$ factors of $\mathrm{Gr}^{\mu_\bullet}_{n,\mathbb{H}}$. Since $m$ is $\mathfrak{L}^+G_{n,\mathbb{H}}$-equivariant, it suffices to show that each fiber $m^{-1}(t^\lambda)$ (for $\lambda\in\Lambda_S^+$) is paved by quaternionic affine spaces. By the above diagram, we have

$$m^{-1}(t^\lambda)=q^{-1}(p^{-1}(t^\lambda)).$$

Let $\mu'_\bullet=(\mu_1,\dots,\mu_{k-1})$. Observe that we have the following commutative diagram.

$$\begin{array}{ccccc} m^{-1}(t^\lambda) & \hookrightarrow & \mathrm{Gr}^{\mu_\bullet}_{n,\mathbb{H}} & \longrightarrow & \mathrm{Gr}^{\mu'_\bullet}_{n,\mathbb{H}} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle a} \\ p^{-1}(t^\lambda) & \hookrightarrow & \mathrm{Gr}_{n,\mathbb{H}}\,\widetilde{\times}\,\mathrm{Gr}^{\mu_k}_{n,\mathbb{H}} & \xrightarrow{\pi} & \mathrm{Gr}_{n,\mathbb{H}} \end{array}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Here, $\pi$ is the projection to the first factor and $a$ is the convolution map. Observe that both horizontal compositions are closed embeddings. Hence, we obtain a Cartesian diagram as follows.

$$\begin{array}{ccc} m^{-1}(t^\lambda) & \hookrightarrow & \mathrm{Gr}^{\mu'_\bullet}_{n,\mathbb{H}} \\ {\scriptstyle \pi\circ q}\downarrow & & \downarrow{\scriptstyle a} \\ \pi(p^{-1}(t^\lambda)) & \hookrightarrow & \mathrm{Gr}_{n,\mathbb{H}} \end{array}$$

By induction, the fibers of $a$ are paved by quaternionic affine spaces. Hence, the same is true of $\pi\circ q$. It therefore suffices to show that $\pi(p^{-1}(t^\lambda))$ is paved by quaternionic affine spaces over which $a$ is a trivial fibration. Since $\mathfrak{L}^+G_{n,\mathbb{H}}$ acts transitively on the fiber $p^{-1}(t^0)$, we have $\pi(p^{-1}(t^0)) = \mathfrak{L}^+G_{n,\mathbb{H}}t^{-\mu_k} = \mathrm{Gr}^{-w_0(\mu_k)}_{n,\mathbb{H}}$ and hence

$$\pi(p^{-1}(t^\lambda)) = t^\lambda\pi(p^{-1}(t^0)) = t^\lambda \mathrm{Gr}^{-w_0(\mu_k)}_{n,\mathbb{H}},$$

where $w_0$ is the longest element of the Weyl group. Multiplication by $t^\lambda$ is an isomorphism commuting with $a$, so it suffices to show that $\mathrm{Gr}^{-w_0(\mu_k)}_{n,\mathbb{H}}$ is paved by quaternionic affine spaces over which $a$ is a trivial fibration. Let $\mu = -w_0(\mu_k)$. The coweight $\mu$ is once again minuscule. Recall that $\mathrm{Gr}^{\mu}_{n,\mathbb{H}}$ is a vector bundle over a partial flag variety of $G_{n,\mathbb{H}}$. On the other hand, $\mathrm{Gr}^{\mu}_{n,\mathbb{H}}$ is closed, so it is a partial flag variety of $G_{n,\mathbb{H}}$. We claim that the orbits of $P^\mu_{n,\mathbb{H}}$ on $\mathrm{Gr}^{\mu}_{n,\mathbb{H}}$ are the desired affine spaces.

Each such orbit has the form $P^\mu_{n,\mathbb{H}}t^{w(\mu)}$ for $w\in W_n$ an element of the Weyl group. Let ${}^wP^\mu_{n,\mathbb{H}} = w(P^\mu_{n,\mathbb{H}})$. Then, by the real Bruhat decomposition, there exists a unipotent subgroup ${}^wN^\mu_{n,\mathbb{H}}$ of ${}^wP^\mu_{n,\mathbb{H}}$ which acts freely and transitively on the orbit $P^\mu_{n,\mathbb{H}}t^{w(\mu)}$. Hence $P^\mu_{n,\mathbb{H}}t^{w(\mu)} = {}^wN^\mu_{n,\mathbb{H}}t^{w(\mu)}$. By the $\mathfrak{L}G_{n,\mathbb{H}}$ equivariance of $a$, we have the following commutative diagram.

$$\begin{array}{ccc} {}^wN^\mu_{n,\mathbb{H}} \times a^{-1}(t^{w(\mu)}) & \longrightarrow & a^{-1}({}^wN^\mu_{n,\mathbb{H}}t^{w(\mu)}) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle a} \\ {}^wN^\mu_{n,\mathbb{H}} & \xrightarrow{\sim} & {}^wN^\mu_{n,\mathbb{H}}t^{w(\mu)} \end{array}$$

As the diagram is Cartesian and the bottom arrow is an isomorphism, the top arrow is an isomorphism as well. By induction, $a^{-1}(t^{w(\mu)})$ is paved by quaternionic affine spaces. Therefore, it suffices to show that the unipotent subgroup ${}^wN^\mu_{n,\mathbb{H}}$ is a quaternionic affine space, which is clear. □

We now recall the terminology of [JMW14] that we will use. For $\lambda\in\Lambda^+_S$, let

$$i_\lambda : \mathrm{Gr}^\lambda_{n,\mathbb{H}} \hookrightarrow \mathrm{Gr}_{n,\mathbb{H}}$$

denote the inclusion.

Definition 4.12. Let $\mathcal{F}\in D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$. We say that $\mathcal{F}$ is $*$-even (respectively, !-even) if for all $\lambda\in\Lambda^+_S$, the $\mathfrak{L}^+G_{n,\mathbb{H}}$-equivariant sheaf $i^*_\lambda\mathcal{F}$ (respectively, $i^!_\lambda\mathcal{F}$) is a direct sum of constant sheaves appearing in even degrees. If $\mathcal{F}$ is both $*$-even and !-even, we simply say that it is even.

We say that $\mathcal{F}$ is $*$-odd (respectively, !-odd) if $\mathcal{F}[1]$ is $*$-even (respectively, !-even). If $\mathcal{F}$ is both $*$-odd and !-odd, we simply say that it is odd.

Proposition 4.13. *For $\lambda\in\Lambda^+_S$, the complex $\mathrm{IC}_\lambda$ is even.*



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

*Proof.* Since $\mathrm{IC}_\lambda$ is self-dual, it suffices to show that it is $*$-even. Recall from Proposition 4.5(1) that we have an equivalence $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}}) \simeq \mathrm{Rep}(G_n)$ taking $\mathrm{IC}_\lambda$ to the highest-weight module $V_\lambda$. Then $V_\lambda$ is a direct summand of a tensor product $V_\epsilon^{\otimes j} \otimes V_{\omega_1}^{\otimes k}$ for some $j, k \geq 0$. Hence $\mathrm{IC}_\lambda$ is a direct summand of the convolution $\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k}$. It therefore suffices to show that $\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k}$ is $*$-even. We now apply Lemma 4.11 with $\mu_1, \ldots, \mu_j = \epsilon$ and $\mu_{j+1} = \ldots = \mu_{j+k} = \omega_1$. Let

$$m : \mathrm{Gr}_{n,\mathbb{H}}^{\mu_\bullet} \to \mathrm{Gr}_{n,\mathbb{H}}$$

denote the convolution map. We have

$$\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k} \simeq m_!(\mathrm{IC}_\epsilon^{\widetilde{\boxtimes} j} \widetilde{\boxtimes} \mathrm{IC}_\epsilon^{\widetilde{\boxtimes} k}).$$

Now let $\nu \in \Lambda_S$ and let $i_\nu : \mathrm{Gr}_{n,\mathbb{H}}^\nu \hookrightarrow \mathrm{Gr}_{n,\mathbb{H}}$ denote the inclusion. Firstly, we have

$$H^*_{\mathfrak{L}^+ G_{n,\mathbb{H}}}(i_\nu^*(\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k})) \simeq H^*_{T_c}(i_\nu^*(\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k}))^W.$$

Next, by proper base change,

$$H^*_{T_c}(i_\nu^*(\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k})) \simeq H^*_{T_c}(m^{-1}(\mathrm{Gr}_{n,\mathbb{H}}^\nu), \mathrm{IC}_\epsilon^{\widetilde{\boxtimes} j} \widetilde{\boxtimes} \mathrm{IC}_{\omega_1}^{\widetilde{\boxtimes} k}).$$

Since $\epsilon$ and $\omega_1$ are minuscule, the orbits $\mathrm{Gr}_{n,\mathbb{H}}^{\omega_1}$ and $\mathrm{Gr}_{n,\mathbb{H}}^\epsilon$ are smooth. Therefore, $\mathrm{IC}_{\omega_1} \simeq \underline{\mathbb{C}}[2(n-1)]$ and $\mathrm{IC}_\epsilon \simeq \underline{\mathbb{C}}$. Hence,

$$H^*_{T_c}(m^{-1}(\mathrm{Gr}_{n,\mathbb{H}}^\nu), \mathrm{IC}_\epsilon^{\widetilde{\boxtimes} j} \widetilde{\boxtimes} \mathrm{IC}_{\omega_1}^{\widetilde{\boxtimes} k}) \simeq H^*_{T_c}(m^{-1}(\mathrm{Gr}_{n,\mathbb{H}}^\nu), \mathbb{C})[2k(n-1)].$$

By Lemma 4.11, the ordinary cohomology $H^*(m^{-1}(\mathrm{Gr}_{n,\mathbb{H}}^\nu), \mathbb{C})$ is concentrated in even degrees (in fact, in degrees divisible by 4). Hence, $m^{-1}(\mathrm{Gr}_{n,\mathbb{H}}^\nu)$ is equivariantly formal with respect to the action of $T_c$. Therefore, $H^*_{T_c}(m^{-1}(\mathrm{Gr}_{n,\mathbb{H}}^\nu), \mathbb{C})$ is concentrated in even degrees.

Now we may express $i_\nu^*(\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k})$ as a direct sum of constant sheaves. We have

$$i_\nu^*(\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k}) \simeq \underline{V}$$

for a complex $V \in D^b(\mathrm{Vect}_\mathbb{C})$. Hence,

$$H^*_{T_c}(i_\nu^*(\mathrm{IC}_\epsilon^{\star j}) \simeq H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}^\lambda, \mathbb{C}) \otimes V.$$

We have shown that $H^*_{T_c}(i_\nu^*(\mathrm{IC}_\epsilon^{\star j})$ is concentrated in even degrees. Since $H^0_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}^\lambda, \mathbb{C}) \neq 0$, we conclude that $V$ is concentrated in even degrees. The result follows. □

As a corollary of the proof we obtain the following parity vanishing result.

Corollary 4.14. *We have* $\mathscr{H}^{i-\langle\lambda,\rho_{2n}\rangle}(\mathrm{IC}_\lambda) = 0$ *for* $i \nmid 4$.

*Proof.* We have shown that any direct summand $\mathrm{IC}_\lambda$ of $\mathrm{IC}_\epsilon^{\star j} \star \mathrm{IC}_{\omega_1}^{\star k}$ satisfies $\mathscr{H}^{i-2k(n-1)}(\mathrm{IC}_\lambda)Z = 0$ for $i \nmid 4$. Since $k\omega_1 - \lambda$ is a non-negative integral sum of positive coroots, we have $\langle k\omega_1 - \lambda, \rho_n\rangle \in \mathbb{Z}$, and hence $2k(n-1) - \langle\lambda, \rho_{2n}\rangle = \langle k\omega_1 - \lambda, \rho_{2n}\rangle = 4\langle k\omega_1 - \lambda, \rho_n\rangle$ is divisible by four. The desired claim follows. □

4.7.2 Our goal is now to apply the parity vanishing result above to deduce the following faithfulness result.

Proposition 4.15. *For any* $\lambda, \mu \in \Lambda_S^+$, *the natural map*

$$\mathrm{Ext}^\bullet_{D^b_{\mathfrak{L}^+ G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \to \mathrm{Hom}^\bullet_{H^*_{\mathfrak{L}^+ G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{\mathfrak{L}^+ G_{n,\mathbb{H}}}(\mathrm{IC}_\lambda), H^*_{\mathfrak{L}^+ G_{n,\mathbb{H}}}(\mathrm{IC}_\mu))$$

*is an isomorphism of graded modules.*



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

We will deduce Proposition 4.15 as a consequence of the following more general result.

Proposition 4.16. *Let* $\mathcal{F}, \mathcal{G} \in D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$. *Assume that* $\mathcal{F}$ *and* $\mathcal{G}$ *are even. Then the natural map*

$$\mathrm{Ext}^\bullet_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}^\bullet_{H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{F}), H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{G}))$$

*is an isomorphism of graded modules.*

*Proof of Proposition 4.15.* By Proposition 4.13, we know that $\mathrm{IC}_\lambda$ and $\mathrm{IC}_\mu$ are even complexes. The claim now follows from Proposition 4.16. ☐

4.7.3 In the proof of Proposition 4.16, we will make use of the following terminology. Consider a triangulated functor

$$\Omega : D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}}) \to D^b(\mathrm{Vect}_{\mathbb{C}}).$$

We say that $\Omega$ is $*$-*parity preserving* (respectively, !-*parity preserving*) if it takes $*$-even (respectively, !-even) complexes of sheaves to even complexes of vector spaces. If $\Omega$ is both $*$-parity preserving and !-parity preserving, we will simply say that it is parity preserving. We use the same terminology for functors

$$\Omega : D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})^{\mathrm{op}} \to D^b(\mathrm{Vect}_{\mathbb{C}}).$$

To check that functors are parity preserving, we will use the following criterion.

Lemma 4.17. *Let*

$$\Omega : D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}}) \to D^b(\mathrm{Vect}_{\mathbb{C}})$$

*be a triangulated functor. Then* $\Omega$ *is* $*$-*even if and only if for each* $\lambda \in \Lambda^+_S$ *the complex* $\Omega(j_{\lambda !}\underline{\mathbb{C}})$ *is even. Here,* $j_\lambda : \mathrm{Gr}^\lambda_{n,\mathbb{H}} \hookrightarrow \mathrm{Gr}_{n,\mathbb{H}}$ *is the natural inclusion.*

*Proof.* We assume that the latter condition holds and prove that $\Omega$ is $*$-parity preserving. Let $\mathcal{F} \in D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$ be $*$-even. We must show that $\Omega(\mathcal{F})$ is even, which we do by induction on the support of $\mathcal{F}$ (which is a finite union of $\mathfrak{L}^+G_{n,\mathbb{H}}$-orbits). Let

$$j : \mathrm{Gr}^\lambda_{n,\mathbb{H}} \hookrightarrow \mathrm{Gr}_{n,\mathbb{H}}$$

denote the inclusion of a $\mathfrak{L}^+G_{n,\mathbb{H}}$-orbit open in the support of $\mathcal{F}$. Let

$$i : \overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}} \setminus \mathrm{Gr}^\lambda_{n,\mathbb{H}} \hookrightarrow \mathrm{Gr}_{n,\mathbb{H}}$$

denote the complementary closed embedding. We have a triangle

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to .$$

Applying $\Omega$ yields

$$\Omega(j_! j^! \mathcal{F}) \to \Omega(\mathcal{F}) \to \Omega(i_* i^* \mathcal{F}) \to .$$

By induction, we may assume that $\Omega(i_* i^* \mathcal{F})$ is even. On the other hand, $\mathcal{F}$ is $\mathfrak{L}^+G_{n,\mathbb{H}}$ equivariant and $*$-even. Hence, $j^!\mathcal{F}$ is a direct sum of complexes of the form $\underline{\mathbb{C}}[m]$ for $m$ even. Therefore, $\Omega(j_! j^! \mathcal{F})$ is $*$-even, $\Omega(\mathcal{F})$ is $*$-even, and $\Omega$ is $*$-parity preserving.

For the converse, observe that each $j_{\lambda !}\underline{\mathbb{C}}$ is $*$-even, as its only non-trivial stalk is isomorphic to $H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{pt})$. Hence $\Omega(j_{\lambda !}\underline{\mathbb{C}})$ is $*$-even. ☐



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Corollary 4.18.

(i) *The functor* $\Gamma_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}},-)$ *is* $*$*-parity preserving.*

(ii) *Let* $\mathcal{G}\in D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$ *be* !*-even. Then the functor* $\mathrm{Hom}_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(-,\mathcal{G})$ *is* $*$*-parity preserving.*

*Proof.*

(i) We claim that the cohomology $H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j_{!\lambda}\underline{\mathbb{C}})$ is even. By the long exact sequence, it suffices to show that each $H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}},\mathbb{C})$ is even. The non-equivariant cohomology $H^*(\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}},\mathbb{C})$ is even because $\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}}$ is paved by quaternionic affine spaces. Therefore $j_{\lambda!}\underline{\mathbb{C}}$ is $\mathfrak{L}^+G_{n,\mathbb{H}}$-equivariantly formal, and so $H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}},\mathbb{C})\simeq H^*(\overline{\mathrm{Gr}^\lambda_{n,\mathbb{H}}},\mathbb{C})\otimes H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{pt},\mathbb{C})$ is even. By Lemma 4.17, we conclude that $\Gamma_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}},-)$ is $*$-parity preserving.

(ii) By Lemma 4.17, we must check that

$$\mathrm{Ext}^i_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(j_{\lambda!}\underline{\mathbb{C}},\mathcal{G})\simeq 0$$

for each $\lambda\in X^+_A$ and $i$ odd. By adjunction,

$$\mathrm{Ext}^i_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(j_{\lambda!}\underline{\mathbb{C}},\mathcal{G})\simeq \mathrm{Ext}^i_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(\underline{\mathbb{C}},j^!_\lambda\mathcal{G})\simeq H^i_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j^!_\lambda\mathcal{G}).$$

The claim follows from the assumption that $\mathcal{G}$ is !-even. □

Lemma 4.19. *Let* $\mathcal{F}\in D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$ *be* $*$*-even. Suppose that* $X\subseteq \mathrm{Gr}_{n,\mathbb{H}}$ *is a closed finite union of* $\mathfrak{L}^+G_{n,\mathbb{H}}$*-orbits such that* $X$ *contains the support of* $\mathcal{F}$*. Let* $Z\subseteq X$ *denote a* $\mathfrak{L}^+G_{n,\mathbb{H}}$*-stable closed subset, and* $U=X\setminus Z$ *its open complement. Let* $j:U\hookrightarrow X$ *and* $i:Z\hookrightarrow X$ *denote the natural inclusions. We have a triangle*

$$j_!j^!\mathcal{F}\to\mathcal{F}\to i_*i^*\mathcal{F}\to .$$

*Now let* $\Omega: D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})\to D^b(\mathrm{Vect}_{\mathbb{C}})$ *denote a* $*$*-parity preserving functor. Then the triangle*

$$\Omega(j_!j^!\mathcal{F})\to\Omega(\mathcal{F})\to\Omega(i_*i^*\mathcal{F})\to$$

*is split.*

*Proof.* We must show that the boundary map $\delta:\Omega(i_*i^*\mathcal{F})\to\Omega(j_!j^!\mathcal{F})[1]$ is zero. Observe that the functors $j_!j^!$ and $i_*i^*$ take $*$-even sheaves to $*$-even sheaves. Since $\Omega$ is $*$-parity preserving, the complexes $\Omega(i_*i^*\mathcal{F})$ and $\Omega(j_!j^!\mathcal{F})$ are even. Hence, $\Omega(j_!j^!\mathcal{F})[1]$ is odd. Therefore, $\delta$ induces the zero map in cohomology and so is zero. □

Lemma 4.20. *Let* $\mathcal{F},\mathcal{G}\in D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$*. We make the following assumptions:*

(1) $\mathcal{F}$ *is* $*$*-even;*

(2) $\mathcal{G}$ *is* !*-even;*

(3) *for any* $\mu\in\Lambda^+_S$*, the map*

$$H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{F})\to H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j^*_\mu\mathcal{F})$$

*is surjective;*

(4) *for any* $\mu\in\Lambda^+_S$*, the map*

$$H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j_{\mu!}j^!_\mu\mathcal{G})\to H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{G})$$

*is injective.*



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

*Then the natural map*

$$\mathrm{Ext}^*_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}^*_{H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{F}), H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{G}))$$

*is an isomorphism of graded modules.*

*Proof.* Let $Z$ denote the union of the supports of $\mathcal{F}$ and $\mathcal{G}$. We proceed by induction on $Z$. Certainly there exists an orbit $\mathrm{Gr}^\lambda_{n,\mathbb{H}}$ open in $Z$. Let $Y = Z \setminus \mathrm{Gr}^\lambda_{n,\mathbb{H}}$, and let $i_\lambda : Y \hookrightarrow Z$ denote the inclusion. We claim that the pair $i^*_\lambda\mathcal{F}$ and $i^!_\lambda\mathcal{G}$ satisfies the assumptions (1)–(4).

That $i^*_\lambda\mathcal{F}$ is $*$-even and that $i^!_\lambda\mathcal{G}$ is !-even is evident. We verify (3) for $i^*_\lambda\mathcal{F}$. If $\mathrm{Gr}^\mu_{n,\mathbb{H}}$ does not lie in the support of $i^*_\lambda\mathcal{F}$, then there is nothing to prove. So we may assume that $\mathrm{Gr}^\mu_{n,\mathbb{H}} \subseteq Y$. Therefore, we have the composition of maps

$$H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathcal{F}) \to H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(i^*_\lambda\mathcal{F}) \to H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j^*_\mu i^*_\lambda\mathcal{F}) \simeq H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j^*_\mu\mathcal{F}).$$

The composite is surjective by assumption. Hence, the map

$$H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(i^*_\lambda\mathcal{F}) \to H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(j^*_\mu i^*_\lambda\mathcal{F})$$

is surjective, as needed. The proof that $i^!_\lambda\mathcal{G}$ satisfies (4) is similar.

Now we proceed with the induction. To avoid overly cumbersone notation, we will suppress the subscripts on $\mathrm{Ext}^*_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}$ and $\mathrm{Hom}^*_{H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}$. Similarly, we will make use of the isomorphism $H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}} \simeq H^*_G$ to simplify notation. Lastly, we let $H = H^*_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})$.

Consider the triangle

$$j_{\lambda!}j^!_\lambda\mathcal{F} \to \mathcal{F} \to i_{\lambda*}i^*_\lambda\mathcal{F} \to . \tag{4.15}$$

By Lemma 4.19, Corollary 4.18, and adjunction, we have an exact sequence

$$0 \to \mathrm{Ext}^*(i^*_\lambda\mathcal{F}, i^!_\lambda\mathcal{G}) \to \mathrm{Ext}^*(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^*(j^!_\lambda\mathcal{F}, j^!_\lambda\mathcal{G}) \to 0. \tag{4.16}$$

We can also apply the functor $H^*$ to (4.15) to obtain the exact sequence

$$0 \to H^*_G(j_{\lambda!}j^!_\lambda\mathcal{F}) \to H^*_G(\mathcal{F}) \to H^*_G(i^*_\lambda\mathcal{F}) \to 0.$$

Similarly, we have the exact sequence

$$0 \to H^*_G(i^!_\lambda\mathcal{G}) \to H^*_G(\mathcal{G}) \to H^*_G(j^*_\lambda\mathcal{G}) \to 0. \tag{4.17}$$

These two exact sequences induce a sequence

$$0 \to \mathrm{Hom}^*(H^*_G(i^*_\lambda\mathcal{F}), H^*_G(i^!_\lambda\mathcal{G})) \to \mathrm{Hom}^*(H^*_G(\mathcal{F}), H^*_G(\mathcal{G})) \to \mathrm{Hom}^*(H^*_G(\mathcal{F}), H^*_G(j^*_\lambda\mathcal{G})).$$

The second map is clearly an injection. We claim that the sequence is also exact in the middle. It suffices to show that any $H$-linear map

$$\alpha : H^*_G(\mathcal{F}) \to H^*_G(i^!_\lambda\mathcal{G})$$

factors through $H^*_G(i^*_\lambda\mathcal{F})$. Consider the compactly supported cohomology $H^*_{G,c}(\mathrm{Gr}^\lambda_{n,\mathbb{H}})$. Let $\mathfrak{c}_\lambda \in H^{\langle 2\rho_{2n},\lambda\rangle}_{G,c}(\mathrm{Gr}^\lambda_{n,\mathbb{H}})$ denote a lift of a generator to $G$-equivariant cohomology; it maps to an element $\mathfrak{c}_\lambda \in H$. Since $\mathfrak{c}_\lambda$ maps to $0 \in H^*_G(Y)$, it acts trivially on $H^*(i^!_\lambda\mathcal{G})$. Since $\alpha$ is $H$-linear, it suffices to show that $H^*(j_{\lambda!}j^!_\lambda\mathcal{F})$ lies in the image of

$$\mathfrak{c}_\lambda : H^*_G(\mathcal{F}) \to H^*_G(\mathcal{F}[-\langle 2\rho_{2n},\lambda\rangle]).$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

To do so, we note that by Poincaré duality for the smooth manifold $\mathrm{Gr}^{\lambda}_{n,\mathbb{H}}$, cupping with $\mathfrak{c}_\lambda$ induces an isomorphism

$$\mathfrak{c}_\lambda : H^*_G(j_{\lambda !}j^!_\lambda \mathcal{F}) \to H^*_G(j_{\lambda *}j^*_\lambda \mathcal{F}[-\langle 2\rho_{2n}, \lambda\rangle]).$$

Thus we obtain the following commutative diagram.

$$\begin{array}{ccc} H^*_G(\mathcal{F}) & \xrightarrow{\mathfrak{c}_\lambda} & H^*_G(\mathcal{F}[-\langle 2\rho_{2n}, \lambda\rangle]) \\ \uparrow & & \downarrow \\ H^*_G(j_{\lambda !}j^!_\lambda \mathcal{F}) & \xrightarrow{\mathfrak{c}_\lambda} & H^*_G(j_{\lambda *}j^*_\lambda \mathcal{F}[-\langle 2\rho_{2n}, \lambda\rangle]) \end{array}$$

As the bottom arrow is an isomorphism, it suffices to show that the right vertical map is surjective. But this is assumed in (3).

Next, we observe that any $H$-linear map $\beta : H^*_G(\mathcal{F}) \to H^*_G(j^*_\lambda \mathcal{G})$ factors through $H^*_G(j^*_\lambda \mathcal{F})$. The proof is similar to that of the previous step, using (4) in place of (3), and is therefore omitted.

Hence, we have an exact sequence

$$0 \to \mathrm{Hom}^*(H^*_G(i^*_\lambda \mathcal{F}), H^*_G(i^!_\lambda \mathcal{G})) \to \mathrm{Hom}^*(H^*_G(\mathcal{F}), H^*_G(\mathcal{G})) \to \mathrm{Hom}^*(H^*_G(j^*_\lambda \mathcal{F}), H^*_G(j^*_\lambda \mathcal{G})).$$

To conclude, we observe that this exact sequence fits into the following commutative diagram with (4.16).

$$\begin{array}{ccccc} \mathrm{Ext}^*(i^*_\lambda \mathcal{F}, i^!_\lambda \mathcal{G}) & \longrightarrow & \mathrm{Ext}^*(\mathcal{F}, \mathcal{G}) & \longrightarrow & \mathrm{Ext}^*(j^*_\lambda \mathcal{F}, j^*_\lambda \mathcal{G}) \\ \downarrow f & & \downarrow g & & \downarrow h \\ \mathrm{Hom}^*_H(H^*_G(i^*_\lambda \mathcal{F}), H^*_G(i^!_\lambda \mathcal{G})) & \longrightarrow & \mathrm{Hom}^*_H(H^*_G(\mathcal{F}), H^*_G(\mathcal{G})) & \longrightarrow & \mathrm{Hom}^*_H(H^*_G(j^*_\lambda \mathcal{F}), H^*_G(j^*_\lambda \mathcal{G})) \end{array}$$

The map $f$ is an isomorphism by induction, and $h$ is easily seen to be an isomorphism. Hence $g$ is an isomorphism, as claimed. □

*Proof of Proposition 4.16.* It suffices to verify that $\mathcal{F}$ and $\mathcal{G}$ satisfy the hypotheses of Lemma 4.20. The properties (1) and (2) are assumed. We will show that (3) holds; the proof of (4) is similar. We must show that for each $\lambda \in \Lambda^+_S$, the map

$$H^*_G(\mathcal{F}) \to H^*_G(j^*_\lambda \mathcal{F})$$

is surjective. It identifies with

$$H^*_{T_c}(\mathcal{F})^{W_n} \to H^*_{T_c}(j^*_\lambda \mathcal{F})^{W_n}.$$

Since the coefficient field has characteristic zero, the functor of $W_n$-invariants is exact, and it suffices to show that the restriction map $H^*_{T_c}(\mathcal{F}) \to H^*_{T_c}(j^*_\lambda \mathcal{F})$ is surjective. We let

$$k_\lambda : (\mathrm{Gr}^{\lambda}_{n,\mathbb{H}})^{T_c} \hookrightarrow \mathrm{Gr}_{n,\mathbb{H}}$$

denote the inclusion of the $T_c$-fixed locus in $\mathrm{Gr}^{\lambda}_{n,\mathbb{H}}$. Now, consider the composition

$$H^*_{T_c}(\mathcal{F}) \to H^*_{T_c}(j^*_\lambda \mathcal{F}) \to H^*_{T_c}(k^*_\lambda \mathcal{F}). \tag{4.18}$$

Observe that $j^*_\lambda \mathcal{F}$ is a constant sheaf and that $\mathrm{Gr}^{\lambda}_{n,\mathbb{H}}$ is an equivariantly formal $T_c$-manifold. Hence, the second map above is injective by the localization theorem. The surjectivity of the first map is then reduced to that of the composition. Now, $k_\lambda$ is a *closed* inclusion, so $k_{\lambda *}k^*_\lambda \mathcal{F}$ is $*$-even. The proof of Lemma 4.19, applied in the $T_c$-equivariant derived category, shows that the restriction map $H^*_{T_c}(\mathcal{F}) \to H^*_{T_c}(k^*_\lambda \mathcal{F})$ is indeed surjective. □



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

### 4.8 Ext algebras

The tensor equivalence $\mathrm{Rep}(G_n)\simeq \mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ gives rise to a monoidal action of $\mathrm{Rep}(G_n)$ on $D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})$. We compute the de-equivariantized extension algebra

$$\mathrm{Ext}^*_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star \mathcal{O}(G_n)).$$

Strictly speaking, $\mathcal{O}(G_n)$ is not an object of $\mathrm{Rep}(G_n)$, but by the Peter–Weyl theorem, it is an increasing direct sum of objects. We understand the above extension algebra to mean the increasing direct sum of extensions.

Proposition 4.21. *There is a $G_n$-equivariant isomorphism of graded algebras*

$$\mathrm{Ext}^*_{D^b_{\mathfrak{L}^+G_{n,\mathbb{H}}}(\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star \mathcal{O}(G_n))\simeq \mathcal{O}(\mathfrak{g}_n[4])\simeq \mathrm{Sym}(\mathfrak{g}_n[-4]).$$

*Proof.* By Proposition 4.15, taking equivariant cohomology induces a $G_n$-equivariant isomorphism of graded algebras

$$\begin{aligned}\mathrm{Ext}^*_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star \mathcal{O}(G_n)) &\simeq (\mathrm{Hom}^*_{H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{T_c}(\mathrm{pt}),H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_0\star \mathcal{O}(G_n))^{\mathrm{W}}\\ &\simeq (\mathrm{Hom}^*_{H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}})}(\mathcal{O}(\mathfrak{t}),\mathcal{O}(G_n\times \mathfrak{t})))^{\mathrm{W}}\\ &\simeq (\mathcal{O}(G_n\times \mathfrak{t})^{\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}}))})^{\mathrm{W}},\end{aligned}$$

where $\mathcal{O}(G_n\times \mathfrak{t})^{\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}}))}\subset \mathcal{O}(G_n\times \mathfrak{t})$ is the subspace consisting of functions that are invariant (relative over $\mathfrak{t}$) with respect to the left action of the group scheme $\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}}))\simeq (G_n\times \mathfrak{t})^{e_X^T}$ on $G_n\times\mathfrak{t}$. Since $\mathcal{O}(\mathfrak{g}_n^{\mathrm{reg}}\times_{\mathfrak{c}_n}\mathfrak{t})=\mathcal{O}(\mathfrak{g}_n\times_{\mathfrak{c}_n}\mathfrak{t})$ and the map

$$\nu: G_n\times \mathfrak{t}\to \mathfrak{g}_n^{\mathrm{reg}}\times_{\mathfrak{c}_n}\mathfrak{t},\quad (g,t)\to(\mathrm{Ad}_{g^{-1}}e_X^T(t),t) \tag{4.19}$$

realizes $G_n\times\mathfrak{t}$ as a $(G_n\times\mathfrak{t})^{e_X^T}$-torsor over $\mathfrak{g}_n^{\mathrm{reg}}\times_{\mathfrak{c}_n}\mathfrak{t}$, we obtain an isomorphism of algebras

$$\begin{aligned}\mathrm{Ext}^*_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star \mathcal{O}(G_n)) &\simeq (\mathcal{O}(G_n\times \mathfrak{t})^{\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}}))})^{\mathrm{W}}\\ &\simeq \mathcal{O}(\mathfrak{g}_n^{\mathrm{reg}}\times_{\mathfrak{c}_n}\mathfrak{t})^{\mathrm{W}}\simeq \mathcal{O}(\mathfrak{g}_n\times_{\mathfrak{c}_n}\mathfrak{t})^{\mathrm{W}}\simeq \mathcal{O}(\mathfrak{g}_n).\end{aligned}$$

It remains to check that the isomorphism above is compatible with the desired gradings. By [Nad05, Theorem 8.5.1], for any $\lambda\in\Lambda_S$ and $\mathcal{F}\in \mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$, the compactly supported cohomology $H_c^*(S^\lambda_{n,\mathbb{H}},\mathcal{F})$ along the real semi-infinite orbit $S^\lambda_{n,\mathbb{H}}$ is non-zero only in degree $\langle\lambda,\rho_{2n}\rangle$. Note that $\langle\lambda,\rho_{2n}\rangle=4\langle\lambda,\rho_n\rangle$, where in the second paring we regard $\lambda$ as an element in $\Lambda_n$. Thus the grading on $H^*(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_\lambda)$ corresponds, under the geometric Satake equivalence, to the grading on $V_\lambda$ given by co-character $4\rho_n$, and it follows that the grading on $H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}},\mathrm{IC}_0\star\mathcal{O}(G_n))\simeq \mathcal{O}(G_n\times\mathfrak{t})$ is induced by the $\mathbb{G}_m$-action on $G_n\times\mathfrak{t}$ given by $x(g,t)=(4\rho_n(x)g,x^{-2}t)$ (note that the generators of $\mathcal{O}(\mathfrak{t})$ are in degree 2). We claim that the map $\nu$ in (4.19) is $\mathbb{G}_m$-equivariant with respect to the above action on $G_n\times\mathfrak{t}$ and the action on $\mathfrak{g}_n^{\mathrm{reg}}\times_{\mathfrak{c}_n}\mathfrak{t}$ given by $x(v,t)=(x^{-4}v,x^{-2}t)$. Indeed, we have

$$\begin{aligned}\mathrm{Ad}_{4\rho_n(x^{-1})}e_X^T(x^{-2}t) &= \mathrm{Ad}_{4\rho_n(x^{-1})}\begin{pmatrix} x^{-4}t_1^2 & 1 & & \\ \vdots & x^{-4}t_2^2 & \ddots & \\ \vdots & & \ddots & 1\\ 0 & 0 & \dots & x^{-4}t_n^2\end{pmatrix}\\ &= \begin{pmatrix} x^{-4}t_1^2 & x^{-4} & & \\ \vdots & x^{-4}t_2^2 & \ddots & \\ \vdots & & \ddots & x^{-4}\\ 0 & 0 & \dots & x^{-4}t_n^2\end{pmatrix}\\ &= x^{-4}e_X^T(t),\end{aligned}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

and hence

$$\nu(x(g,t)) = \nu(4\rho_n(x)g, x^{-2}t) = (\mathrm{Ad}_{g^{-1}}\mathrm{Ad}_{4\rho_n(x)^{-1}} e_X^T(x^{-2}t), x^{-2}t)$$
$$= (x^{-4}\mathrm{Ad}_{g^{-1}} e_X^T(t), x^{-2}t) = x(\mathrm{Ad}_{g^{-1}} e_X^T(t), t) = x\nu(g,t).$$

Thus the pullback along the map $\nu$ induces an isomorphism of graded algebras

$$(\mathcal{O}(G_n \times \mathfrak{t})^{\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}}))})^{\mathrm{W}} \simeq \mathcal{O}(\mathfrak{g}_n^{\mathrm{reg}}[4] \times_{\mathfrak{g}_n[4]//G_n} \mathfrak{t}[2])^{\mathrm{W}} \simeq \mathcal{O}(\mathfrak{g}_n[4]).$$

This finishes the proof of the theorem. □

### 4.9 IC-stalks, $q$-analogue of weight multiplicity, and Kostka–Foulkes polynomials

In this section we shall prove Theorem 1.9(2). We will follow Ginzburg's approach [Gin95] (see also [Zhu15, § 5]) using techniques of equivariant cohomology.

Let $V \in \mathrm{Rep}(G_n)$. Consider the Brylinski–Kostant filtration $F_iV := \ker \mathbf{e}_n^{i+1}, i \geq 0$ on $V$ associated to the regular nilpotent element $\mathbf{e}_n$. For any $\mu \in \Lambda_n$, we denote by $V(\mu)$ the $\mu$-weight space of $V$ (since $G_n$ is self-dual, we can view $\Lambda_n$ as the weight lattice of $G_n$). The filtration $F_iV$ induces a filtration on the weight space:

$$F_iV(\mu) = F_iV \cap V(\mu).$$

Let

$$P_\mu(V,q) = \sum_i \dim(F_iV(\mu)/F_{i-1}V(\mu))q^i$$

be the $q$-analogue of the weight multiplicity polynomial.

From now on we will identify $\Lambda_n$ with the set $\Lambda_S$ of real coweights and denote by $s_\mu : \{\mu\} \to \Lambda_n \simeq \Lambda_S \subset \mathrm{Gr}_{n,\mathbb{H}}$ the inclusion map.

Theorem 4.22. *Let $\mathcal{F} \in \mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$ and let $V = H^*(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F})$ be the corresponding representation of $G_n$. We have*

$$P_\mu(V,q) = \sum_i \dim H^{-4i-4\langle\mu,\rho_n\rangle}(s_\mu^*\mathcal{F})q^i = \sum_i \dim H^{4i+4\langle\mu,\rho_n\rangle}(s_\mu^!\mathcal{F})q^i.$$

The theorem above implies Theorem 1.9(2) in the case of the quaternionic affine Grassmannian. Indeed, if $\mu, \lambda \in \Lambda_n^+$ and $V = V_\lambda$ is the irreducible representation of highest weight $\lambda$, then it is known that $P_\mu(V_\lambda, q) = K_{\lambda,\mu}(q)$ is the Kostka–Foulkes polynomial associated to $\lambda$ and $\mu$ (see e.g. [Bry89]). Thus, for any $x \in \mathrm{Gr}_{n,\mathbb{H}}^\mu$, we have

$$K_{\lambda,\mu}(q) = \sum_i \dim H^{-4i-4\langle\mu,\rho_n\rangle}(s_\mu^*\mathcal{F})q^i = \sum_i \dim \mathscr{H}_x^{-4i-4\langle\mu,\rho_n\rangle}(\mathrm{IC}_\lambda)q^i,$$

and it follows that

$$q^{\langle\lambda-\mu,\rho_n\rangle}K_{\lambda,\mu}(q^{-1}) = \sum_i \dim \mathscr{H}_x^{-4i-4\langle\mu,\rho_n\rangle}(\mathrm{IC}_\lambda)q^{-i-\langle\mu,\rho_n\rangle+\langle\lambda,\rho_n\rangle} = \sum_i \dim \mathscr{H}_x^{4i-4\langle\lambda,\rho_n\rangle}(\mathrm{IC}_\lambda)q^i.$$

The case of $\mathfrak{L}K$-orbits on $\mathrm{Gr}_{2n}$ follows from the fact [CN24, Theorem 7.5] that there is a stratified $K_c$-equivariant homeomorphism between $\Omega K_c\backslash\mathrm{Gr}_{2n}$ and $\mathrm{Gr}_{n,\mathbb{H}}$ (where $\Omega K_c$ is the based loop group of $K_c$) with stratifications given by images of $\mathfrak{L}K$-orbits on $\mathrm{Gr}_{2n}$ in the quotient $\Omega K_c\backslash\mathrm{Gr}_{2n}$ and the $\mathfrak{L}^+G_{n,\mathbb{H}}$-orbits on $\mathrm{Gr}_{n,\mathbb{H}}$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

4.9.1 *Proof of Theorem 4.22.* We follow closely the presentation in [Zhu15, § 5]. For any $t \in \mathfrak{t}$ we denote by $\kappa(t)$ the residue field of $t$. The specialized cohomology

$$H_t(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) := H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \otimes_{R_T} \kappa(t)$$

carries a canonical filtration

$$H_t^{\leq i}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) := \mathrm{Im}\bigg( \sum_{j \leq i} H^j_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \to H_t(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \bigg).$$

Let us identify $H_t(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \simeq (H^*(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \otimes R_T) \otimes_{R_T} \kappa(t) \simeq V$ via the canonical splitting in (4.11). As explained in the proof of Proposition 4.21, the cohomological grading on $H^*(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F})$ corresponds to the grading on the representation $V$ given by the eigenvalues of $4\rho_n$. It follows that the filtration $H_t^{\leq i}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F})$ corresponds to the increasing filtration on $V$ given by the eigenvalues of $4\rho_n$ (see e.g. [Gin95, Theorem 5.2.1]).

Fix a generic element $t = (t_1, \dots, t_n) \in \mathfrak{t}$ away from the root hyperplanes. The localization theorem implies that there is an isomorphism

$$\bigoplus_{\mu \in \Lambda_n} H_t(s_\mu^! \mathcal{F}) \simeq H_t(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}). \tag{4.20}$$

Recall the description of the equivariant homology $\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}})) \simeq (G_n \times \mathfrak{t})^{e_X^T}$ in Lemma 4.9. The fiber of the group scheme $(G_n \times \mathfrak{t})^{e_X^T}$ over $t$ is the centralizer subgroup $(G_n)^{e_X^T(t)} \subset G_n$ of the element $e_X^T(t) \in \mathfrak{g}_n$ in (3.14) . Let $B_n \to T_n$ be the natural projection. It is shown in [O'Br23, Lemma 4] (generalizing [YZ11, Remark 3.4]) that the composition

$$\mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}})) \simeq (G_n \times \mathfrak{t})^{e_X^T} \subset B_n \times \mathfrak{t} \to T_n \times \mathfrak{t}$$

can be identified with the map coming from equivariant localization,

$$\mathrm{Spec}(\mathrm{Loc}_*) : \mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}})) \to \mathrm{Spec}(R_T[\Lambda_n]) = T_n \times \mathfrak{t}.$$

Over $t$, this is an isomorphism and therefore we obtain a canonical isomorphism $(G_n)^{e_X^T(t)} \simeq T_n \times \{t\} \simeq T_n$. In addition, the action of $(G_n)^{e_X^T(t)}$ on $H_t(S^\mu_{n,\mathbb{H}}, \mathcal{F})$ via $(G_n)^{e_X^T(t)} \simeq T_n$ is identified with the natural action of $(G_n)^{e_X^T(t)}$ on $H_t(S^\mu_{n,\mathbb{H}}, \mathcal{F}) \simeq H_t(s_\mu^! \mathcal{F})$. Thus we conclude that the decomposition in (4.20) corresponds to the weight decomposition under $(G_n)^{e_X^T(t)}$ (generalizing [Zhu15, Proposition 5.2]).

Lemma 4.23. *The decomposition in (4.20) corresponds, under the canonical isomorphism $H_t(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \simeq V$, to the weight decomposition $V = \oplus_{\mu \in \Lambda_n} V(\mu_t)$ with respect to the action of the maximal torus $(G_n)^{e_X^T(t)}$. Here $V(\mu_t)$ is the weight space associated to the character $\mu_t : (G_n)^{e_X^T(t)} \simeq T_n \xrightarrow{\mu} \mathbb{C}^\times$.*

Choose $t \in \mathfrak{t}$ such that $e_X^T(t) = \mathbf{e}_n + 2\rho_n$. Let $u$ be the unique element in $N_n$ such that $\mathrm{Ad}_u(\mathbf{e}_n + 2\rho_n) = 2\rho_n$.

Lemma 4.24. *We have*

$$H_t^{\leq 4i+2m}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \cap \bigoplus_{\mu \in \Lambda_n, 2\langle \mu, \rho_n \rangle = m} H_t(s_\mu^! \mathcal{F}) = F_i V \cap \bigoplus_{\mu \in \Lambda_n, 2\langle \mu, \rho_n \rangle = m} V(\mu_t).$$

*Proof.* Let $V = \bigoplus V^1(i)$ and $V = \bigoplus V^2(i)$ be two gradings on $V$ given by the cocharacters $2\rho_n$ and $\mathrm{Ad}_{u^{-1}} 2\rho_n$, respectively. Let $F_i^1 V$ and $F_i^2 V$ be the two filtrations on $V$ given by $F_i^1 V = \oplus_{j \leq i} V^1(j)$ and $F_i^2 V = \ker(\mathbf{e}_n^{i+1})$. We have

$$F_i V \cap \bigoplus_{\mu \in \Lambda_n, 2\langle \mu, \rho_n \rangle = m} V(\mu_t) = F_i^2 V \cap V^2(m)$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

and

$$H_t^{\leq 4i+2m}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \cap \bigoplus_{\mu\in\Lambda_n, 2\langle\mu,\rho_n\rangle=m} H_t(s_\mu^! \mathcal{F}) = F^1_{2i+m}(V) \cap V^2(m),$$

and the desired claim follows from [Zhu15, Lemma 5.5]. □

Note that we have shown in (4.18) that the natural map $H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \to H^*_{T_c}(s_\mu^* \mathcal{F})$ is a surjective map of free $R_T$-modules, and it implies that the dual map $H^*_{T_c}(s_\mu^! \mathcal{F}) \to H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F})$ is a splitting injective map of free $R_T$-modules. Thus we have

$$H_t^{\leq i}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \cap H_t(s_\mu^! \mathcal{F}) = H_t^{\leq i}(s_\mu^! \mathcal{F}).$$

On the other hand, the element $u \in N_n$ above maps $V(\mu_t)$ to $V(\mu)$ and preserves the filtration $F_iV$, and hence $\dim(F_iV(\mu)/F_{i-1}V(\mu)) = \dim(F_iV(\mu_t)/F_{i-1}V(\mu_t))$. Now the lemma above implies

$$\begin{aligned} P_\mu(V, q) &= \sum_i \dim(F_iV(\mu)/F_{i-1}V(\mu))q^i = \sum_i \dim(F_iV(\mu_t)/F_{i-1}V(\mu_t))q^i \\ &= \sum_i \dim(H_t^{\leq 4i+4\langle\mu,\rho_n\rangle}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \cap H_t(s_\mu^! \mathcal{F})/H_t^{\leq 4(i-1)+4\langle\mu,\rho_n\rangle}(\mathrm{Gr}_{n,\mathbb{H}}, \mathcal{F}) \cap H_t(s_\mu^! \mathcal{F}))q^i \\ &= \sum_i \dim(H_t^{\leq 4i+4\langle\mu,\rho_n\rangle}(s_\mu^! \mathcal{F})/H_t^{\leq 4(i-1)+4\langle\mu,\rho_n\rangle}(s_\mu^! \mathcal{F}))q^i. \end{aligned}$$

To conclude the proof, we observe that under the canonical isomorphism $H_t(s_\mu^! \mathcal{F}) \simeq H^*(s_\mu^! \mathcal{F})$, the canonical filtration on the left-hand side corresponds to the cohomological degree filtration on the right-hand side, and hence we obtain

$$P_\mu(V, q) = \sum_i \dim(H_t^{\leq 4i+4\langle\mu,\rho_n\rangle}(s_\mu^! \mathcal{F})/H_t^{\leq 4(i-1)+4\langle\mu,\rho_n\rangle}(s_\mu^! \mathcal{F}))q^i = \sum_i \dim\, H^{4i+4\langle\mu,\rho_n\rangle}(s_\mu^! \mathcal{F})q^i.$$

## 5. Main results

### 5.1 Formality

The goal of this section is to show that the dg-algebra

$$\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \mathcal{O}(G_n))$$

is formal.

The proof is based on the following key proposition. The existence of the left adjoint of the nearby cycles functor in Lemma 4.8 gives rise to a map between $K_c$-equivariant cohomology

$$H^*_{K_c}(\mathrm{Gr}_{n,\mathbb{H}}) \simeq \mathrm{Ext}^*(\mathbb{C}_{\mathrm{Gr}_{n,\mathbb{H}}}, \mathbb{C}_{\mathrm{Gr}_{n,\mathbb{H}}}) \xrightarrow{{}^L\mathrm{R}'} \mathrm{Ext}^*(\mathbb{C}_{\mathrm{Gr}_{2n}}, \mathbb{C}_{\mathrm{Gr}_{2n}}) \simeq H^*_{K_c}(\mathrm{Gr}_{2n}). \tag{5.1}$$

By taking the graded dual (see § 4.6.2), we get a map between equivariant homology

$$H_*^{K_c}(\mathrm{Gr}_{2n}) \to H_*^{K_c}(\mathrm{Gr}_{n,\mathbb{H}}). \tag{5.2}$$

Proposition 5.1. *We have the commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(H_*^{K_c}(\mathrm{Gr}_{n,\mathbb{H}})) & \longrightarrow & \mathrm{Spec}(H_*^{K_c}(\mathrm{Gr}_{2n})) \\ \downarrow \simeq & & \downarrow \simeq \\ J_n & \longrightarrow & J_{2n}|_{\mathfrak{c}_n} \end{array}$$

*where the bottom arrow $J_n \to J_{2n}|_{\mathfrak{c}_n}$ is the morphism introduced in (3.6).*



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

*Proof.* We shall verify the statement for $T_c$-equivariant homology, that is, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}})) & \longrightarrow & \mathrm{Spec}(H_*^{T_c}(\mathrm{Gr}_{2n})) \\ \downarrow \simeq & & \downarrow \simeq \\ (G_n \times \mathfrak{t})^{e_X^T} \simeq J_n \times_{\mathfrak{c}_n} \mathfrak{t} & \longrightarrow & (G_{2n} \times \mathfrak{t})^{e^T} \simeq J_{2n} \times_{\mathfrak{c}_{2n}} \mathfrak{t} \end{array}$$

where the bottom arrow is the map (3.16). All the maps above are compatible with the natural W-actions, and upon taking W-invariants we get the desired claim.

Let $V_{\omega_1}$ and $V_{\omega_1'}$ be the standard representations of $G_n$ and $G_{2n}$, respectively. Recall the isomorphisms

$$H^*_{T_c}(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1'}) \simeq H^*(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1'}) \otimes R_T \simeq V_{\omega_1'} \otimes R_T, \tag{5.3}$$

$$H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'})) \simeq H^*(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'})) \otimes R_T \simeq V_{\omega_1'} \otimes R_T \tag{5.4}$$

induced by the complex and real MV-filtrations. Together with the canonical isomorphism

$$H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'})) \simeq H^*_{T_c}(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1'}), \tag{5.5}$$

we get an automorphism

$$V_{\omega_1'} \otimes R_T \simeq H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'})) \simeq H^*_{T_c}(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1}) \simeq V_{\omega_1'} \otimes R_T \tag{5.6}$$

and hence an element

$$\Phi' \in \mathrm{GL}(V_{\omega_1'}) \otimes R_T \simeq G_{2n} \otimes R_T. \tag{5.7}$$

Note that the isomorphisms (5.3) and (5.4) map the standard basis $\{e_1 \otimes 1, \dots, e_{2n} \otimes 1\}$ of $V_{\omega_1'} \otimes R_T$ to the basis

$$\{b_1, \dots, b_{2n}\} = \{[\mathbb{P}^0], \dots, [\mathbb{P}^{2n-1}]\}$$

of $H^*_{T_c}(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1'}) \simeq H^*_{T_c}(\mathbb{P}^{2n-1})$ (up to a constant degree shift) and the basis

$$\{c_1, \dots, c_{2n}\} = \{[\mathbb{HP}^0][2], [\mathbb{HP}^0], [\mathbb{HP}^1][2], [\mathbb{HP}^1], \dots, [\mathbb{HP}^{n-1}][2], [\mathbb{HP}^{n-1}]\}$$

of $H^*_T(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'})) \overset{\mathrm{Cor}4.6}{\simeq} H^*_{T_c}(\mathbb{HP}^{n-1}) \oplus H^{*-2}_{T_c}(\mathbb{HP}^{n-1})$ (up to a constant degree shift), respectively, and the element $\Phi'$ is the matrix for the linear map sending $c_i \to b_i$ in the basis $c_1, \dots, c_{2n}$ (which is not the identity element).

By Lemma 4.9, there is a commutative diagram

$$\begin{array}{ccccccc} (G_n \times \mathfrak{t})^{e_X^T} & \xrightarrow{\simeq} & \mathrm{Spec}\, H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}) & \longrightarrow & \mathrm{GL}(H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'}))) & \xrightarrow[(5.4)]{\simeq} & G_{2n} \times \mathfrak{t} \\ & & \downarrow & & (5.5)\downarrow \simeq & & \mathrm{Ad}_{\Phi'} \downarrow \simeq \\ (G_{2n} \times \mathfrak{t})^{e^T} & \xrightarrow{\simeq} & \mathrm{Spec}\, H^*_{T_c}(\mathrm{Gr}_{2n}) & \longrightarrow & \mathrm{GL}(H^*_{T_c}(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1'})) & \xrightarrow[(5.3)]{\simeq} & G_{2n} \times \mathfrak{t} \end{array} \tag{5.8}$$

where the upper and lower middle arrows are given by the co-action of $H_*^{T_c}(\mathrm{Gr}_{n,\mathbb{H}})$ and $H_*^{T_c}(\mathrm{Gr}_{2n})$ on $H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}}, \mathrm{R}(\mathrm{IC}_{\omega_1'}))$ and $H^*_{T_c}(\mathrm{Gr}_{2n}, \mathrm{IC}_{\omega_1'})$, and the right vertical isomorphism is given by the conjugation action

$$\mathrm{Ad}_{\Phi'} : G_{2n} \times \mathfrak{t} \to G_{2n} \times \mathfrak{t}, \quad (g, t) \to (\mathrm{Ad}_{\Phi'(t)} g, t).$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Note that in the above diagram the lower composed map $(G_{2n}\times\mathfrak{t})^{e^T}\to G_{2n}\times\mathfrak{t}$ is the natural embedding and the upper composed map $(G_n\times\mathfrak{t})^{e_X^T}\to G_{2n}\times\mathfrak{t}$ is the restriction of the map

$$\mathrm{Ad}_P\circ\delta: G_n\times\mathfrak{t}\to G_{2n}\times\mathfrak{t}\to G_{2n}\times\mathfrak{t},\quad (g,t)\to(P\delta(g)P^{-1},t)$$

to $(G_n\times\mathfrak{t})^{e_X^T}$, where $P\in G_{2n}$ is the permutation matrix which sends the the ordered basis $\{e_1,e_3,\dots,e_{2n-1},e_2,e_4,\dots,e_{2n}\}$ to the ordered basis $\{e_1,\dots,e_{2n}\}$ (see § 3.4.3).

Thus, in view of the description of the map $(G_n\times\mathfrak{t})^{e_X^T}\to(G_{2n}\times\mathfrak{t})^{e^T}$ in (3.16), we need to show that the element

$$\Phi:=\Phi'\circ P\in G_{2n}\otimes R_T$$

satisfies

$$e^T=\Phi(\tau\circ e_X^T)\Phi^{-1}\in\mathfrak{g}_{2n}\otimes R_T. \tag{5.9}$$

To this end, we observe that, by Lemma 4.4, the elements $\tau\circ e_X^T$ and $e^T$ in $\mathfrak{g}_{2n}\otimes R_T$ are the matrices of the cup product map $c_1^T(\mathcal{L})\cup(-): H^*_{T_c}(\mathrm{Gr}_{2n},\mathrm{IC}_{\omega_1'})\to H^*_{T_c}(\mathrm{Gr}_{2n},\mathrm{IC}_{\omega_1'})$ in the bases $\{d_1,\dots,d_{2n}\}=\{[\mathbb{HP}^0][2],\dots,[\mathbb{HP}^{n-1}][2],[\mathbb{HP}^0],\dots,[\mathbb{HP}^{n-1}]\}$ and $\{b_1,\dots,b_{2n}\}$, respectively. On the other hand, the element $\Phi=\Phi'\circ P$ is the matrix for the linear map sending $d_i\to c_i\to b_i$ in the basis $d_1,\dots,d_{2n}$, and hence (5.9) holds. This completes the proof of the proposition. □

*Remark* 5.2. The proof gives a canonical construction of the element $\Phi$ in (3.17).

In the statement and proof of the next results, we continue with the following convention: the corresponding objects $\mathcal{O}(G_n)\leftrightarrow\mathrm{IC}_{\mathcal{O}(G_n)}$ and their variants may not lie in $\mathrm{Rep}(G_n)\simeq\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$, but they are an increasing direct sum of objects, and we understand all calculations to mean the increasing direct sum of calculations.

Proposition 5.3. *The dg-algebra* $\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n))$ *is formal.*

*Proof.* Consider the dg-algebras

$$A=\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_{2n})),$$
$$B=\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n\times\mathbb{G}_m)).$$

Here we consider $\mathcal{O}(G_n\times\mathbb{G}_m)\simeq\bigoplus_{j\in\mathbb{Z}}\mathrm{IC}_{\mathcal{O}(G_n)}[j]$ via the monoidal functor $\mathrm{Rep}(G_n\times\mathbb{G}_m)\simeq\bigoplus_{j\in\mathbb{Z}}\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})[j]\subset D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})$. Proposition 4.5(3) implies that the nearby cycle functor gives rise to a map of dg-algebras

$$\begin{aligned}\phi: A&=\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{2n}\backslash\mathrm{Gr}_{2n})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_{2n}))\\&\xrightarrow{\mathrm{R}}\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star(\mathrm{Res}^{G_{2n}}_{G_n\times\mathbb{G}_m}\mathcal{O}(G_{2n})))\\&\to B=\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n\times\mathbb{G}_m)),\end{aligned}$$

where the last arrow is induced by the quotient map $\mathrm{Res}^{G_{2n}}_{G_n\times\mathbb{G}_m}\mathcal{O}(G_{2n})\to\mathcal{O}(G_n\times\mathbb{G}_m)$ (in the category of $\mathrm{Rep}(G_n\times\mathbb{G}_m)$). The right regular representations of $G_{2n}$ on $G_n\times\mathbb{G}_m$ induce natural $G_{2n}$- and $G_n\times\mathbb{G}_m$-actions on $A$ and $B$, and their restrictions to the subgroup $\mathbb{G}_m\subset G_n\times\mathbb{G}_m\subset G_{2n}$ give rise to $\mathbb{G}_m$-weight decompositions $A=\oplus_{j\in\mathbb{Z}}A_j$ and $B=\oplus_{j\in\mathbb{Z}}B_j$. Note that the zero-weight spaces $A_0$ and $B_0$ are dg-subalgebras of $A$ and $B$ and that $B_0=\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash\mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0,\mathrm{IC}_0\star\mathcal{O}(G_n))$.

According to [BF08], the dg-algebra $A$ is formal; moreover, we have $A\simeq H^*(A)\simeq\mathcal{O}(\mathfrak{g}_{2n}[2])$. Note that the map $\phi: A\to B$ above respects the $\mathbb{G}_m$-action and hence restricts to a map $\phi_0: A_0\to B_0$ fitting into the following diagram.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

$$\begin{array}{ccc} A_0 & \xrightarrow{\phi_0} & B_0 \\ \downarrow & & \downarrow \\ A & \xrightarrow{\phi} & B \end{array}$$

We claim that the map $H^*(\phi_0): H^*(A_0) \to H^*(B_0)$ between cohomology is surjective. Since $A_0$ is formal with generators in even degree and $H^*(B_0) \simeq H^*(\mathrm{RHom}_{D^b(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \mathcal{O}(G_n))) \simeq \mathcal{O}(\mathfrak{g}_n[4])$, which is a polynomial ring with generators in even degree (see Lemma 4.21), Lemma 5.4 below implies that $B_0$ is formal. The proposition follows.

To proof the claim and show the surjectivity of $H^*(\phi_0): H^*(A_0) \to H^*(B_0)$, we can ignore the grading and view $H^*(\phi_0)$ as maps between ungraded algebras. We have a commutative diagram

$$\begin{array}{ccc} H^*(A) & \xrightarrow{\simeq} & \mathrm{Hom}^*_{H^*_{G_{2n}}(\mathrm{Gr}_{2n})}(H^*_{G_{2n}}(\mathrm{IC}_0), H^*_{G_{2n}}(\mathrm{IC}_0 \star \mathcal{O}(G_{2n}))) \\ \downarrow{\scriptstyle H^*(\phi)} & & \downarrow \\ H^*(B) & \xrightarrow{\simeq} & \mathrm{Hom}^*_{H^*_{K_c}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{K_c}(\mathrm{IC}_0), H^*_{K_c}(\mathrm{IC}_0 \star \mathcal{O}(G_n \times \mathbb{G}_m))) \end{array}$$

where the horizontal isomorphisms are given by the functor of equivariant cohomology; see Proposition 4.15. Note that $\mathrm{IC}_0 \star \mathcal{O}(G_n \times \mathbb{G}_m) \simeq \bigoplus_{j \in \mathbb{Z}} \mathrm{IC}_{\mathcal{O}(G_n)}[j]$ is a direct sum of shifts of IC-complexes and hence Proposition 4.15 is applicable. On the other hand, using Proposition 5.1, we can identify the right vertical arrow as

$$\begin{array}{ccc} \mathrm{Hom}^*_{H^*_{G_{2n}}(\mathrm{Gr}_{2n})}(H^*_{G_{2n}}(\mathrm{IC}_0), H^*_{G_{2n}}(\mathrm{IC}_0 \star \mathcal{O}(G_{2n}))) & \longrightarrow & \mathcal{O}(G_{2n} \times \mathfrak{c}_{2n})^{J_{2n}} \\ \downarrow & & \downarrow \\ \mathrm{Hom}^*_{H^*_{K_c}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{K_c}(\mathrm{IC}_0), H^*_{K_c}(\mathrm{IC}_0 \star \mathcal{O}(G_n \times \mathbb{G}_m))) & \longrightarrow & \mathcal{O}(G_n \times \mathbb{G}_m \times \mathfrak{c}_n)^{J_n} \end{array}$$

where the right vertical map above is induced by the embeddings $\tau: \mathfrak{c}_n \to \mathfrak{c}_{2n}$ in (3.4) and $\delta \times 2\rho_L: G_n \times \mathbb{G}_m \to G_{2n}$ in (3.19). The group schemes $J_{2n}$ and $J_n$ act on $\mathcal{O}(G_{2n} \times \mathfrak{c}_{2n})$ and $\mathcal{O}(G_n \times \mathbb{G}_m \times \mathfrak{c}_n)$ via the identifications $J_{2n} \simeq (G_{2n} \times \mathfrak{c}_{2n})^{\mathrm{Ad}_P^{-1} \circ \kappa_{2n}}$ and $J_n \simeq (G_n \times \mathfrak{c}_n)^{\tau \circ \kappa_n}$, where $\mathrm{Ad}_P^{-1} \circ \kappa_{2n}: \mathfrak{c}_{2n} \xrightarrow{\kappa_{2n}} \mathfrak{g}_{2n}^{\mathrm{reg}} \xrightarrow{\mathrm{Ad}_P^{-1}} \mathfrak{g}_{2n}^{\mathrm{reg}}$ and $\tau \circ \kappa_n: \mathfrak{c}_n \xrightarrow{\kappa_n} \mathfrak{g}_n^{\mathrm{reg}} \xrightarrow{\tau} \mathfrak{g}_{2n}^{\mathrm{reg}}$ are the maps in (3.9). Thus we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(G_{2n} \times \mathfrak{c}_n)^{J_{2n}} & \xrightarrow{\simeq} & \mathcal{O}(\mathfrak{g}_{2n}^{\mathrm{reg}}) \simeq \mathcal{O}(\mathfrak{g}_{2n}) \\ \downarrow & & \downarrow \\ \mathcal{O}(G_n \times \mathbb{G}_m \times \mathfrak{c}_n)^{J_n} & \xrightarrow{\simeq} & \mathcal{O}(\mathfrak{g}_n^{\mathrm{reg}} \times \mathbb{G}_m) \simeq \mathcal{O}(\mathfrak{g}_n \times \mathbb{G}_m) \end{array}$$

where the right vertical arrow is given by pullback of functions along the map

$$\mathfrak{g}_n \times \mathbb{G}_m \to \mathfrak{g}_{2n}, \qquad (C, t) \to \mathrm{Ad}_{2\rho_L(t)^{-1}} \tau(C) = \begin{pmatrix} 0 & t^{-2}\mathrm{Id}_n \\ t^2 C & 0 \end{pmatrix}. \tag{5.10}$$

All together we can identify $H^*(\phi): H^*(A) \to H^*(B)$ with the map $\mathcal{O}(\mathfrak{g}_{2n}) \to \mathcal{O}(\mathfrak{g}_n \times \mathbb{G}_m)$ (as map between non-graded algebras), and we need to show that the induced map

$$\mathcal{O}(\mathfrak{g}_{2n})_0 \to \mathcal{O}(\mathfrak{g}_n \times \mathbb{G}_m)_0 = \mathcal{O}(\mathfrak{g}_n) \tag{5.11}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

between the zero-$\mathbb{G}_m$-weight spaces is surjective. For this we observe that the map

$$\mathfrak{g}_{2n} \to \mathfrak{g}_n, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to BC \tag{5.12}$$

is $\mathbb{G}_m$-equivariant ($\mathbb{G}_m$ acts trivially on $\mathfrak{g}_n$) and the composition $\mathfrak{g}_n \times \mathbb{G}_m \overset{(5.10)}{\to} \mathfrak{g}_{2n} \overset{(5.12)}{\to} \mathfrak{g}_n$ is the projection map $(C,t) \to C$. Thus the pullback map $\mathcal{O}(\mathfrak{g}_n) \to \mathcal{O}(\mathfrak{g}_{2n})_0$ along (5.12) defines a section of (5.11). We are done. □

LEMMA 5.4. *Let $\phi: A_1 \to A_2$ be a map of dg-algebras. Assume that* (1) *$H^*(A_1)$ is commutative,* (2) *$H^*(A_2)$ is isomorphic to a polynomial ring with generators in even degree, and* (3) *the map $H^*(\phi): H^*(A_1) \to H^*(A_2)$ is surjective. Then $A_1$* being *formal implies that $A_2$ is formal.*

*Proof.* Let $x_1, \ldots, x_l$ be the set of generators of $H^*(A_2)$ in even degree such that $\mathbb{C}[x_1, \ldots, x_l] \simeq H^*(A_2)$. Since $H^*(\phi): H^*(A_1) \to H^*(A_2)$ is surjective, one can find homogeneous elements $y_1, \ldots, y_l \in H^*(A_1)$ such that $H^*(\phi)(y_i) = x_i$ for $i = 1, \ldots, l$. Assume that $H^*(A_1) \simeq A_1$ is formal; then we have map of dg-algebras $\mathbb{C}[z_1, \ldots, z_l] \to H^*(A_1) \simeq A_1$ sending $z_i$ to $y_i$. Then the composition $\gamma: \mathbb{C}[z_1, \ldots, z_l] \to H^*(A_1) \simeq A_1 \overset{\phi}{\to} A_2$ defines a dg-algebra morphism such that $H^*(\gamma): \mathbb{C}[z_1, \ldots, z_l] \simeq \mathbb{C}[x_1, \ldots, x_l] \simeq H^*(A_2)$ is the isomorphism sending $z_i$ to $x_i$. The lemma follows. □

### 5.2 Derived geometric Satake equivalence for the quaternionic groups

Denote by $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ the dg-category of $G_n$-equivariant dg-modules over the dg-algebra $\mathrm{Sym}(\mathfrak{g}_n[-4])$ (equipped with trivial differential). It is known that $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ is compactly generated and the full subcategory $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))^c$ of compact objects coincides with the full subcategory $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))^c = D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ consisting of perfect modules. Denote by $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$ and $D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$ the full subcategories of $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ and $D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$, respectively, consisting of modules that are set-theoretically supported on the nilpotent cone $\mathrm{Nilp}(\mathfrak{g}_n)$ of $\mathfrak{g}_n$.

Note that the category $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ (respectively, $D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$, $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$, or $D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$) has a natural monoidal structure given by the (derived) tensor product: $(\mathcal{F}_1, \mathcal{F}_2) \to \mathcal{F}_1 \otimes \mathcal{F}_2 := \mathcal{F}_1 \otimes^L_{\mathrm{Sym}(\mathfrak{g}_n[-4])} \mathcal{F}_2$.

THEOREM 5.5.

(1) *There is a canonical equivalence of monoidal categories*

$$\mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}})) \simeq D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$$

*which induces a monoidal equivalence*

$$D^b(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}}) \simeq D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$$

*between the corresponding (non-cocomplete) full subcategory of compact objects.*

(2) *There is a canonical equivalence of monoidal categories*

$$D(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}}) \simeq D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$$

*which induces a monoidal equivalence*

$$D(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}})^c \simeq D^{G_n}_{\mathrm{perf}}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$$

*between the corresponding (non-cocomplete) full subcategory of compact objects.*



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

*Proof.* To prove (1), write $\mathcal{C} := \mathrm{Ind}(\mathrm{D}^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))$. The dg-category $\mathcal{C}$ is a module category for the dg-category $D(\mathrm{QCoh}(BG_n))$ of quasi-coherent sheaves on $BG_n$, and we can form the de-equivariantized category $\mathcal{C}_{\mathrm{deeq}} := \mathcal{C} \times_{BG} \{\mathrm{pt}\}$ with objects $\mathrm{Ob}(\mathcal{C}_{\mathrm{deeq}}) = \mathrm{Ob}(\mathcal{C})$ and (dg-)morphisms

$$\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathcal{F}_1, \mathcal{F}_2) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_1, \mathcal{F}_2 \star \mathcal{O}(G_n)) = \mathrm{RHom}_{\mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))}(\mathcal{F}_1, \mathcal{F}_2 \star \mathcal{O}(G_n)).$$

Every object $\mathcal{F} \in \mathcal{C}_{\mathrm{deeq}}$ carries a natural action of $G_n$, and we can recover $\mathcal{C}$ by taking $G_n$-equivariant objects in $\mathcal{C}_{\mathrm{deeq}}$. The fact that $\mathrm{IC}_0$ is compact and generates $\mathcal{C}$ under the action of $D\mathrm{QCoh}(BG_n)$ implies that $\mathrm{IC}_0$, viewed as an object in $\mathcal{C}_{\mathrm{deeq}}$, is a compact generator. Hence the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5] implies that the assignment $\mathcal{F} \to \mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathcal{F})$ defines an equivalence of categories

$$\mathcal{C}_{\mathrm{deeq}} \simeq D(\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{\mathrm{op}}) \quad (\text{respectively, } \mathcal{C} \simeq D^{G_n}(\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{\mathrm{op}})),$$

where $\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{op}$ is the opposite of the dg-algebra of endomorphisms of $\mathrm{IC}_0$ and $D(\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{\mathrm{op}})$ (respectively, $D^{G_n}(\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{\mathrm{op}})$) are the corresponding dg-categories of dg-modules (respectively, $G_n$-equivariant dg-modules). Now Proposition 4.21 and Proposition 5.3 imply that the dg-algebra $\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{\mathrm{op}}$ is formal and there is a $G_n$-equivariant isomorphism[8]

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_{\mathrm{deeq}}}(\mathrm{IC}_0, \mathrm{IC}_0)^{\mathrm{op}} &\simeq \mathrm{RHom}_{\mathrm{Ind}(\mathrm{D}^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))}(\mathrm{IC}_0, \mathrm{IC}_0 \star \mathcal{O}(G_n))^{\mathrm{op}} \\ &\simeq \mathrm{Ext}^*_{\mathrm{Ind}(\mathrm{D}^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))}(\mathrm{IC}_0, \mathrm{IC}_0 \star \mathcal{O}(G_n))^{\mathrm{op}} \\ &\simeq \mathrm{Sym}(\mathfrak{g}_n[-4])^{\mathrm{op}} \simeq \mathrm{Sym}(\mathfrak{g}_n[-4])^8,\end{aligned}$$

and hence we conclude that there is an equivalence

$$\begin{gathered}\mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})) \simeq D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4])), \\ \mathcal{F} \to \mathrm{RHom}_{\mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))}(\mathrm{IC}_0, \mathcal{F} \star \mathcal{O}(G_n)).\end{gathered}$$

The monoidal structure on the constructed equivalence will be proved in § 5.4. This finishes the proof of part (1).

Part (2) follows from the general discussion in [AG15, § 12] on derived geometric Satake for complex reductive groups. Let $\widetilde{\mathrm{IC}}_0 = p_!(\mathbb{C}_{\mathrm{pt}}) \in D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})$ where $p : \mathrm{pt} \to \mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{pt} \simeq \mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}^0_{n,\mathbb{H}} \to \mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}$. By [AG15, § 12.6.6], $\widetilde{\mathrm{IC}}_0$ is in fact a compact object $\widetilde{\mathrm{IC}}_0 \in D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})^c$ and the category $D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})$ is compactly generated by objects of the form $\widetilde{\mathrm{IC}}_0 * V$ for $V \in \mathrm{Rep}(G_n)$. We claim that under the fully faithful embedding

$$D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}) \simeq \mathrm{Ind}(D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})^c) \to \mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})) \simeq D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4])), \tag{5.13}$$

the compact generator $\widetilde{\mathrm{IC}}_0$ goes to $\mathrm{Sym}(\mathfrak{g}_n[-4]) \otimes_{\mathrm{Sym}(\mathfrak{g}_n[-4])^{G_n}} \mathbb{C}$, where $\mathbb{C}$ is the augmentation module of $\mathrm{Sym}(\mathfrak{g}_n[-4])^{G_n}$. Since $\mathrm{Sym}(\mathfrak{g}_n) \otimes_{\mathrm{Sym}(\mathfrak{g}_n)^{G_n}} \mathbb{C} \simeq \mathcal{O}(\mathrm{Nilp}(\mathfrak{g}_n))$, we conclude that $D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}})$ is equivalent to the full subcategory of $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ generated by objects of the form $\mathcal{O}(\mathrm{Nilp}(\mathfrak{g}_n)) \otimes V$ for $V \in \mathrm{Rep}(G_n)$. It is clear that this subcategory is exactly $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))_{\mathrm{Nilp}(\mathfrak{g}_n)}$. To prove the claim, we observe that Proposition 4.15 implies that the image of $\widetilde{\mathrm{IC}}_0$ under (5.13) is given by

$$\mathrm{RHom}_{\mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))}(\mathrm{IC}_0, \widetilde{\mathrm{IC}}_0 \star \mathcal{O}(G_n)) \simeq \mathrm{Hom}^*_{H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{T_c}(\mathrm{IC}_0), H^*_{T_c}(\widetilde{\mathrm{IC}}_0) \otimes \mathcal{O}(G_n))^{\mathrm{W}}.$$

[8]The last isomorphism follows from the fact that $\mathrm{Sym}(\mathfrak{g}_n[-4])$ is commutative with grading in even degree.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Note that $H^*_{T_c}(\widetilde{\mathrm{IC}}_0) \simeq \mathbb{C}$ is isomorphic to the augmented module of $H^*_{T_c}(\mathrm{IC}_0) \simeq \mathcal{O}(\mathfrak{t}[2])$, and the same computation as in Proposition 4.21 shows that

$$\begin{aligned}\mathrm{Hom}^*_{H^*_{T_c}(\mathrm{Gr}_{n,\mathbb{H}})}(H^*_{T_c}(\mathrm{IC}_0), H^*_{T_c}(\widetilde{\mathrm{IC}}_0) \otimes \mathcal{O}(G_n))^{\mathrm{W}} &\simeq \mathcal{O}(\mathfrak{g}_n^{\mathrm{reg}}[4] \times_{\mathfrak{g}_n[4]//G_n} \{0\})^{\mathrm{W}} \\ &\simeq \mathcal{O}(\mathfrak{g}_n[4] \times_{\mathfrak{g}_n[4]//G_n} \{0\}) \\ &\simeq \mathrm{Sym}(\mathfrak{g}_n[-4]) \otimes_{\mathrm{Sym}(\mathfrak{g}_n[-4])^{G_n}} \mathbb{C}.\end{aligned}$$

The claim follows. □

### 5.3 Spectral description of nearby cycles functors

5.3.1 *Shift of grading.* Let $(A = \oplus A^i, d)$ be a dg-algebra equipped with an action of $G = H \times \mathbb{G}_m$. We will write $A^i = \oplus A^i_j$ where the lower index $j$ refers to the $\mathbb{G}_m$-weights coming from the $\mathbb{G}_m$-action. Assume that the $\mathbb{G}_m$-weights are even, that is, we have $A^i_j = 0$ if $j \in 2\mathbb{Z}+1$. Following [AG15, Appendix A.2], one can introduce a new dg-algebra $(\tilde{A} = \bigoplus \tilde{A}^i_j, d)$ where

$$\tilde{A}^i_j = A^{i+j}_j,$$

such that the map sending a $G$-equivariant dg-module $(M = \bigoplus M^i_j, d)$ over $A$ to the dg-module $(\tilde{M} = \bigoplus \tilde{M}^i_j, d)$ over $\tilde{A}$ with

$$\tilde{M}^i_j = M^{i+j}_j$$

induces an equivalence of triangulated categories

$$D^G(A) \simeq D^G(\tilde{A}) \quad (\text{respectively, } D^G_{\mathrm{perf}}(A) \simeq D^G_{\mathrm{perf}}(\tilde{A})).$$

*Example* 5.6. Consider the dg-algebra $(A = \mathrm{Sym}(\mathfrak{g}_{2n}), d = 0)$. The subgroup $G_n \times \mathbb{G}_m \subset G_{2n}$ as in § 3.5 acts on the generators $\mathfrak{g}_{2n}$ of $A$ via the adjoint action, and if we write the elements in $\mathfrak{g}_{2n}$ in the form

$$\mathfrak{g}_{2n} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A, B, C, D \in \mathfrak{g}_n \right\},$$

then $A$ and $D$ are of weight zero, $B$ is of weight 2, and $C$ is of weight $-2$. It follows that

$$\tilde{A} \simeq \mathrm{Sym}(\tilde{\mathfrak{g}}_{2n})$$

where $\tilde{\mathfrak{g}}_{2n}$ consists of elements of the form

$$\tilde{\mathfrak{g}}_{2n} = \left\{ \begin{pmatrix} A[0] & B[-2] \\ C[2] & D[0] \end{pmatrix} \middle| A, B, C, D \in \mathfrak{g}_n \right\}.$$

5.3.2 It follows from Example 5.6 that we have an equivalence of categories

$$D^{G_n \times \mathbb{G}_m}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) \simeq D^{G_n \times \mathbb{G}_m}(\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])). \tag{5.14}$$

On the other hand, the natural $G_n$-equivariant map $\mathfrak{g}_n[4] \to \tilde{\mathfrak{g}}_{2n}[2]$ sending

$$C[4] \to \begin{pmatrix} 0 & \mathrm{Id}_n \\ C[4] & 0 \end{pmatrix} \quad \text{for } C \in \mathfrak{g}_n$$

gives rise to a map of dg-algebras

$$\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2]) \simeq \mathcal{O}(\tilde{\mathfrak{g}}_{2n}[2]) \longrightarrow \mathcal{O}(\mathfrak{g}_n[4]) \simeq \mathrm{Sym}(\mathfrak{g}_n[-4]) \tag{5.15}$$



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

(here we identify the graded duals of $\tilde{\mathfrak{g}}_{2n}[-2]$ and $\mathfrak{g}_n[4]$ with $\tilde{\mathfrak{g}}_{2n}[2]$ and $\mathfrak{g}_n[-4]$ via the trace form) and hence a functor

$$D^{\mathrm{GL}_n}(\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])) \to D^{\mathrm{GL}_n}(\mathrm{Sym}(\mathfrak{g}_n[-4])), \quad M \to \mathrm{Sym}(\mathfrak{g}_n[-4]) \otimes^L_{\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])} M. \tag{5.16}$$

Finally, let us consider the functor

$$\begin{aligned}\Phi : D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) &\xrightarrow{F} D^{G_n\times\mathbb{G}_m}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) \overset{(5.14)}{\simeq} D^{\mathrm{GL}_n\times\mathbb{G}_m}(\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])) \\ &\xrightarrow{F} D^{G_n}(\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])) \overset{(5.16)}{\to} D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4])),\end{aligned} \tag{5.17}$$

where $F$ are the natural forgetful functors.

Theorem 5.7. *The following square is commutative*

$$\begin{array}{ccc} \mathrm{Ind}\, D^b(\mathfrak{L}^+G_{2n}\backslash \mathrm{Gr}_{2n}) & \xrightarrow{\mathrm{R}} & \mathrm{Ind}\, D^b(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}) \\ {\scriptstyle\Psi}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\Psi_{\mathbb{H}}}\downarrow{\scriptstyle\simeq} \\ D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) & \xrightarrow{\Phi} & D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4])) \end{array}$$

*where $\Psi$ and $\Psi_{\mathbb{H}}$ are the complex and quaternionic Satake equivalences, respectively. It induces a similar commutative diagram for the subcategories of compact objects.*

*Proof.* We shall construct a natural transformation $\Phi\circ\Psi \to \Psi_{\mathbb{H}}\circ \mathrm{R}$. Write $A = \mathrm{RHom}(\mathrm{IC}_0, \mathrm{IC}_{\mathcal{O}(G_{2n})}) \simeq \mathrm{Sym}(\mathfrak{g}_{2n}[-2])$, $B = \mathrm{RHom}(\mathrm{IC}_0, \mathrm{IC}_{\mathcal{O}(G_n)}) \simeq \mathrm{Sym}(\mathfrak{g}_n[-4])$, and $A' = \mathrm{RHom}(\mathrm{IC}_0, \mathrm{R}(\mathrm{IC}_{\mathcal{O}(G_{2n})}))$. Since $\mathrm{R}(\mathrm{IC}_{\mathcal{O}(G_{2n})})$ is an algebra object in $\mathrm{Ind}(D(\mathfrak{L}^+G_{n,\mathbb{H}}\backslash \mathrm{Gr}_{n,\mathbb{H}}))$, the (dg) Hom space $A'$ is naturally a dg-algebra.

For any $\mathcal{F}$, we have a map of dg-modules for the dg-algebra $A$,

$$\Psi(\mathcal{F}) \simeq \mathrm{RHom}(\mathrm{IC}_0, \mathcal{F}\star \mathrm{IC}_{\mathcal{O}(G_{2n})}) \xrightarrow{\mathrm{R}} \mathrm{RHom}(\mathrm{IC}_0, \mathrm{R}(\mathcal{F})\star \mathrm{R}(\mathrm{IC}_{\mathcal{O}(G_{2n})})) := \Psi(\mathcal{F})',$$

where $A$ acts on $\Psi(\mathcal{F})'$ via the dg-algebra map

$$A = \mathrm{RHom}(\mathrm{IC}_0, \mathrm{IC}_{\mathcal{O}(G_{2n})}) \xrightarrow{\mathrm{R}} A' = \mathrm{RHom}(\mathrm{IC}_0, \mathrm{R}(\mathrm{IC}_{\mathcal{O}(G_{2n})})).$$

The right regular $\mathbb{G}_m$-action on $G_{2n}$ via the co-character $2\rho_L : \mathbb{G}_m \to G_n\times\mathbb{G}_m \subset G_{2n}$ induces a $\mathbb{G}_m$-action on the dg-algebras $A$ and $A'$ (with even weights) and also the dg-modules $\Psi(\mathcal{F})$ and $\Psi(\mathcal{F})'$. Thus we can perform the shift of grading operation in § 5.3.1 and obtain a map of dg-modules for the dg-algebra $\tilde{A}$,

$$\widetilde{\Psi(\mathcal{F})} \to \widetilde{\Psi(\mathcal{F})'}, \tag{5.18}$$

where $\tilde{A}$ acts on $\widetilde{\Psi(\mathcal{F})'}$ via the map $\tilde{A}\to\tilde{A}'$. By Example 5.6, we have

$$\tilde{A} \simeq \mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2]).$$

On the other hand, by Proposition 4.5(3), we have[9]

$$\mathrm{R}(\mathrm{IC}_{\mathcal{O}(G_{2n})}) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{IC}_{\mathrm{Res}^{G_{2n}}_{G_n}\mathcal{O}(G_{2n})_j}[j]$$

where

$$\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n})) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{Res}^{G_{2n}}_{G_n\times\mathbb{G}_m}\mathcal{O}(G_{2n})_j$$

[9] We have the shift $[j]$ instead of $[-j]$ because we consider right regular action of $\mathbb{G}_m$.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

is the $\mathbb{G}_m$-weight decomposition of $\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n}))$, and it follows that

$$\tilde{A}' \simeq \mathrm{RHom}(\mathrm{IC}_0, \mathrm{IC}_{\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n}))}), \quad \widetilde{\Psi(\mathcal{F})'} \simeq \mathrm{RHom}(\mathrm{IC}_0, \mathrm{R}(\mathcal{F}) \star \mathrm{IC}_{\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n}))}).$$

Since the natural algebra map $\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n})) \to \mathcal{O}(G_n)$ of algebra objects in $\mathrm{Rep}(G_n)$ coming from the embedding $G_n \to G_n \times \mathbb{G}_m, g \to (g,e)$ induces a map

$$\iota : \mathrm{IC}_{\mathrm{Res}^{G_{2n}}_{G_n \times \mathbb{G}_m}(\mathcal{O}(G_{2n}))} \to \mathrm{IC}_{\mathcal{O}(G_n)}$$

between the corresponding algebra objects in $\mathrm{Perv}(\mathrm{Gr}_{n,\mathbb{H}})$, we obtain a map of dg-algebras,

$$\tilde{A} \to \tilde{A}' \simeq \mathrm{RHom}(\mathrm{IC}_0, \mathrm{IC}_{\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n}))}) \xrightarrow{\iota} \mathrm{RHom}(\mathrm{IC}_0, \mathrm{IC}_{\mathcal{O}(G_n)}) = B, \tag{5.19}$$

and a map of dg-modules over the dg-algebra $\tilde{A}$,

$$\begin{aligned}\widetilde{\Psi(\mathcal{F})} \overset{(5.18)}{\to} \widetilde{\Psi(\mathcal{F})'} &\simeq \mathrm{RHom}(\mathrm{IC}_0, \mathrm{R}(\mathcal{F}) \star \mathrm{IC}_{\mathrm{Res}^{G_{2n}}_{G_n}(\mathcal{O}(G_{2n}))}) \xrightarrow{\iota} \mathrm{RHom}(\mathrm{IC}_0, \mathrm{R}(\mathcal{F}) \star \mathrm{IC}_{\mathcal{O}(G_n)}) \\ &\simeq \Psi_{\mathbb{H}} \circ \mathrm{R}(\mathcal{F}),\end{aligned} \tag{5.20}$$

where $\tilde{A}$ acts on $\Psi_{\mathbb{H}} \circ \mathrm{R}(\mathcal{F})$ via the morphism (5.19). Moreover, the proof of Proposition 5.3 implies that the map (5.19) is equal to the map in (5.15). Thus, by the universal property of the tensor product, the map (5.20) gives rise to a map of dg-modules over the dg-algebra $B$,

$$\Phi \circ \Psi(\mathcal{F}) \simeq B \otimes^L_{\tilde{A}} \widetilde{\Psi(\mathcal{F})} \to \Psi_{\mathbb{H}} \circ \mathrm{R}(\mathcal{F}). \tag{5.21}$$

This finishes the construction of the desired natural transformation map.

Now, to finish the proof, it suffices to check that (5.21) is an isomorphism when $\mathcal{F}$ is of the form $\mathcal{F} \simeq \mathrm{IC}_{\mathrm{V}}$ with $\mathrm{V} \in \mathrm{Rep}(G_{2n})$. For this we observe that if $\mathrm{V} = \oplus_{j\in\mathbb{Z}} \mathrm{V}_j$ is the $\mathbb{G}_m$-weight decomposition, then we have

$$\Psi_{\mathbb{H}} \circ \mathrm{R}(\mathrm{IC}_{\mathrm{V}}) \simeq \bigoplus_{j\in\mathbb{Z}} \Psi_{\mathbb{H}}(\mathrm{IC}_{\mathrm{V}_j})[-j] \simeq \bigoplus_{j\in\mathbb{Z}} B \otimes_{\mathbb{C}} \mathrm{V}_j[-j]. \tag{5.22}$$

On the other hand, we have

$$\widetilde{\Psi(\mathrm{IC}_{\mathrm{V}})} \simeq (\widetilde{A \otimes_{\mathbb{C}} \mathrm{V}}) \simeq \bigoplus_{j\in\mathbb{Z}} \tilde{A} \otimes_{\mathbb{C}} \mathrm{V}_j[-j]$$

and hence

$$\Phi \circ \Psi(\mathrm{IC}_{\mathrm{V}}) \simeq B \otimes_{\tilde{A}} \widetilde{\Psi(\mathrm{IC}_{\mathrm{V}})} \simeq B \otimes_{\tilde{A}} \left( \bigoplus_{j\in\mathbb{Z}} \tilde{A} \otimes_{\mathbb{C}} \mathrm{V}_j[-j] \right) \simeq \bigoplus_{j\in\mathbb{Z}} B \otimes_{\mathbb{C}} \mathrm{V}_j[-j]. \tag{5.23}$$

It follows from the construction that the map (5.21) is given by

$$\Phi \circ \Psi(\mathrm{IC}_{\mathrm{V}}) \overset{(5.23)}{\simeq} \bigoplus_{j\in\mathbb{Z}} B \otimes_{\mathbb{C}} \mathrm{V}_j[-j] \overset{(5.22)}{\simeq} \Psi_{\mathbb{H}} \circ \mathrm{R}(\mathrm{IC}_{\mathrm{V}})$$

and hence is an isomorphism. This completes the proof of the proposition. □

### 5.4 Monoidal structures

We construct a monoidal structure on the equivalence $\Psi_{\mathbb{H}} : \mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}})) \simeq D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ in Theorem 5.5. Consider the monoidal structure on $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$,

$$M_1 \otimes' M_2 := \Psi_{\mathbb{H}}(\Psi_{\mathbb{H}}^{-1}(M_1) \star \Psi_{\mathbb{H}}^{-1}(M_2)),$$

induced from the monoidal structure on $\mathrm{Ind}(D^b(\mathfrak{L}^+G_{n,\mathbb{H}} \backslash \mathrm{Gr}_{n,\mathbb{H}}))$ via the equivalence $\Psi_{\mathbb{H}}$. We would like to show that $\otimes'$ is isomorphic to the natural tensor monoidal structure. The square



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

in Theorem 5.7, together with the fact that the derived Satake equivalence $\Psi$ is monoidal, implies that the functor $\Phi : D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) \to D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ in *loc. cit.* is monoidal with respect to the natural tensor monoidal structure on $D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])$ and the above monoidal structure $\otimes'$ on $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$. Now the desired claim follows from the following lemma.

LEMMA 5.8. *Equip $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ with its natural tensor monoidal structure.*

*Then the natural tensor monoidal structure on $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ is the unique (up to equivalence) monoidal structure on $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ such that the* $\mathrm{Rep}(G_{2n})$*-module functor*

$$\Phi : D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) \longrightarrow D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$$

*may be compatibly lifted to a monoidal functor. Moreover, the compatible monoidal structure on $\Phi$ is unique (up to equivalence).*

*Proof.* Returning to its construction in (5.17), recall that $\Phi$ factors into the sheared forgetful functor

$$D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2])) \to D^{G_n}(\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2]))$$

followed by the restriction

$$D^{G_n}(\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])) \to D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4])), \quad M \mapsto \mathrm{Sym}(\mathfrak{g}_n[-4]) \otimes^L_{\mathrm{Sym}(\tilde{\mathfrak{g}}_{2n}[-2])} M.$$

First, $\mathrm{Sym}(\mathfrak{g}_{2n}[-2]))$ is the unit of $D^{G_{2n}}(\mathrm{Sym}(\mathfrak{g}_{2n}[-2]))$, so $\mathrm{Sym}(\mathfrak{g}_n[-4]) \simeq \Phi(\mathrm{Sym}(\mathfrak{g}_{2n}[-2]))$ must be the unit of $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$.

Next, recall that $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ is compactly generated by $V \otimes \mathrm{Sym}(\mathfrak{g}_n[-4])$, where $V$ is a finite-dimensional representation of $G_n$. Note that every such $V$ is a direct summand in the restriction of a finite-dimensional representation of $G_{2n}$. Since $\Phi$ is a $\mathrm{Rep}(G_{2n})$-module map, this determines the monoidal product on $D^{G_n}(\mathrm{Sym}(\mathfrak{g}_n[-4]))$ as well as its coherent associativity structure.

Finally, since the monoidal structures on $\Phi$ must be compatible with its $\mathrm{Rep}(G_{2n})$-module structure, it is determined by its restriction to the unit $\mathrm{Sym}(\mathfrak{g}_{2n}[-2]))$ where there are no choices. □

## Acknowledgments

We would like to thank David Ben-Zvi, Jonathan Wang, and Geordie Williamson for useful discussions. We also would like to thank the anonymous referee for their thoughtful and detailed suggestions. T-H Chen would also like to thank the Institute for Advanced Study in Princeton for support, hospitality, and a nice research environment.

## Conflicts of interest

None.

## Financial support

T-H Chen is supported by NSF grant DMS-2001257 and the S. S. Chern Foundation. D. Nadler is supported by NSF grant DMS-2101466. MM is supported by an NSF Graduate Research Fellowship.



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press

Journal information

*Compositio Mathematica* is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

## References

AR15 P. Achar and L. Rider, *Parity sheaves on the affine Grassmannian and the Mirkovic-Vilonen conjecture*, Acta Math. **215** (2015), 183–216.

AG15 D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, Selecta Math. (N.S.) **21** (2015), 1–199.

BZN13 D. Ben-Zvi and D. Nadler, *Loop spaces and representations*, Duke Math. J. **162** (2013), 1587–1619.

BZN15 D. Ben-Zvi and D. Nadler, *Betti geometric Langlands*, in *Algebraic geometry: Salt Lake City 2015*, Proceedings of Symposia in Pure Mathematics, vol. 97, part 2 (American Mathematical Society, Providence, RI, 2015).

BZSV24 D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh, *Relative Langlands duality*, Preprint (2024), arXiv:2409.04677.

BF08 R. Bezrukavnikov and M. Finkelberg, *Equivariant Satake category and Kostan-Whittaker reduction*, Mosc. Math. J. **8** (2008), 39–72.

BFM05 R. Bezrukavnikov, M. Finkelberg and I. Mirkovic, *Equivariant homology and K-theory of affine Grassmannians and Toda lattices*, Compositio Math. **141** (2005), 746–768.

Bry89 R. K. Brylinski, *Limits of weight spaces, Lusztig's q-analogs, and fiberings of adjoint orbits*, J. Amer. Math. Soc. **2** (1989), 517–533.

CN18 T.-H. Chen and D. Nadler, *Affine Matsuki correspondence for sheaves*, Preprint (2018), arXiv:1805.06514.

CN24 T.-H. Chen and D. Nadler, *Real groups, symmetric varieties and Langlands duality*, Preprint (2024), arXiv:2403.13995.

GR17 D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry. Volume I: Correspondences and duality*, Mathematical Surveys and Monographs, vol. 221 (American Mathematical Society, Providence, RI, 2017).

Gin91 V. Ginzburg, *Perverse sheaves and* $\mathbb{C}^*$*-actions*, J. Amer. Math. Soc. **4** (1991), 483–490.

Gin95 V. Ginzburg, *Perverse sheaves on a loop group and Langlands' duality*, Preprint (1995), arXiv:alg-geom/9511007.

GM83 M. Goresky and R. MacPherson, *Morse theory and intersection homology theory*, in *Analyse et topologie sur les espaces singuliers (II–III) – 6–10 juillet 1981*, Astérisque, vol. 101–102 (Société Mathématique de France, 1983), 135–192.

Hai06 T. Haines, *Equidimensionality of convolution morphisms and applications to saturation problems*, Adv. Math. **207** (2006), 297–327.

JS90 H. Jacquet and J. Shalika, *Exterior square L-functions*, in *Automorphic forms, Shimura varieties, and L-functions. Volume II: Proceedings of a conference held at the University of Michigan, Ann Arbor, July 6–16, 1988*, Perspectives in Mathematics, vol. 11 (Academic Press, Boston, MA, 1990), 143–226.

JR92 H. Jacquet and S. Rallis, *Symplectic periods*, J. Reine Angew. Math. **423** (1992), 175–197.

JMW14 D. Juteau, C. Mautner and G. Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), 1169–1212.

Laf09 V. Lafforgue, *Quelques calculs reliés à la correspondance de Langlands geometrique sur* $P^1$, Preprint (2009), https://vlafforg.perso.math.cnrs.fr/files/geom.pdf.

Lur17 J. Lurie, *Higher algebra* (2017), https://www.math.ias.edu/~lurie/papers/HA.pdf.

Lus81 G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. Math. **42** (1981), 169–178.
Nad05 D. Nadler, *Perverse heaves on real loop Grassmannians*, Invent. Math. **159** (2005), 1–73.
NY21 D. Nadler and Z. Yun, *Automorphic gluing functor in Betti geometric Langlands*, Preprint (2021), arXiv:2105.12318.
Ngô06 B. C. Ngô, *Fibration de Hitchin et endoscopie*, Invent. Math. **164** (2006), 399–453.
O'Br23 J. O'Brien, *Homology of loops on a splitting-rank symmetric space via the geometric Satake for real groups*, Preprint (2023), arXiv:2309.09935.
PW19 M. Pflaum and G. Wilkin, *Equivariant control data and neighborhood deformation retractions*, Methods Appl. Anal. **26** (2019), 13–36.
Wan J. Wang, *Examples of relative duality*, https://www.jonathanpwang.com/notes/RelativeDualitydb.html.
YZ11 Z. Yun and X. Zhu, *Integral homology of loop groups via Langlands dual groups*, Contemp. Math. **15** (2011), 347–369.
Zhu15 X. Zhu, *The geometric Satake correspondence for ramified groups*, Ann. Sci. Éc. Norm. Supér. **48** (2015), 409–451.

Tsao-Hsien Chen chenth@umn.edu
School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA

Mark Macerato macerato@berkeley.edu
Department of Mathematics, UC Berkeley, Evans Hall, Berkeley, CA 94720, USA

David Nadler macerato@berkeley.edu
Department of Mathematics, UC Berkeley, Evans Hall, Berkeley, CA 94720, USA

John O'Brien chenth@umn.edu
School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA



https://doi.org/10.1112/S0010437X25007146 Published online by Cambridge University Press