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EXTENDED QUANTUM ENVELOPING ALGEBRAS OF $\mathfrak{sl}(2)$

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Abstract. In present paper we define a new kind of quantized enveloping algebra of $\mathfrak{sl}(2)$. We denote this algebra by $U_{r,t}$, where r, t are two non-negative integers. It is a non-commutative and non-cocommutative Hopf algebra. If $r = 0$, then the algebra $U_{r,t}$ is isomorphic to a tensor product of the algebra of infinite cyclic group and the usual quantum enveloping algebra of $\mathfrak{sl}(2)$ as Hopf algebras. The representation of this algebra is studied.

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1. Introduction. Quantized enveloping algebras for Kac–Moody algebras were introduced independently by Drinfel'd and Jimbo [**1**, **3**] in studying the quantum Yang–Baxter equation and two-dimensional solvable lattice models. There is a rich mathematical theory developed for these objects and their representations with connections to many areas of both mathematics and physics.

Suppose the Kac–Moody algebra is $\mathfrak{sl}(2)$. Then the usual quantum enveloping algebra is generated by E, F, K, K^{-1}. The four generators satisfy some relations. We obtain the extended quantum enveloping algebra $U_{r,t}$ of $\mathfrak{sl}(2)$ by adding new generators J, J^{-1}. $U_{r,t}$ is an algebra generated as an algebra over a field by six generators $E, F, K, K^{-1}, J, J^{-1}$. They satisfy the following relations:

$$K^{-1}K = KK^{-1} = JJ^{-1} = J^{-1}J = 1, \tag{1.1}$$

$$KEK^{-1} = q^2E, \tag{1.2}$$

$$KFK^{-1} = q^{-2}F, \tag{1.3}$$

$$EF - FE = \frac{K - K^{-1}J^r}{q - q^{-1}}, \tag{1.4}$$

This algebra can be obtained from the weak quantum enveloping algebra of $\mathfrak{sl}(2)$ defined in [**11**]. We can introduce co-multiplication and counit on the $U_{r,t}$ to make it into a Hopf algebra. It is a non-commutative and non-cocommutative Hopf algebra. If $r = 0$, then the algebra $U_{r,t}$ is isomorphic to a tensor product of the algebra of an infinite cyclic group and the usual quantum enveloping algebra of $\mathfrak{sl}(2)$ as Hopf algebras. We will study the representation of this algebra in this paper.

Let us outline the structure of this paper. In Section 2, we give the definition of $U_{r,t}$ and obtain some properties of $U_{r,t}$. For example, we prove that $U_{r,t}$ is a Noetherian domain, a Hopf algebra. In Section 3, we study the representation of $U_{r,t}$. Using the theory developed in Section 3, we character the centre of $U_{r,t}$ in Section 4. Unlike the representation theory of usual quantum enveloping $U_q(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$, there exist finite-dimensional non-semisimple $U_{r,t}$-modules. But we can prove that the tensor product of two simple $U_{r,t}$-modules is semisimple, in Section 5. We also obtain a decomposition theory about the tensor product of two simple $U_{r,t}$-modules. In Section 6, we briefly discuss the representation of $U_{r,t}$ in the case where q is a root of unity. In Section 7, we use the $U_{r,t}$ to construct a Hopf algebra with dimension le^3 for any positive integers l, e, where $e \geq 2$.

Throughout this paper $\mathbf{k}$ is a fixed algebraically closed field with characteristic zero; $\mathbf{N}$ is the set of natural numbers; $\mathbf{Z}$ is the set of all integers. For the other undefined terms we refer to [**5**–**7**, **9**].

2. The definition of $U_{r,t}$ and its basic properties. In this section, we will define the extended quantum enveloping algebra $U_{r,t}$ of the Lie algebra $\mathfrak{sl}(2)$ and study its basic properties. Recall that the three matrices $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ consist of a basis of $\mathfrak{sl}(2)$. Before giving the definition of extended quantum enveloping algebra of $\mathfrak{sl}(2)$, we introduce some notations first. Let us fix two indeterminates q, J.

For any integer n, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$

We have the following version of factorials and binomial coefficients. For integers $0 \leq k \leq n$, set $[0]! = 1$,

$$[k]! = [1][2] \ldots [k],$$

if $k > 0$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

With this new notation we can prove the following proposition by induction:

Lemma 2.1. *If x and y are variables subject to the relation $yx = q^2xy$, then*

$$(x + y)^n = \sum_{k=0}^{n} q^{(n-k)k} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

for any positive integer n.

Let $\mathbf{k}$ be an algebraically closed field with characteristic zero. We use $\mathbf{k}_q$ to denote the fraction field of the domain $\mathbf{k}[q, q^{-1}]$.

Definition 2.2. Let r, t be two fixed non-negative integers. We define $U_{r,t} = U_{r,t}(\mathfrak{sl}(2))$ as the $\mathbf{k}_q$-algebra generated by six variables $E, F, K, K^{-1}, J^{-1}, J$, where J and J^{-1} are in the centre of $U_{r,t}$, with the relations

$$K^{-1}K = KK^{-1} = JJ^{-1} = J^{-1}J = 1, \tag{2.1}$$

$$KEK^{-1} = q^2 E, \tag{2.2}$$

$$KFK^{-1} = q^{-2} F, \tag{2.3}$$

$$EF - FE = \frac{K - K^{-1}J^r}{q - q^{-1}}. \tag{2.4}$$

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From the definition, we can prove that there is an algebra automorphism ω_s of $U_{r,t}$ such that $\omega_s(E) = FJ^s$, $\omega_s(F) = EJ^{-s}$, $\omega_s(K) = K^{-1}J^r$, $\omega_s(K^{-1}) = KJ^{-r}$, $\omega_s(J) = J$, $\omega_s(J^{-1}) = J^{-1}$ for any integer s. Moreover, we have the following proposition:

PROPOSITION 2.1. *There exists a unique algebra anti-automorphism ω of $U_{r,t}$ such that $\omega(E) = KF$, $\omega(F) = EK^{-1}$, $\omega(K) = K$, $\omega(K^{-1}) = K^{-1}$, $\omega(J) = J$, $\omega(J^{-1}) = J^{-1}$.*

Proof. To show this proposition, we only need to check the following relations:

$$\omega(K)\omega(E) = q^{-2}\omega(E)\omega(K), \qquad \omega(K)\omega(F) = q^2\omega(F)\omega(K),$$

$$[\omega(F), \omega(E)] = \frac{\omega(K) - \omega(K^{-1})\omega(J^r)}{q - q^{-1}} = \frac{K - K^{-1}J^r}{q - q^{-1}}.$$

The first two relations result directly from definition. We compute the third one as

$$[\omega(F), \omega(E)] = EK^{-1}KF - KFEK^{-1} = EF - FE = \frac{K - K^{-1}J^r}{q - q^{-1}},$$

by relations (2.2) and (2.3). □

LEMMA 2.3. *Let $m \geq 0$, and $n \in \mathbf{Z}$. The following relations hold in $U_{r,t}$:*

$$E^mK^n = q^{-2mn}K^nE^m, \qquad F^mK^n = q^{2mn}K^nF^m, \tag{2.5}$$

$$\begin{aligned} EF^m - F^mE &= [m]F^{m-1}\frac{q^{-(m-1)}K - q^{m-1}K^{-1}J^r}{q - q^{-1}} \\ &= [m]\frac{q^{m-1}K - q^{-(m-1)}K^{-1}J^r}{q - q^{-1}}F^{m-1}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} E^mF - FE^m &= [m]\frac{q^{-(m-1)}K - q^{m-1}K^{-1}J^r}{q - q^{-1}}E^{m-1} \\ &= [m]E^{m-1}\frac{q^{m-1}K - q^{-(m-1)}K^{-1}J^r}{q - q^{-1}}. \end{aligned} \tag{2.7}$$

Proof. The first two relations result trivially from relations (2.2) and (2.3). The third one is proved by induction on m using

$$[E, F^m] = [E, F^{m-1}]F + F^{m-1}[E, F].$$

Similarly, we can prove (2.7). □

THEOREM 2.4. *The algebra $U_{r,t}$ is Noetherian and has no zero divisor. The set $\{E^iF^jK^lJ^s\}_{i,j\in\mathbf{N},l,s\in\mathbf{Z}}$ is a basis of $U_{r,t}$.*

Proof. Let $A_0 = \mathbf{k}_q[K, K^{-1}, J, J^{-1}]$. Since A_0 is a homomorphic image of a Noetherian algebra, it is a Noetherian algebra. Moreover, the family $\{K^lJ^s | l, s \in \mathbf{Z}\}$ is a basis of A_0.

Consider the automorphism α_1 of A_0 determined by $\alpha_1(K) = q^2K$, $\alpha_1(J) = J$ and the corresponding Ore extension $A_1 = A_0[F, \alpha_1, 0]$: the latter has a basis consisting of the monomials $\{F^jK^lJ^s | j \in \mathbf{N}, l, s \in \mathbf{Z}\}$.

It is easy to prove that A_1 is the algebra generated by $F, F^{-1}, K, K^{-1}, J, J^{-1}$ and the relations

$$FK = q^2KF, \qquad FJ = JF.$$

Define

$$\alpha(F^jK^lJ^s) = q^{-2l}F^jK^lJ^s, \tag{2.8}$$

$$\delta(K^l) = \delta(J^s) = 0, \tag{2.9}$$

$$\delta(F^jK^lJ^s) = \sum_{i=0}^{j-1} F^{j-1}\delta(F)(q^{-2i}K)K^lJ^s, \tag{2.10}$$

where $\delta(F)(q^{-2i}K) = \frac{q^{-2i}K - q^{2i}K^{-1}J^r}{q-q^{-1}}$, and $j \geq 1$. We claim that δ extends to an α-derivation of A_1. We must check that for all $j, m \in \mathbf{N}$, and $l_1, l_2, s_1, s_2 \in \mathbf{Z}$, we have

$$\delta(F^jK^{l_1}J^{s_1} \cdot F^mK^{l_2}J^{s_2}) = \alpha(F^jK^{l_1}J^{s_1})\delta(F^mK^{l_2}J^{s_2}) + \delta(F^jK^{l_1}J^{s_1})F^mK^{l_2}J^{s_2}. \tag{2.11}$$

Let us compute the right-hand side of the above equation. We have

$$\begin{aligned}
&\alpha(F^jK^{l_1}J^{s_1})\delta(F^mK^{l_2}J^{s_2}) + \delta(F^jK^{l_1}J^{s_1})F^mK^{l_2}J^{s_2}\\
&= q^{-2l_1}F^jK^{l_1}J^{s_1}\sum_{i=0}^{m-1} F^{m-1}\delta(F)(q^{-2i}K)K^{l_2}J^{s_2}\\
&\quad + \sum_{i=0}^{j-1} F^{j-1}\delta(F)(q^{-2i}K)K^{l_1}J^{s_1}F^mK^{l_2}J^{s_2}\\
&= \sum_{i=0}^{m-1} q^{-2l_1m}F^{j+m-1}\delta(F)(q^{-2i}K)K^{l_1+l_2}J^{s_1+s_2}\\
&\quad + \sum_{i=m}^{m+j-1} q^{-2l_1m}F^{j+m-1}\delta(F)(q^{-2i}K)K^{l_1+l_2}J^{s_1+s_2}\\
&= q^{-2l_1m}\delta(F^{m+l}K^{l_1+l_2}J^{s_1+s_2})\\
&= \delta(F^jK^{l_1}J^{s_1}F^mK^{l_2}J^{s_2}).
\end{aligned}$$

We now build an Ore extension $A_2 = A_1[E, \alpha, \delta]$. Then the following relations hold in A_2:

$$EK = \alpha(K)E + \delta(K) = q^{-2}KE,$$
$$EJ = \alpha(J)E + \delta(J) = JE,$$

and

$$EF = \alpha(F)E + \delta(F) = FE + \frac{K - K^{-1}J^r}{q - q^{-1}}.$$

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From these one easily concludes that A_2 is isomorphic to $U_{r,t}$. Then the properties of $U_{r,t}$ are warranted by the properties of the Ore extension. □

To make the algebra $U_{r,t}$ into the Hopf algebra, we define the following three maps

$$\Delta(E) = J^{-rt} \otimes E + E \otimes KJ^{rt}, \tag{2.12}$$
$$\Delta(F) = K^{-1}J^{r(t+1)} \otimes F + F \otimes J^{-rt}, \tag{2.13}$$
$$\Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \tag{2.14}$$
$$\Delta(J) = J \otimes J, \qquad \Delta(J^{-1}) = J^{-1} \otimes J^{-1}, \tag{2.15}$$
$$\varepsilon(K) = \varepsilon(K^{-1}) = \varepsilon(J) = \varepsilon(J^{-1}) = 1, \tag{2.16}$$
$$\varepsilon(E) = \varepsilon(F) = 0, \tag{2.17}$$

and

$$S(E) = -EK^{-1}, \qquad S(F) = -KFJ^{-r}, \qquad S(J) = J^{-1}, \tag{2.18}$$
$$S(J^{-1}) = J, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K. \tag{2.19}$$

THEOREM 2.5. *Relations* (2.12)–(2.19) *endow* $U_{r,t}$ *with a Hopf algebra.*

Proof. (a) We first show that Δ defines a morphism of algebras from $U_{r,t}$ into $U_{r,t} \otimes U_{r,t}$. It is enough to check that

$$\Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1 \otimes 1,$$
$$\Delta(J)\Delta(J^{-1}) = \Delta(J^{-1})\Delta(J) = 1 \otimes 1,$$
$$\Delta(K)\Delta(E)\Delta(K^{-1}) = q^2\Delta(E),$$
$$\Delta(K)\Delta(F)\Delta(K^{-1}) = q^{-2}\Delta(F),$$
$$\Delta(E)\Delta(F) - \Delta(F)\Delta(E) = \frac{\Delta(K) - \Delta(K^{-1})\Delta(J^r)}{q - q^{-1}},$$

and

$$\Delta(X)\Delta(J) = \Delta(J)\Delta(X),$$

for $X = E, F, K, K^{-1}$. We give a sample calculation for $\Delta(E)\Delta(F) - \Delta(F)\Delta(E) = \frac{\Delta(K)-\Delta(K^{-1})\Delta(J^r)}{q-q^{-1}}$ as follows:

$$\begin{aligned}
[\Delta(E), \Delta(F)] &= (J^{-rt} \otimes E + E \otimes KJ^{rt})(K^{-1}J^{r(t+1)} \otimes F + F \otimes J^{-rt}) \\
&\quad - (K^{-1}J^{r(t+1)} \otimes F + F \otimes J^{-rt})(J^{-rt} \otimes E + E \otimes KJ^{rt}) \\
&= K^{-1}J^r \otimes \frac{K - K^{-1}J^r}{q - q^{-1}} + \frac{K - K^{-1}J^r}{q - q^{-1}} \otimes K \\
&= \frac{\Delta(K) - \Delta(K^{-1}J^r)}{q - q^{-1}}.
\end{aligned}$$

(b) Next, we show that Δ is coassociative. It suffices to do it on the six generators. We give a sample calculation for E. On the one hand, we have

$$\begin{aligned}
(\Delta \otimes id)\Delta(E) &= (\Delta \otimes id)(J^{-rt} \otimes E + E \otimes KJ^{rt}) \\
&= J^{-rt} \otimes J^{-tr} \otimes E + J^{-rt} \otimes E \otimes KJ^{tr} + E \otimes KJ^{tr} \otimes KJ^{rt}.
\end{aligned}$$

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On the other hand, we have

$$(id \otimes \Delta)\Delta(E) = (id \otimes \Delta)(J^{-rt} \otimes E + E \otimes KJ^{rt})$$
$$= J^{-rt} \otimes J^{-tr} \otimes E + J^{-rt} \otimes E \otimes KJ^{tr} + E \otimes KJ^{tr} \otimes KJ^{rt},$$

which is the same.

(c) It is easy to prove that ε defines a morphism of algebras from $U_{r,t}$ to $\mathbf{k}_q$ and satisfies the counit axiom.

(d) It remains to see that S defines an antipode of $U_{r,t}$. We have first to check that S is a morphism of algebras from $U_{r,t}$ into $U_{r,t}^{\mathrm{opp}}$, namely the following relations hold:

$$S(K)S(K^{-1}) = S(K^{-1})S(K) = 1, \qquad S(J)S(J^{-1}) = S(J^{-1})S(J) = 1,$$
$$S(K^{-1})S(E)S(K) = q^2S(E), \qquad S(K^{-1})S(F)S(K) = q^{-2}S(F),$$
$$[S(F), S(E)] = \frac{S(K) - S(K^{-1})S(J^r)}{q - q^{-1}}, \tag{2.20}$$

and $S(X)S(J) = S(J)S(X)$ for $X = E, F, K, K^{-1}, J^{-1}$.

We only give the computation for (2.20). We have

$$[S(F), S(E)] = KFJ^{-r}EK^{-1} - EFJ^{-r}$$
$$= (FE - EF)J^{-r}$$
$$= \frac{S(K) - S(K^{-1})S(J^r)}{q - q^{-1}}.$$

It is easy to check that

$$\sum_{(x)} x_{(1)}S(x_{(2)}) = \sum_{(x)} S(x_{(1)})x_{(2)} = \varepsilon(x)1$$

holds when x is any of the generators $E, F, K^{-1}, K, J, J^{-1}$. Since S is an anti-automorphism of $U_{r,t}$, S is an antipode. □

PROPOSITION 2.2. (1) *If $r = 0$, then $U_{0,t}$ is isomorphic to $\mathbf{k}_q[\mathbf{Z}] \otimes U_q(\mathfrak{sl}(2))$ as Hopf algebras, where $\mathbf{k}_q[\mathbf{Z}]$ is the group algebra of infinite cyclic group $\mathbf{Z}$, $U_q(\mathfrak{sl}(2))$ is the usual quantum enveloping algebra of $\mathfrak{sl}(2)$.*

(2) *We have $S^2(u) = KuK^{-1}$ for any $u \in U_{r,t}$.*

Proof. Obvious. □

PROPOSITION 2.3. *For all $i, j \in \mathbf{N}$ and all $l, s \in \mathbf{Z}$, we have*

$$\Delta(E^iF^jK^lJ^s) = \sum_{u=0}^{i}\sum_{v=0}^{j} q^{u(i-u)+v(j-v)-2(i-u)(j-v)} \begin{bmatrix} i \\ u \end{bmatrix} \begin{bmatrix} j \\ v \end{bmatrix}$$
$$\times (J^{r(t+1)v(j-v)-rut+s} \otimes J^{rt(i-u-v)+s})$$
$$\times (E^{i-u}F^vK^{l-j+v} \otimes E^uF^{j-v}K^{l+i-u}).$$

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Proof. First observe that

$$\begin{aligned}\Delta(E^iF^jK^lJ^s) &= \Delta(E)^i\Delta(F)^j\Delta(K)^l\Delta(J)^s\\ &= (J^{-rt}\otimes E + E\otimes KJ^{rt})^i(K^{-1}J^{-r(t+1)}\otimes F + F\otimes J^{-rt})^j(K^lJ^s\otimes K^lJ^s).\end{aligned}$$

Since

$$(J^{-rt}\otimes E)(E\otimes KJ^{rt}) = q^{-2}(E\otimes KJ^{rt})(J^{-rt}\otimes E),$$

$$\begin{aligned}\Delta(E)^i &= (J^{-rt}\otimes E + E\otimes KJ^{rt})^i\\ &= \sum_{u=0}^{i} q^{u(i-u)}\begin{bmatrix} i\\ u\end{bmatrix}(J^{-rt}\otimes E)^u(E\otimes KJ^{rt})^{i-u}\\ &= \sum_{u=0}^{i} q^{u(i-u)}\begin{bmatrix} i\\ u\end{bmatrix}(J^{-rtu}\otimes 1)(E^{i-u}\otimes E^uK^{i-u})(1\otimes J^{r(i-u)t}),\end{aligned}$$

by Lemma 2.1. Similarly, we have

$$\begin{aligned}\Delta(F)^j &= (K^{-1}J^{r(t+1)}\otimes F + F\otimes J^{-rt})^i\\ &= \sum_{v=0}^{j} q^{v(j-v)}\begin{bmatrix} j\\ v\end{bmatrix}(F\otimes J^{-rt})^v(K^{-1}J^{r(t+1)}\otimes F)^{j-v}\\ &= \sum_{v=0}^{j} q^{v(j-v)}\begin{bmatrix} j\\ v\end{bmatrix}(J^{r(t+1)(j-v)}\otimes J^{-rtv})(F^vK^{j-v}\otimes F^{j-v}).\end{aligned}$$

Hence

$$\begin{aligned}\Delta(E^iF^jK^lJ^s) &= \sum_{u=0}^{i}\sum_{v=0}^{j} q^{u(i-u)+v(j-v)}\begin{bmatrix} i\\ u\end{bmatrix}\begin{bmatrix} j\\ v\end{bmatrix}\\ &\quad\times(J^{-rut+r(j-v)(t+1)}\otimes J^{-vrt+rt(i-u)})\\ &\quad\times(E^{i-u}\otimes E^uK^{i-u})(F^vK^{-(j-v)}\otimes F^{j-v})(K^lJ^s\otimes K^lJ^s)\\ &= \sum_{u=0}^{i}\sum_{v=0}^{j} q^{u(i-u)+v(j-v)}\begin{bmatrix} i\\ u\end{bmatrix}\begin{bmatrix} j\\ v\end{bmatrix}(J^{-r(ut-(j-v)(t+1))+s}\\ &\quad\otimes J^{rt(i-u-v)+s})(E^{i-u}F^vK^{-(j-v)}K^l\otimes E^uK^{i-u}F^{j-v}K^l)\\ &= \sum_{u=0}^{i}\sum_{v=0}^{j} q^{u(i-u)+v(j-v)-2(i-u)(j-v)}\begin{bmatrix} i\\ u\end{bmatrix}\begin{bmatrix} j\\ v\end{bmatrix}\\ &\quad\times(J^{r(t+1)v(j-v)-rut+s}\otimes J^{rt(i-u-v)+s})\\ &\quad\times(E^{i-u}F^vK^{l-j+v}\otimes E^uF^{j-v}K^{l+i-u}).\end{aligned}$$

By now the proof is completed. □

Finally in this section, we give some remarks.

REMARK 2.6. Suppose G is an abelian group, and $g, h \in G$ are two fixed elements. Then we can define a Hopf algebra $U_{g,h}$ as follows:

(1) As vector spaces $U_{g,h}$ is isomorphic to the tensor product of $\mathbf{k}[G]$, the group algebra of G over the field $\mathbf{k}$, and $U_q(\mathfrak{sl}(2))$, the usual quantum enveloping algebra of

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$\mathfrak{sl}(2)$, which is generated by four variables E, F, K, K^{-1}. Any element of $\mathbf{k}[G]$ is in the centre of $U_{g,h}$. The other generators satisfy the following relations:

$$K^{-1}K = KK^{-1} = 1, \tag{2.21}$$

$$KEK^{-1} = q^2E, \tag{2.22}$$

$$KFK^{-1} = q^{-2}F, \tag{2.23}$$

$$EF - FE = \frac{K - K^{-1}g}{q - q^{-1}}. \tag{2.24}$$

(2) The other operations of Hopf algebra $U_{g,h}$ are defined as follows:

$$\Delta(E) = h^{-1} \otimes E + E \otimes hK \tag{2.25}$$

$$\Delta(F) = K^{-1}hg \otimes F + F \otimes h^{-1} \tag{2.26}$$

$$\Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \tag{2.27}$$

$$\Delta(a) = a \otimes a, \qquad a \in G, \tag{2.28}$$

$$\varepsilon(K) = \varepsilon(K^{-1}) = \varepsilon(a) = 1, \qquad a \in G, \tag{2.29}$$

$$\varepsilon(E) = \varepsilon(F) = 0, \tag{2.30}$$

and

$$S(E) = -EK^{-1}, \qquad S(F) = -KFg^{-1}, \tag{2.31}$$

$$S(a) = a^{-1}, \qquad a \in G, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K. \tag{2.32}$$

REMARK 2.7. By using the above method, we can construct extensions of quantum enveloping algebras of others Lie algebras (or Kac–Moody algebras [**4**]) by group algebras.

REMARK 2.8. We can assume that q is an element of $\mathbf{k}$. If $q^2 \neq 1$, then $U_{r,t}$ is a Hopf algebra over $\mathbf{k}$. In the remainder of this paper we always assume that q is an element in $\mathbf{k}$ and $q^2 \neq 1$.

REMARK 2.9. One can study the dual algebra $U_{r,t}^*$ of $U_{r,t}$. In the case $r = 0$,

$$U_{0,t}^* = Hom_{\mathbf{k}}(U_{0,t}, \mathbf{k}) \simeq Hom_{\mathbf{k}}(\mathbf{k}[\mathbf{Z}], U_q(\mathfrak{sl}(2))^*),$$

by Proposition 2.2. Moreover, one can determine whether $U_{r,t}$ is quasi-triangular or not.

3. The representation of $U_{r,t}$. In this section, let q be an element in the algebraically closed field $\mathbf{k}$ with characteristic zero. Moreover, we assume that q is not a root of unity. We shall determine all finite-dimensional simple $U_{r,t}$-modules in this section.

For any two elements $\lambda, \alpha \in \mathbf{k}$ and any $U_{r,t}$-module V, we denote by

$$V^{\lambda,\alpha} = \{v \in V | Kv = \lambda v, Jv = \alpha^2 v\}.$$

The pair (λ, α) is called a weight of V if $V^{\lambda,\alpha} \neq 0$.

LEMMA 3.1. *We have* $EV^{\lambda,\alpha} \subseteq V^{q^2\lambda,\alpha}$ *and* $FV^{\lambda,\alpha} \subseteq V^{q^{-2}\lambda,\alpha}$.

Proof. For any $v \in V^{\lambda,\alpha}$, we have

$$\begin{cases} KEv = q^2EKv = q^2\lambda Ev \\ JEv = EJv = \alpha^2 Ev \end{cases},$$

and

$$\begin{cases} KFv = q^{-2}FKv = q^{-2}\lambda Fv \\ JFv = FJv = \alpha^2 Fv \end{cases}.$$

So this lemma holds. □

DEFINITION 3.2. Let V be a $U_{r,t}$-module and (λ, α) is a pair of scalars. An element $v \neq 0$ of V is the highest weight vector of weight (λ, α) if $Ev = 0$, $Kv = \lambda v$ and $Jv = \alpha^2 v$. A $U_{r,t}$-module is the highest weight module of highest weight (λ, α) if it is generated by the highest vector v of weight (λ, α).

PROPOSITION 3.1. *Any non-zero finite-dimensional $U_{r,t}$-module contains a highest weight vector. Moreover the endomorphisms induced by E and F are nilpotent.*

Proof. Since **k** is algebraically closed, V is finite-dimensional and $JK = KJ$, there exists a non-zero vector w and (μ, α) such that

$$Kw = \mu w, \qquad Jw = \alpha^2 w.$$

If $Ew = 0$, then the vector w is the highest weight vector and we are done. If not, let us consider the sequence of vectors $E^n w$, where n runs over the non-negative integers. According to Lemma 3.1, it is a sequence of eigenvectors with distinct eigenvalues. Consequently, there exists an integer n such that $E^n w \neq 0$ and $E^{n+1} w = 0$. The vector $E^n w$ is the highest weight vector.

In order to prove that the action of E on V is nilpotent, it suffices to check that 0 is the only eigenvalue of E. Now, if v is a non-zero eigenvector for E with eigenvalue $\lambda \neq 0$, then so is $K^n v$ with eigenvalue $q^{-2n}\lambda$. The endomorphism E would then have infinitely many distinct eigenvalues which is impossible. The same argument works for F. □

LEMMA 3.3. *Let v be a highest weight vector of weight (λ, α). Set $v_0 = v$ and $v_p = \frac{1}{[p]!}F^p v$ for $p > 0$. Then*

$$Kv_p = q^{-2p}\lambda v_p, \qquad Jv_p = \alpha^2 v_p, \qquad Fv_{p-1} = [p]v_p,$$

and

$$Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}\alpha^{2r}}{q - q^{-1}} v_{p-1}. \tag{3.1}$$

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Proof. We only check equation (3.1). By Lemma 2.3, we have

$$Ev_p = \frac{1}{[p]!}\left(F^pE + [p]F^{p-1}\frac{q^{-(p-1)}K - q^{p-1}K^{-1}J^r}{q - q^{-1}}\right)v_0$$
$$= \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}\alpha^{2r}}{q - q^{-1}}v_{p-1}.$$

□

THEOREM 3.4. *(a) Let V be a finite-dimensional $U_{r,t}$-module generated by the highest weight vector v of weight (λ, α). Then*

(i) $\lambda = \epsilon q^n\alpha^n$, where $\epsilon = \pm 1$ and n is the integer defined by $\dim V = n + 1$.

(ii) Setting $v_p = \frac{1}{[p]!}F^p v$, we have $v_p = 0$ for $p > n$ and in addition the set $\{v = v_0, v_1, \ldots, v_n\}$ is a basis of V.

(iii) The operator K acting on V is diagonalizable with $(n + 1)$ distinct eigenvalues

$$\{\epsilon q^n\alpha^r, \epsilon q^{n-2}\alpha^r, \ldots, \epsilon q^{-n+2}\alpha^r, \epsilon q^{-n}\alpha^r\},$$

and the operator J acts on V by a scalar α^2.

(iv) Any other highest weight vector in V is a scalar multiple of v and is of weight (λ, α).

(v) The module is simple.

(b) Any simple finite-dimensional $U_{r,t}$-module is generated by the highest weight vector. Two finite-dimensional $U_{r,t}$-modules generated by highest vectors of the same weight are isomorphic.

Proof. According to Lemma 3.3, the sequence $\{v_p | p \geq 0\}$ is a sequence of eigenvectors for K with distinct eigenvalues. Since V is finite-dimensional, there is an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$. Then from the formulas

$$Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}\alpha^{2r}}{q - q^{-1}}v_{p-1},$$

we obtain $v_m = 0$ for all $n > m$ and $v_m \neq 0$ for all $m \leq n$. Moreover,

$$0 = Ev_{n+1} = \frac{q^{-n}\lambda - q^n\lambda^{-1}\alpha^{2r}}{q - q^{-1}}v_n.$$

Hence $\lambda^2 = q^{2n}\alpha^{2r}$, which is equivalent to $\lambda = \epsilon q^n\alpha^r$. The rest of the proof of (i)–(iii) is easy. So we omit it.

(iv) Let v' be another highest weight vector. It is an eigenvector for the action of K and J; hence it is a scalar multiple of some vector v_i. But the vector v_i is killed by E if and only if $i = 0$.

(v) Let V' be a non-zero $U_{r,t}$-submodule of V and let v' be the highest weight vector of V'. Then v' also is the highest weight vector for V. By (iv), v' has to be a non-zero scalar multiple of v. Therefore v is in V'. Since v generates V, we must have $V = V'$, which proves that V is simple.

(b) By Proposition 3.1, any simple finite-dimensional $U_{r,t}$-module V contains a highest weight vector v. Let V' be the submodule of V generated by v. Since V is simple, $V = V'$ and hence V is generated by the highest weight vector v. The rest results of (b) follow from (a). □

We denote the $(n+1)$-dimensional simple $U_{r,t}$-module-generated highest weight vector v by $V_{\epsilon,n,\alpha}$, where v satisfies

$$Ev = 0, \qquad Jv = \alpha^2 v, \qquad Kv = \epsilon q^n \alpha^r v.$$

Let $\rho_{\epsilon,n,\alpha}$ be the corresponding morphism of algebras from $U_{r,t}$ to $End(V_{\epsilon,n,\alpha})$.

Observe that the formulas of Lemma 3.3 may be rewritten as follows for $V_{\epsilon,n,\alpha}$:

$$Kv_p = \epsilon q^{n-2p}\alpha^r v_p, \qquad Jv_p = \alpha^2 v_p, \qquad Fv_{p-1} = [p]v_p,$$

and

$$Ev_p = \epsilon \frac{q^{n-(p-1)}\alpha^r - q^{p-1-n}\alpha^r}{q-q^{-1}} v_{p-1} = \epsilon\alpha^r[n-p+1]v_{p-1}. \tag{3.2}$$

As a special case, we have $V_{\epsilon,0,\alpha} = \mathbf{k}$. The morphism $\rho_{\epsilon,0,\alpha}$ is given by

$$\rho_{\epsilon,0,\alpha}(K) = \epsilon\alpha^r, \qquad \rho_{\epsilon,0,\alpha}(E) = \rho_{\epsilon,0,\alpha}(F) = 0, \qquad \rho_{\epsilon,0,\alpha}(J) = \alpha^2.$$

Lemma 3.5. *There exists an element C of the centre of $U_{r,t}$ acting by 0 on $V_{\epsilon,0,\alpha}$ and by a non-zero scalar on $V_{\epsilon',n,\alpha}$ when n is an integer greater than zero, and $\epsilon, \epsilon' = \pm 1$.*

Proof. Define $C = C_p - \epsilon\frac{\alpha^r(q+q^{-1})}{(q-q^{-1})^2}$, where $C_p = EF + \frac{q^{-1}K+qK^{-1}J^r}{(q-q^{-1})^2}$. First we show that C_p is in the centre of $U_{r,t}$. Let us calculate KC_pK^{-1} and EC_p.

$$\begin{aligned} KC_pK^{-1} &= KEFK^{-1} + \frac{q^{-1}K + qK^{-1}J^r}{(q-q^{-1})^2} \\ &= EF + \frac{q^{-1}K + qK^{-1}J^r}{(q-q^{-1})^2} \\ &= C_p. \end{aligned}$$

Since

$$[E,F] = \frac{K - K^{-1}J^r}{q-q^{-1}}, \qquad C_p = FE + \frac{qK + q^{-1}K^{-1}J^r}{(q-q^{-1})^2}.$$

Hence

$$\begin{aligned} EC_p &= EFE + E\frac{qK + q^{-1}K^{-1}J^r}{(q-q^{-1})^2} \\ &= EFE + \frac{q^{-1}K + qK^{-1}J^r}{(q-q^{-1})^2}E \\ &= C_pE. \end{aligned}$$

Similarly we can prove $FC_p = C_pF$. So C_p is in the centre of $U_{r,t}$. Consequently C is in the centre of $U_{r,t}$.

C acts on $V_{\epsilon,0,\alpha}$ by

$$\frac{q\epsilon\alpha^r + q^{-1}\epsilon\alpha^r}{(q-q^{-1})^2} - \epsilon\frac{q\alpha^r + q^{-1}\alpha^r}{(q-q^{-1})^2} = 0.$$

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Since C acts on $V_{\epsilon',n,\alpha}$ by

$$\beta = \frac{q^{n+1}\epsilon'\alpha^r + q^{-1-n}\epsilon'\alpha^r}{(q-q^{-1})^2} - \epsilon\frac{q\alpha^r + q^{-1}\alpha^r}{(q-q^{-1})^2} = 0,$$

we have to show that $\beta \neq 0$ when $n > 0$. If $\beta = 0$, we would have $(q^{n+2} - \epsilon\epsilon')(q^n - \epsilon\epsilon') = 0$, which would be contrary to the assumption, that q is not a root of unity. □

THEOREM 3.6. *When q is not a root of unity, any two-dimensional $U_{r,t}$-module V is isomorphic to either $V_{\epsilon,0,\alpha} \oplus V_{\epsilon',0,\beta}$, or $V_{\epsilon,1,\alpha}$, or a module $V(\alpha, \epsilon, y)$ with basis $\{v_1, v_2\}$ such that $\rho(E) = \rho(F) = 0$, and $\rho(J) = \begin{pmatrix}\alpha^2 & y\\ 0 & \alpha^2\end{pmatrix}$, $\rho(K) = \begin{pmatrix}\epsilon\alpha^r & \frac{ry}{2}\epsilon\alpha^{r-2}\\ 0 & \epsilon\alpha^r\end{pmatrix}$, where ρ is the algebra homomorphism determined by $V(\alpha, \epsilon, y)$.*

Proof. Suppose V is simple. Then V is isomorphic to $V_{\epsilon,1,\alpha}$ by Theorem 3.4. Otherwise there exists a proper submodule V' of V. Since the dimension of V' is equal to one, we can assume that $\{v_1, v_2\}$ is a basis of V satisfying

$$Kv_1 = \epsilon\alpha^r v_1, \qquad Kv_2 = \epsilon'\beta^r v_2 + xv_1,$$
$$Jv_1 = \alpha^2 v_1, \qquad Jv_2 = \beta^2 v_2 + yv_1.$$

Since $\epsilon'\beta^r(\beta^2 v_2 + yv_1) + x\alpha^2 v_1 = JKv_2 = KJv_2 = \beta^2(\epsilon'\beta^r v_2 + xv_1) + y\epsilon\alpha^r v_1$, $x(\alpha^2 - \beta^2) = y(\epsilon'\beta^r - \epsilon\alpha^r)$.

If $\epsilon\alpha^r \neq \epsilon'\beta^r$ and $\alpha^2 \neq \beta^2$, then $v_1, v_2' = v_2 + \frac{x}{\epsilon'\beta^r - \epsilon\alpha}v_1 = v_2 + \frac{y}{\beta^2-\alpha^2}v_1$ is another basis of V. Since $Kv_2' = \epsilon'\beta v_2'$ and $Jv_2' = \beta^2 v_2'$, $V = \mathbf{k}v_1 \oplus \mathbf{k}v_2'$ is a direct sum of $U_{r,t}$-modules.

If $\alpha^2 = \beta^2$ and $\epsilon'\beta^r \neq \epsilon\alpha^r$, then $y = 0$. Let $v_2' = v_2 + \frac{x}{\epsilon'\beta^r - \epsilon\alpha^r}v_1$. Then $Jv_2' = \beta^2 v_2'$ and $Kv_2' = \epsilon'\beta^r v_2'$. Consequently $V = \mathbf{k}v_1 \oplus \mathbf{k}v_2'$ is a direct sum of $U_{r,t}$-modules.

If $\alpha^2 \neq \beta^2$ and $\epsilon'\beta^r = \epsilon\alpha^r$, then $x = 0$. Let $v_2' = v_2 + \frac{y}{\beta^2-\alpha^2}v_1$. Then $Jv_2' = \beta^2 v_2'$ and $Kv_2' = \epsilon'\beta^r v_2'$. Consequently $V = \mathbf{k}v_1 \oplus \mathbf{k}v_2'$ is a direct sum of $U_{r,t}$-modules.

Next we assume that $\epsilon\alpha^r = \epsilon'\beta^r$, and $\alpha^2 = \beta^2$. Since Ev_1 is an eigenvector for K with eigenvalue $\epsilon q^2\alpha^r \neq \epsilon\alpha^r$, it is zero. Let us prove that Ev_2 is zero too. Indeed, writing $Ev_2 = \lambda v_1 + \mu v_2$, we have

$$\epsilon\alpha^r\lambda v_1 + \mu(\epsilon\alpha^r v_2 + xv_1) = KEv_2 = q^2EKv_2 = q^2E(\epsilon\alpha^r v_2 + xv_1) = q^2\epsilon\alpha^r(\lambda v_1 + \mu v_2).$$

Hence

$$\begin{cases}\epsilon\alpha^r\lambda + x\mu = q^2\epsilon\alpha^r\lambda\\ \mu\epsilon\alpha^r = q^2\mu\epsilon\alpha^r.\end{cases} \tag{3.3}$$

Since $q^2 \neq 1$, we obtain $\lambda = \mu = 0$ from (3.3). One can show in a similar way that F acts as zero on V. Since $[E, F]$ acts as zero, we have $K = K^{-1}J^r$ on V. In particular, since $K^{-1}v_2 = \epsilon\alpha^{-r}v_2 - x\alpha^{-2r}v_1$,

$$J^rK^{-1}v_2 = \epsilon\alpha^{-r}J^rv_2 - x\epsilon\alpha^r v_1 = \epsilon\alpha^r v_2 + (\epsilon ry\alpha^{r-2} - x)v_1.$$

Hence $\epsilon ry\alpha^{r-2} - x = x$ and $x = \frac{ry}{2}\epsilon\alpha^{r-2}$. So $\rho(E) = \rho(F) = 0$, and $\rho(J) = \begin{pmatrix}\alpha^2 & y\\ 0 & \alpha^2\end{pmatrix}$, $\rho(K) = \begin{pmatrix}\epsilon\alpha^r & \frac{ry}{2}\epsilon\alpha^{r-2}\\ 0 & \epsilon\alpha^r\end{pmatrix}$, where ρ is the algebra homomorphism determined by $V(\alpha, \epsilon, y)$. □

REMARK 3.7. If $y \neq 0$, then $V(\alpha, \epsilon, y)$ is not a semisimple $U_{r,t}$-module.

REMARK 3.8. Suppose that the submodule V' of a module V is simple of dimension greater than 1 and the dimension of V/V_1 is 1. Then there exists a one-dimensional module V_2 such that $V = V_1 \oplus V_2$. In fact, let the one-dimensional quotient module V/V' has weight $(\epsilon\alpha^r, \alpha)$. Let us consider the operator

$$C = C_p - \epsilon\frac{q\alpha^r + q^{-1}\alpha^r}{(q-q^{-1})^2},$$

it acts by zero on V/V'. Consequently, we have $CV \subseteq V'$. On the other hand, C acts on V' as multiplication by a scalar $y \neq 0$. It follows that $\frac{1}{y}C$ is the identity on V'. Therefore the map $\frac{1}{y}C$ is a projector of V onto V'. This projector is a $U_{r,t}$-linear since C is central. Let $V_2 = ker(\frac{1}{y}C)$. Then $V = V' \oplus V_2$.

THEOREM 3.9. *The dual module $V^*_{\epsilon,n,\alpha}$ of the simple $U_{r,t}$-module $V_{\epsilon,n,\alpha}$ is a simple module, and $V^*_{\epsilon,n,\alpha} \simeq V_{\epsilon,n,\alpha^{-1}}$.*

Proof. Since $U_{r,t}$ is a Hopf algebra, the dual of any $U_{r,t}$-module is still a $U_{r,t}$-module. First we prove that V is a simple module if and only if $V^* := Hom_{\mathbf{k}}(V, \mathbf{k})$ is a simple module. Since V is finite dimensional, $V \simeq V^{**}$. We only need to verify the implication that V^* is simple if V is simple. Let L be a non-zero submodule of V^*. If $L \neq V^*$, then $W = \{x \in V | f(x) = 0 \quad for \quad all \quad x \in L\} \neq 0$. For any $x \in W$ and any $f \in L$, we have $f(Kx) = (K^{-1}f)(x) = 0$, $f(Jx) = (J^{-1}f)(x) = 0$, $0 = (-Ef)(Kx) = f(Ex)$ and $0 = (-FKf)(J^r x) = f(Fx) = 0$. Hence W is a submodule of V. Consequently, $W = V$. So $L = 0$. This is contrary to our original assumption. Hence V^* is simple. Now suppose $V_{\epsilon,n,\alpha}$ is spanned by $\{v_0, \dots, v_n\}$ with relations

$$Kv_p = \epsilon q^{n-2p}\alpha^r v_p, \qquad Jv_p = \alpha^2 v_p, \qquad Fv_{p-1} = [p]v_p,$$

and

$$Ev_p = \epsilon\frac{q^{n-(p-1)}\alpha^r - q^{p-1-n}\alpha^r}{q-q^{-1}}v_{p-1} = \epsilon\alpha^r[n-p+1]v_{p-1}.$$

Let $\{v_0^*, \dots, v_n^*\}$ be the dual basis of $\{v_0, \dots, v_n\}$. Then

$$(Ev_n^*)(v_i) = -v_n^*(EK^{-1}v_i) = \epsilon\alpha^r q^{2i-n}[n-i+1]v_n^*(v_{i-1}) = 0,$$

$$(Kv_n^*)(v_i) = v_n^*(K^{-1}v_i) = q^{2i-n}\epsilon\alpha^{-r}v_n^*(v_i) = q^n\epsilon\alpha^{-r}v_n^*(v_i)$$

and

$$(Jv_n^*)(v_i) = v_n^*(J^{-1}v_i) = \alpha^{-2}v_n^*(v_i).$$

Thus, v_n^* is the highest weight vector with weight $(q^n\alpha^{-r}, \alpha^{-1})$ of $V^*_{\epsilon,n,\alpha}$ and hence $V^*_{\epsilon,n,\alpha} \simeq V_{\epsilon,n,\alpha^{-1}}$. □

Finally in this section, for any given finite-dimensional semisimple $U_{r,t}$-module V, we construct a scalar product, i.e. a non-degenerated symmetric bilinear form $(,)$ on V such that

$$(xv, v') = (v, \omega(x)v') \tag{3.4}$$

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for all $x \in U_{r,t}$ and $v, v' \in V$. The linear map ω has been defined in Proposition 2.1. This is done in the following theorem:

THEOREM 3.10. *On the simple $U_{r,t}$-module $V_{\epsilon,n,\alpha}$ generated by the highest weight vector v, there exists a unique scalar product such that $(v, v) = 1$. If we define the vectors $v_i := \frac{1}{[i]}F^i v$ for all $i \geq 0$, then they are pairwise orthogonal and we have*

$$(v_i, v_j) = q^{i(i+1-n)} \begin{bmatrix} n \\ i \end{bmatrix} \delta_{ij}.$$

Proof. Let us first assume that there exists a scalar product on $V_{\epsilon,n,\alpha}$ such that $(v, v) = 1$. Next we will show that $(v_i, v_j) = q^{i(i+1)-ni}[{n \atop i}]\delta_{ij}$. By definition and (3.4) we have

$$(v_i, v_j) = \frac{1}{[i]!}(F^i v, v_j) = \frac{1}{[i]!}(v, \omega(F^i)v_j) = \frac{1}{[i]!}(v, (EK^{-1})^i v_j).$$

By (2.5) we can prove that $(EK^{-1})^i = q^{i(i+1)}K^{-i}E^i$ for any $i > 0$. Consequently, the vector $\omega(F^i)v_j$ is a scalar multiple of $E^i v_j$, which is equal to zero as soon as $i > j$. Therefore $(v_i, v_j) = 0$ if $i > j$. By symmetry, we also have $(v_i, v_j) = 0$ if $i < j$.

We need the formula

$$E^i v_j = (\epsilon\alpha^r)^i \frac{[n-j+i]}{[n-j]} v_{j-i}$$

to compute (v_i, v_i). We have

$$\begin{aligned}(v_i, v_i) &= \frac{1}{[i]!}q^{i(i+1)}(v, K^{-i}E^i v_i) \\ &= (\epsilon\alpha^r)^i q^{i(i+1)} \frac{[n]!}{[i]![n-i]!}(v, K^{-i}v) \\ &= q^{i(i+1)-ni}\frac{[n]!}{[i]![n-i]!}.\end{aligned}$$

This proves the uniqueness of the scalar product. Let us now prove its existence.

Clearly, there exists a non-degenerate symmetric bilinear form such that

$$(v_i, v_j) = q^{i(i+1-n)} \begin{bmatrix} n \\ i \end{bmatrix} \delta_{ij}. \tag{3.5}$$

We have to check that it satisfies relation (3.4). It is enough to check this for $x = E, F, K, K^{-1}, J$ and J^{-1}. We shall do this for $x = E$ and $x = F$, since the other computations are easy.

For the case $x = E$. On the one hand, we have

$$(Ev_i, v_j) = \epsilon\alpha^r[n-i+1](v_{i-1}, v_j) = \epsilon\alpha^r q^{(i-1)(i-n)} \frac{[n]!}{[i-1]![n-i]!}\delta_{i-1j}.$$

One the other hand, by Proposition 2.1 and by (3.4), we have

$$\begin{aligned}(v_i, \omega(E)v_j) &= (v_i, KFv_j)\\ &= [j+1](v_i, Kv_{j+1})\\ &= \epsilon\alpha^r q^{i(i+1-n)+n-2(j+1)}[j+1]\frac{[n]!}{[i]![n-i]!}\delta_{ij+1}\\ &= \epsilon\alpha^r q^{(i-1)(i-n)}\frac{[n]!}{[i-1]![n-i]!}\delta_{ij+1}\\ &= (Ev_i, v_j).\end{aligned}$$

For the case $x = F$. On the one hand, we have

$$(Fv_i, v_j) = [i+1](v_{i+1}, v_j) = q^{(i+1)(i+2-n)}\frac{[n]!}{[i]![n-i-1]!}\delta_{i+1j}.$$

One the other hand, by Proposition 2.1 and by (3.4), we have

$$\begin{aligned}(v_i, \omega(F)v_j) &= (v_i, EK^{-1}v_j)\\ &= \epsilon\alpha^{-r}q^{2j-n}(v_i, Ev_j)\\ &= q^{2j-n}[n-j+1](v_i, v_{j-1})\\ &= q^{i(i+1-n)+2(i+1)-n}[n-i]\frac{[n]!}{[i]![n-i]!}\delta_{ij-1}\\ &= q^{(i+1)(i+2-n)}\frac{[n]!}{[i]![n-i-1]!}\delta_{ij-1}\\ &= (Fv_i, v_j).\end{aligned}$$

This completes the proof of this theorem. □

4. The Harish-Chandra homomorphism and the centre of $U_{r,t}$. Our objective in this section is to describe the centre Z of $U_{r,t}$ in case q is not a root of unity. We assume this throughout this section.

Let us fix (λ, α), where $\alpha\lambda \neq 0$. Consider an infinite-dimensional vector space $V(\lambda, \alpha)$ with denumerable basis $\{v_i | i \in \mathbf{N}\}$. For $p \geq 0$, set

$$\begin{cases} Kv_p = q^{-2p}\lambda v_p, & Jv_p = \alpha^2 v_p,\\ Ev_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_p, & \\ Ev_0 = 0, & Fv_p = [p+1]v_{p+1}.\end{cases} \tag{4.1}$$

$$K^{-1}v_p = q^{2p}v_p, \qquad J^{-1}v_p = \alpha^{-2}v_p. \tag{4.2}$$

LEMMA 4.1. *Relations in* (4.1) *and* (4.2) *define a $U_{r,t}$-module structure on* $V(\lambda, \alpha)$. *The element v_0 generates $V(\lambda, \alpha)$ as a $U_{r,t}$-module and is the highest weight vector of weight* (λ, α).

Proof. Immediate computation yield

$$K^{-1}Kv_p = KK^{-1}v_p = v_p, \qquad J^{-1}Jv_p = JJ^{-1}v_p = v_p,$$

$$KEK^{-1}v_p = q^2 Ev_p, \qquad KFK^{-1}v_p = q^{-2}Fv_p,$$

$$\begin{aligned}[E, F]v_p &= ([p+1]\frac{q^{-p}\lambda - q^{p}\lambda^{-1}\alpha^{2r}}{q-q^{-1}} - [p]\frac{q^{-p+1}\lambda - q^{p-1}\lambda^{-1}\alpha^{2r}}{q-q^{-1}})v_p \\ &= \frac{q^{-2p}\lambda - q^{2p}\lambda^{-1}\alpha^{2r}}{q-q^{-1}}v_p \\ &= \frac{K - K^{-1}J^{r}}{q-q^{-1}}v_p. \end{aligned} \tag{4.3}$$

This show that the relations in (4.1) and (4.2) define a $U_{r,t}$-module structure on $V(\lambda, \alpha)$. The proof is complete. □

Let U^K be the subalgebra of $U_{r,t}$ of all elements commuting with K.

LEMMA 4.2. *An element of $U_{r,t}$ belongs to U^K if and only if it is of the form*

$$\sum_{i\geq 0} F^i P_i E^i,$$

where $P_0, P_1, \ldots$ are elements of $\mathbf{k}[K, K^{-1}; J, J^{-1}]$.

Proof. This is a consequence of the fact that $\{F^i K^l J^s E^j | i, j \in \mathbf{N}, l, s \in \mathbf{Z}\}$ is a basis of $U_{r,t}$ and that

$$K(F^i K^l J^s E^j)K^{-1} = q^{2(j-s)} F^i K^l J^s E^i.$$

□

LEMMA 4.3. *We have $I = U_{r,t}E \cap U^K = FU_{r,t} \cap U^K$ and*

$$U^K = \mathbf{k}[K, K^{-1}; J, J^{-1}] \oplus I.$$

Proof. Let $u = \sum_{i\geq 0} F^i P_i E^i \in U_{r,t}$ be an element of U^K. If u also lies in $U_{r,t}E$, then $P_0 = 0$. Hence u belongs to $FU_{r,t} \cap U^K$ and conversely. Since the form $\sum_{i\geq 0} F^i P_i E^i$ is unique for any element of U^K, we get the desired direct sum. □

It results from $I = U_{r,t}E \cap U^K = FU_{r,t} \cap U^K$ that I is a two-sided ideal and the projector φ from U^K onto $\mathbf{k}[K, K^{-1}; J, J^{-1}]$ is a morphism of algebras. The map φ is called the Harish-Chandra homomorphism. It permits one to express the action of the centre Z on the highest weight module.

PROPOSITION 4.1. *Let $V(\lambda, \alpha)$ be the highest weight module of $U_{r,t}$ with highest weight (λ, α). Then, for any central element $z \in Z$ and any $v \in V$, we have*

$$zv = \varphi(z)(\lambda, \alpha^2)v.$$

Recall that $\varphi(z)$ is a Laurent polynomial in K, J, and $\varphi(z)(\lambda, \alpha^2)$ is its value at $K = \lambda$ and $J = \alpha^2$.

Proof. Let v_0 be the highest weight vector generating $V(\lambda, \alpha)$ and z a central element of $U_{r,t}$. The element z can be written in the form

$$z = \varphi(z) + \sum_{i>0} F^i P_i E^i.$$

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Since

$$\begin{cases} Ev_0 = 0, & Jv_0 = \alpha^2 v_0, \\ Kv_0 = \lambda v_0, \end{cases}$$

we get $zv_0 = \varphi(z)(\lambda, \alpha^2)v_0$. If v is an arbitrary element of $V(\lambda, \alpha)$, we have $v = xv_0$ for some $x \in U_{r,t}$, hence $zv = xzv_0 = \varphi(z)(\lambda, \alpha^2)v$. □

LEMMA 4.4. *Let $z \in Z$. If $\varphi(z) = 0$, then $z = 0$.*

Proof. Let z be an element in the centre such that $\varphi(z) = 0$. Assume $z \neq 0$. Since $z \in U^K$, we can assume that $z = \sum_{i=k}^{l} F^i P_i E^i \in FU_{r,t}$ for some $k \geq 1$, where $P_k, P_{k+1}, \ldots, P_l$ are non-zero Laurant polynomials in K and J. Consider a Verma module $V(\lambda, \alpha)$, The relations in (4.1) and (4.2) show that $Ev_p = 0$ if and only if $p = 0$. Let us apply z to the vector v_k of $V(\lambda, \alpha)$. On the one hand

$$zv_k = \varphi(z)(\lambda, \alpha^2)v_k = 0.$$

On the other hand, we get

$$zv_k = F^k P_k E^k v_k = cP_k(\lambda, \alpha^2)v_k,$$

where c is a non-zero constant. It follows that $P(\lambda, \alpha^2) = 0$ for any non-zero λ and α. Thus $P_k = 0$. This is impossible. □

THEOREM 4.5. *When q is not a root of unity, the centre Z of $U_{r,t}$ is a polynomial algebra generated by the element C_p over the algebra $\mathbf{k}[J, J^{-1}]$. The restriction of Harish-Chandra homomorphism to Z is an isomorphism onto the subalgebra of $\mathbf{k}[K, K^{-1}, J^{-1}, J]$ generated by $qK + q^{-1}K^{-1}J^r$.*

Proof. For any integer $n > 0$, consider the Verma module $V(q^{n-1}\alpha^r, \alpha)$ for any non-zero element α. By (4.1) we have $Ev_n = 0$. Thus v_n is the highest weight vector of weight $(q^{-n-1}\alpha^r, \alpha)$. By Proposition 4.1 a central element z acts on the module generated by v_n as the multiplication by scalar $\varphi(z)(q^{-(n-1)}\alpha^r, \alpha^2)$; but since v_n is in $V(q^{n-1}\alpha^r, \alpha)$, the element z also acts as the scalar $\varphi(q^{n-1}\alpha^r, \alpha^2)$. Thus

$$\varphi(z)(q^{n-1}\alpha^r, \alpha^2) = \varphi(z)(q^{-(n+1)}\alpha^r, \alpha^2) \tag{4.4}$$

for any $\alpha \neq 0$ and any $n > 0$. Suppose $\varphi(z) = P(K, K^{-1}, J, J^{-1})$. Then (4.4) implies

$$P(q^{n-1}\alpha^r, q^{-(n-1)}\alpha^{-r}, \alpha^2, \alpha^{-2}) = P(q^{-(n+1)}\alpha^r, q^{n+1}\alpha^{-r}, \alpha^2, \alpha^{-2}). \tag{4.5}$$

Let

$$\psi_\alpha(x) = P(q^{-1}\alpha^r x, q\alpha^r x^{-1}, \alpha^2, \alpha^{-2}).$$

Then $\psi_\alpha(q^n) = \psi_\alpha(q^{-n})$ for any integer n by (4.5). Hence

$$\psi_\alpha(x) = \sum_{i \geq 0} a_i(\alpha)(x + x^{-1})^i,$$

where $a_i(\alpha) \in \mathbf{k}[\alpha, \alpha^{-1}]$. Therefore

$$\psi_\alpha(qK\alpha^{-r}) = \sum_{i \geq 0} a_i(\alpha)(qK\alpha^{-r} + q^{-1}K^{-1}\alpha^r)^i = P(K, K^{-1}, \alpha^2, \alpha^{-2}), \tag{4.6}$$

for any non-zero α. Since

$$P(K, K^{-1}, (-\alpha)^2, (-\alpha)^{-2}) = P(K, K^{-1}, \alpha^2, \alpha^{-2}),$$

$$\sum_{i\geq 0} a_i(\alpha)\alpha^{-ri}(qK + q^{-1}K^{-1}\alpha^{2r})^i = \sum_{i\geq 0} a_i(-\alpha)\alpha^{-ri}(qK + q^{-1}K^{-1}\alpha^{2r})^i.$$

Hence $a_i(\alpha) = \alpha^{ir}b_i(\alpha^2)$. So

$$\sum_{i\geq 0} b_i(\alpha^2)(qK + q^{-1}K^{-1}\alpha^{2r})^i = P(K, K^{-1}, \alpha^2, \alpha^{-2}). \tag{4.7}$$

Consequently,

$$\varphi(z) = \sum_{i\geq 0} c_i(J, J^{-1})(qK + q^{-1}K^{-1}J^r)^i.$$

Since $\varphi(C_p) = \frac{qK+q^{-1}K^{-1}J^r}{(q-q^{-1})^2}$, $\varphi(J) = J$ and $\varphi(J^{-1}) = J^{-1}$, φ is a surjective map from Z to the subalgebra of $\mathbf{k}[K, K^{-1}, J^{-1}, J]$ generated by $qK + q^{-1}K^{-1}J^r$. Using Lemma 4.4, we obtain the proof of the remaining results of this theorem. □

5. The generalized quantum Clebsch–Gordan formula. We now prove a generalized quantum Clebsch–Gordan formula for the finite-dimensional simple $U_{r,t}$-modules. Since

$$V_{\epsilon,n,\alpha} \simeq V_{\epsilon,0,\alpha} \otimes V_{1,n,1},$$

and $V_{1,n,1}$ can view a module over $U_{r,t}/(J-1) \simeq U_q(\mathfrak{sl}(2))$, we get the following lemma by using the quantum Clebsch–Gordan formula for the usual quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$.

LEMMA 5.1. *Let $n \geq m$ be two non-negative integers. There exists an isomorphism of $U_{r,t}$-modules*

$$V_{\epsilon,n,\alpha} \otimes V_{\epsilon',n,\beta} \simeq V_{\epsilon\epsilon',n+m,\alpha\beta} \oplus V_{\epsilon\epsilon',n+m-2,\alpha\beta} \oplus \cdots \oplus V_{\epsilon\epsilon',n-m,\alpha\beta}.$$

Proof. It is obvious that $V_{\epsilon,0,\alpha} \otimes V_{\epsilon',0,\beta} \simeq V_{\epsilon\epsilon',0,\alpha\beta}$. Thus this lemma follows from the above remark. □

In the remainder of this section, we always assume that $n \geq m$ and $\epsilon = \epsilon' = 1$. In the case $\alpha^r = 1$, we can determine the all highest weight vectors of $V_{\epsilon,n,\alpha} \otimes V_{\epsilon',n,\beta}$ in the following lemma.

LEMMA 5.2. *Let $v^{(n)}$ be the highest weight vector of weight $(q^n\alpha^r, \alpha)$ in $V_{1,n,\alpha}$ and $v^{(m)}$ be the highest weight vector of weight $(q^m\beta^r, \beta)$ in $V_{1,m,\beta}$. Let us define $v_p^{(n)} = \frac{1}{[p]!}F^p v^{(n)}$, $v_p^{(m)} = \frac{1}{[p]!}F^p v^{(m)}$, for all $p \geq 0$. Suppose $\alpha^r = 1$. Then*

$$v^{(n+m-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]![n-i]!}{[m-p]![n]!} q^{-i(m-2p+i+1)} \beta^{2rt(n-i)} v_i^{(n)} \otimes v_{p-i}^{(m)}$$

is the highest weight vector of weight $(q^{n+m-2p}\beta^r, \alpha\beta)$.

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Proof. It is clear that $v_i^{(n)} \otimes v_{p-i}^{(m)}$ has weight $(q^{n+m-2p}\beta^r, \alpha\beta)$. Let us check that $Ev^{(n+m-2p)} = 0$. Recall that

$$\Delta(E) = J^{-rt} \otimes E + E \otimes KJ^{rt}.$$

It follows that

$$\begin{aligned}
Ev^{(n+m-2p)} &= \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]![n-i]!}{[m-p]![n]!} q^{-i(m-2p+i+1)}[m-p+i+1] \\
&\quad \times \beta^{2rt(n-i)+r} v_i^{(n)} \otimes v_{p-i-1}^{(m)} \\
&\quad + \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]![n-i]!}{[m-p]![n]!} q^{-i(m-2p+i+1)+(m-2p+2i)}[n-i+1] \\
&\quad \times \beta^{2rt(n-i+1)+r} v_{i-1}^{(n)} \otimes v_{p-i}^{(m)} \\
&= \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]![n-i+1]!}{[m-p]![n]!} q^{-(i-1)(m-2p+i)} (\beta^{2rt(n-i+1)+r} \\
&\quad - \beta^{2rt(n-i+1)+r}) v_i^{(n)} \otimes v_{p-i}^{(m)} \\
&= 0.
\end{aligned}$$

Thus this lemma is true. □

We wish to go one step further and address the following problem. We now have two bases of $V_{1,n,\alpha} \otimes V_{1,m,\beta}$ at our disposal. They are of different natures, the first one, adapted to the tensor product, is the set

$$\{v_i^{(n)} \otimes v_j^{(m)} | 0 \le i \le n, 0 \le j \le m\};$$

the second one, formed by the vectors

$$v_k^{(n+m-2p)} = \frac{1}{[k]!} F^k v^{(n+m-2p)}$$

with $0 \le p \le m$ and $0 \le k \le n+m-2p$, is better adapted to the $U_{r,t}$-module structure. Comparing both bases leads us to the so-called generalized quantum Clebsch–Gordan coefficients $\left\{\begin{smallmatrix} n & m & n+m-2p \\ i & j & k \end{smallmatrix}\right\}$ defined for $0 \le p \le m$, and $0 \le k \le n+m-2p$ by

$$v_k^{(n+m-2p)} = \sum_{0 \le i \le n; 0 \le j \le m} \begin{Bmatrix} n & m & n+m-2p \\ i & j & k \end{Bmatrix} v_i^{(n)} \otimes v_j^{(m)}.$$

In particular,

$$\begin{aligned}
\begin{Bmatrix} n & m & n+m-2p \\ i & j & 0 \end{Bmatrix} &= (-1)^i \frac{[m-p+i]![n-i]!}{[m-p]![n]!} q^{-i(m-2p+i+1)} \beta^{2rt(n-i)} \\
&= \begin{bmatrix} n & m & n+m-2p \\ i & j & 0 \end{bmatrix} \beta^{2rt(n-i)},
\end{aligned}$$

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where $\left[\begin{smallmatrix} n & m & n+m-2p \\ i & j & 0 \end{smallmatrix}\right]$ is the usual quantum Clebsch–Gordan coefficients, is also called quantum $3j$-symbols in the physics literature.

PROPOSITION 5.1. *Fix p and k. The vector $v_k^{(n+m-2p)}$ is a linear combination of vectors of the form $v_i^{(n)} \otimes v_{p-i+k}^{(m)}$. Therefore we have $\left\{\begin{smallmatrix} n & m & n+m-2p \\ i & j & k \end{smallmatrix}\right\} = 0$ when $i+j \neq p+k$. We also have the induction relation*

$$\begin{Bmatrix} n & m & n+m-2p \\ i & j+1 & k+1 \end{Bmatrix} = \frac{[j+1]q^{2i-n}}{[k+1]} \begin{Bmatrix} n & m & n+m-2p \\ i & j & k \end{Bmatrix} + \frac{[i]}{[k+1]} \begin{Bmatrix} n & m & n+m-2p \\ i-1 & j+1 & k \end{Bmatrix} \beta^{-2rt}.$$

Proof. This goes by induction on k. The assertion holds for $k = 0$ by Lemma 5.2. Supposing

$$v_k^{(n+m-2p)} = \sum_i x_i v_i^{(n)} \otimes v_{p-i+k}^{(m)},$$

we have

$$\begin{aligned}
[k+1]v_{k+1}^{(n+m-2p)} &= F v_k^{(n+m-2p)} \\
&= \sum_i x_i (J^{r(t+1)} K^{-1} v_i^{(n)} \otimes F v_{p-i+k}^{(m)} + F v_i^{(n)} \otimes J^{-rt} v_{p-i+k}^{(m)}) \\
&= \sum_i x_i ([p-i+k+1] q^{2i-n} v_i^{(n)} \otimes v_{p-i+k+1}^{(m)} \\
&\quad + [i+1]\beta^{-2rt} v_{i+1}^{(n)} \otimes v_{p-i+k}^{(m)}) \\
&= \sum_i (x_i [p-i+k+1] q^{2i-n} + x_{i-1}[i]\beta^{-2rt}) \\
&\quad \times v_i^{(n)} \otimes v_{p-i+k+1}^{(m)}.
\end{aligned}$$

The rest follows easily. □

We now prove some orthogonality relations for the generalized quantum Clebsch–Gordan coefficients. Let us equip $V_{1,n,\alpha}$ and $V_{1,m,\beta}$ with the scalar product $(,)$ defined in Section 4. Consider the symmetric bilinear form on $V_{1,n,\alpha} \otimes V_{1,m,\beta}$ given by

$$(v_1 \otimes v_1', v_2 \otimes v_2') = (v_1, v_2)(v_1', v_2'),$$

where $v_1, v_2 \in V_{1,n,\alpha}$ and $v_1', v_2' \in V_{1,m,\beta}$.

PROPOSITION 5.2. (*a*) *We have*

$$v_k^{(n+m-2p)} = \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i \frac{[m-p+i]![n-i]![s+i]![p+k-i-s]!}{[m-p]![n]![i]![p-i]!} \times q^{-i(m-2p+i+1)+(k-s)(s+2i-n)} \beta^{2rt(n-i-s)} v_{i+s}^{(n)} \otimes v_{p+k-i-s}^{(m)}.$$

(*b*) $(v_k^{(n+m-2p)}, v_l^{(n+m-2q)}) = 0$ *whenever* $p+k \neq q+l$.

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Proof. Since $\Delta(F) = J^{r(t+1)}K^{-1} \otimes F + F \otimes J^{-rt}$,

$$\Delta(F^k) = \sum_{s=0}^{k} q^{s(k-s)} \begin{bmatrix} k \\ s \end{bmatrix} (J^{r(t+1)(k-s)}F^s K^{-(k-s)} \otimes J^{-rts}F^{k-s}).$$

Hence

$$\begin{aligned} v_k^{(n+m-2p)} &= \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i \begin{bmatrix} k \\ s \end{bmatrix} \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \\ &\quad \times q^{-i(m-2p+i+1)+(k-s)s} \beta^{2rt(n-i)} \\ &\quad \times F^s K^{-(k-s)} v_i^{(n)} \otimes J^{-rts} F^{k-s} v_{p-i}^{(m)} \\ &= \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i \begin{bmatrix} k \\ s \end{bmatrix} \frac{[m-p+i]![n-i]!}{[m-p]![n]!} \times \\ &\quad q^{-i(m-2p+i+1)+(2i-n+s)(k-s)} \beta^{2rt(n-i)-2rts} F^s v_i^{(n)} \otimes F^{k-s} v_{p-i}^{(m)} \\ &= \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i \begin{bmatrix} k \\ s \end{bmatrix} \frac{[m-p+i]![n-i]![i+s]![p+k-i-s]!}{[m-p]![n]![i]![k-s]!} \\ &\quad \times q^{-i(m-2p+i+1)+(2i-n+s)(k-s)} \beta^{2rt(n-i-s)} v_{i+s}^{(n)} \otimes v_{p+k-s-i}^{(m)}. \end{aligned}$$

By Theorem 3.10, $(v_{i+s}^{(n)}, v_{j+u}^{(n)})(v_{p+k-i-s}^{(m)}, v_{q+l-j-u}^{(m)}) = 0$ either $i+s \neq j+u$ or $p+k-i-s \neq q+l-j-u$. If $i+s = j+u$ and $p+k-i-s = q+l-j-u$, then $p+k = q+l$. Hence $(v_k^{(n+m-2p)}, v_l^{(n+m-2q)}) = 0$ whenever $p+k \neq q+l$. □

REMARK 5.3. Similarly to [3], one can study the categorification of tensor products of arbitrary finite-dimensional irreducible modules over the $U_{r,t}$.

6. In the case q is a root of unity. Our main aim is to find all finite-dimensional simple $U_{r,t}$ in the case when the parameter q is a root of unity $\neq \pm 1$. Denote by d the order of q, i.e. the smallest integer greater than 1 such that $q^d = 1$. Since we assume $q^2 \neq 1$, $d > 2$. Define

$$e = \begin{cases} d & \text{if } d \text{ is odd} \\ \frac{d}{2} & \text{when } d \text{ is even.} \end{cases}$$

It is easy to check that $[n] = 0$ if and only if $n \equiv 0 (mod\ e)$.

LEMMA 6.1. *The elements E^e, F^e and K^e belong to the centre of $U_{r,t}$.*

Proof. K^e commutes with E and F because $q^{2e} = 1$. So K^e is in the centre of $U_{r,t}$. Since $[e] = 0$,

$$[E^e, F] = [e] \frac{q^{-(e-1)}K - q^{e-1}K^{-1}J^r}{q - q^{-1}} E^{e-1} = 0.$$

Moreover $KE^eK^{-1} = (KEK^{-1})^e = (q^2E)^e = E$. So E^e belongs to the centre of $U_{r,t}$. Similar arguments can be applied to F^e. □

LEMMA 6.2. *There is no simple finite-dimensional $U_{r,t}$ module of dimension greater than e.*

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Proof. Let us assume that there exists a simple finite-dimensional module greater than e. We shall prove that V has a non-zero submodule of dimension less than or equal to e. Hence, a contradiction.

(a) Suppose there exists a non-zero vector $v \in V$ such that $Kv = \lambda v$, $Jv = \alpha^2 v$ and $Fv = 0$. We claim that the subspace V' spanned by $v, Ev, \ldots, E^{e-1}v$ is a submodule of dimension less than or equal to e. It is enough to check that V' is stable under the action of generators E, F, K, J. This is clear for K, J. Let us prove that V' is stable under the action of E. The vector $E(E^p v) = E^{p-1}v$ belongs to V' if $p < e-1$. If $p = e-1$, then the action of E^e on the irreducible module V is given by a scalar c, as E^e is in the centre of $U_{r,t}$. So $E(E^{e-1})v = cv$ belongs to V'. Finally, V' is stable under the F by $Fv = 0$ and Lemma 2.3.

(b) Suppose there is no common eigenvector v of K and J satisfying $Fv = 0$. We claim that the subspace V' spanned by $v, Fv, \ldots, F^{e-1}v$ is a submodule of V, where v satisfies $Kv = \lambda v, Jv = \alpha^2 v$. Since F^e is in the centre of $U_{r,t}$, $F^e v = cv$ for some $c \in \mathbf{k}$ and $c \neq 0$. Thus V' is stable under the action of F. It is easy to prove that V' is stable under the actions of J, K. Let us show that V' is stable under the action of E. Recall that $C_p = EF + \frac{q^{-1}K+qK^{-1}J^r}{(q+q^{-1})^2} = FE + \frac{qK+q^{-1}K^{-1}J^r}{(q+q^{-1})^2}$ is in the centre of $U_{r,t}$. Hence there exists $a \in \mathbf{k}$ such that $C_p w = aw$ for any vector $w \in V$. Hence $Ev = \frac{1}{c}EF^e v = \frac{1}{c}(C_p - \frac{q^{-1}K+qK^{-1}J^r}{(q+q^{-1})^2})F^{e-1}v = \frac{1}{c}(a - \frac{q\lambda+q-1\lambda^{-1}\alpha^{2^r}}{(q+q^{-1})^2})F^{e-1}v$. For any $p \geq 0$, $EF^{p+1}v = ([p+1]\frac{q^pK+q^{-p}K^{-1}J^r}{q-q^{-1}}F^p + F^{p+1}E)v = (\frac{q^{-p}\lambda+q^p\lambda^{-1}\alpha^{2r}}{q-q^{-1}}[p+1] + a - \frac{q\lambda+q-1\lambda^{-1}\alpha^{2^r}}{(q+q^{-1})^2})F^p v$. From the above computation, we show that V' is stable under the action of E. Hence V' is a submodule of V. □

THEOREM 6.3. *Any non-zero simple finite-dimensional $U_{r,t}$ is isomorphic to a module of the form*

(i) $V_{\epsilon,n,\alpha}$ *with* $0 \leq n < e-1$,

(ii) $V_{\lambda,\alpha,a}$, *where* $V_{\lambda,a}$ *has a basis* $\{v_0, v_1, \ldots, v_{e-1}\}$ *such the action of the generators of* $U_{r.t}$ *given by*

$$Kv_p = q^{2p}v_p,\ 0 \leq p \leq e-1, \tag{6.1}$$
$$Jv_p = \alpha^2 v_p,\ 0 \leq p \leq e-1, \tag{6.2}$$
$$Fv_{p+1} = \frac{q^{-p}\lambda^{-1}\alpha^{2r} - q^p\lambda}{q-q^{-1}}[p+1]v_p,\ 0 \leq p < e-1, \tag{6.3}$$
$$Ev_p = v_{p+1},\ 0 \leq p < e-1, \tag{6.4}$$
$$Fv_0 = 0,\ Ev_{e-1} = av_0, \tag{6.5}$$

(iii) $V_{\lambda,\alpha,a,b}$, *where* $b \neq 0$ *and* $V_{\lambda,\alpha,a,b}$ *has a basis* $\{v_0, v_1, \ldots, v_{e-1}\}$ *such the action of the generators of* $U_{r.t}$ *given by*

$$Kv_p = q^{-2p}v_p,\ 0 \leq p \leq e-1, \tag{6.6}$$
$$Jv_p = \alpha^2 v_p,\ 0 \leq p \leq e-1, \tag{6.7}$$
$$Ev_{p+1} = \left(\frac{q^p\lambda - q^{-p}\lambda^{-1}\alpha^{2r}}{q-q^{-1}}[p+1] + ab\right)v_p,\ 0 \leq p < e-1, \tag{6.8}$$
$$Fv_p = v_{p+1},\ 0 \leq p < e-1, \tag{6.9}$$
$$Fv_{e-1} = bv_0,\ Ev_0 = av_{e-1}, \tag{6.10}$$

Proof. Suppose the simple module V with $dimV < e$. Then we can prove V is isomorphic to $V_{\epsilon,n,\alpha}$, as we have done in the proof of Theorem 3.4.

Suppose the simple module V with $dimV = e$. Then we can obtain that V is isomorphic to either $V_{\lambda,\alpha,a}$, or $V_{\lambda,\alpha,a,b}$ from the proof of Lemma 6.2 □

REMARK 6.4. In Sections 3 and 6, we describe the irreducible representations of $U_{r,t}$. An irreducible representation of the quantum group $U_q(\mathfrak{sl}(2))$ can be realized in terms of the space of functions on some algebraic varieties [2]. We will study the representations of $U_{r,t}$ on some spaces of functions, and establish the relations between the representations of $U_{r,t}$ and hypergeometric series as in refs. [**7**, **10**] in the future paper.

7. Finite-dimensional Hopf algebra. The basic problem in the theory of Hopf algebras is to classify finite-dimensional Hopf algebras (see [8] and references therein). So one need to construct various Hopf algebras. Our main aim in this section is to construct a kind of finite-dimensional Hopf algebras by using the algebra $U_{r,t}$. We assume that the parameter q is a root of unity $\neq \pm 1$. The definitions of e and q were given in Section 6.

LEMMA 7.1. *Let $U' = U_{r,t}/(E^e, F^e)$. Then U' has a basis $\{E^iF^jK^mJ^n | 0 \leq i, j \leq e - 1, m, n \in \mathbf{Z}\}$.*

Proof. From Theorem 2.4, we know that U' is generated by $\{E^iF^jK^mJ^n | 0 \leq i, j \leq e - 1, m, n \in \mathbf{Z}\}$. We only need to prove the elements in $\{E^iF^jK^mJ^n | 0 \leq i, j \leq e - 1, m, n \in \mathbf{Z}\}$ are linearly independent. Suppose

$$Z = \sum_{0\leq i,j\leq e-1, r_1\leq m\leq s_1, r_2\leq n\leq s_2} a_{ijmn} E^iF^jK^mJ^n = 0.$$

Let V be a $U_{r,t}$-module with basis $\{v_0, v_1, \ldots, v_{e-1}\}$ such that $Ev_{e-1} = 0$, $Ev_i = v_{i+1}$ for $0 \leq i < e - 1$, $Fv_{p+1} = \frac{q^{-p}\lambda^{-1}\alpha^{2r} - p^q\lambda}{q - q^{-1}}[p + 1]v_p$ for $0 \leq p < e - 1$, and $Fv_0 = 0$, $Kv_p = q^{2p}\lambda v_p$, $Jv_p = \alpha^2 v_p$, where λ is neither zero nor a root of unity. Then

$$Zv_{e-1} = \sum_{1\leq i\leq e-1, r_1\leq m\leq s_1, r_2\leq n\leq s_2} a_{ie-1mn}\alpha^2 n\lambda^m v_i = 0.$$

Hence

$$\sum_{r_1\leq m\leq s_1} \left(\sum_{r_2\leq n\leq s_2} a_{ie-1mn}\alpha^{2r} \right) \lambda^m = 0, \tag{7.1}$$

for any $0 \leq i \leq e - 1$. Writing (7.1) for $s_1 - r_1 + 1$ distinct elements $\lambda \in \mathbf{k}$, we get a linear system whose determinant is not equal to zero. Consequently,

$$\sum_{r_2\leq n\leq s_2} a_{ie-1mn}\alpha^{2n} = 0, \tag{7.2}$$

for any m. Similarly we can prove $a_{ie-1mn} = 0$ for any n from (7.2).

Next, we apply Z to the vector v_{e-2}. We get $a_{ie-2mn} = 0$ for all i, m, n by the same argument as above. Applying Z successively to the vector v_{e-2} down to v_0, one shows that all coefficients a_{ijmn} vanish. □

LEMMA 7.2. *Let $U'' = U_{r,t}/(E^e, F^e, K^e - 1)$. Then U'' has a basis $\{E^iF^jK^mJ^n | 0 \leq i, j, m \leq e-1, n \in \mathbf{Z}\}$.*

Proof. We use $d(Z)$ (resp. $\delta(Z)$) to denote the degree in K (resp. K^{-1}) of the element $Z \in U'$. It is clear that the set $\{E^iF^jK^mJ^n | 0 \leq i, j, m \leq e-1, n \in \mathbf{Z}\}$ span the algebra U''. It remains to check that they are linearly independent. If

$$Z = \sum_{0 \leq i,j,m \leq e-1, r_1 \leq n \leq s_1} a_{ijmn} E^iF^jK^mJ^n = 0$$

in U'', then in U'

$$\begin{aligned} Z &= (K^e - 1)Y \\ &= \sum_{0 \leq i,j \leq e-1, m,n \in \mathbf{Z}} b_{ijmn} E^iF^jK^{m+e}J^n \\ &\quad - \sum_{0 \leq i,j \leq e-1, m,n \in \mathbf{Z}} b_{ijmn} E^iF^jK^mJ^n, \end{aligned} \tag{7.3}$$

where $Y = \sum_{0 \leq i,j \leq e-1, m,n \in \mathbf{Z}} b_{ijmn} E^iF^jK^mJ^n$. Since

$$Z = \sum_{0 \leq i,j,m \leq e-1, r_1 \leq n \leq s_1} a_{ijmn} E^iF^jK^mJ^n,$$

$0 \leq \delta(Z) \leq d(Z) < e$. From (7.3) we obtain $d(Z) = d(Y) + e$ and $\delta(Z) = \delta(Y)$. Thus $d(Y) = d(Z) - e < 0 \leq \delta(Z) = \delta(Y)$. This is impossible, hence $Z = 0$ in U'. Therefore all coefficients a_{ijmn} vanish. □

LEMMA 7.3. *Let $U_{r,t,l} = U_{r,t}/(E^e, F^e, K^e - 1, J^l - 1)$. Then $U_{r,t,l}$ has a basis $\{E^iF^jK^mJ^n | 0 \leq i, j, m \leq e-1, 0 \leq n \leq l-1\}$.*

Proof. The proof is similar to that of Lemma 7.2. □

THEOREM 7.4. *Let $U_{r,t,l} = U_{r,t}/(E^e, F^e, K^e - 1, J^l - 1)$. Then $U_{r,t,l}$ has a unique Hopf algebra structure such that the canonical projection from $U_{r,t}$ to $U_{r,t,l}$ is a morphism of Hopf algebras. Moreover the dimension of $U_{r,t,l}$ is equal to le^3.*

Proof. We only need to check that

$$\Delta(E^e) = \Delta(F^e) = \Delta(K^e) - 1 = \Delta(J^l) - 1 = 0, \tag{7.4}$$

$$\varepsilon(E^e) = \varepsilon(F^e) = \varepsilon(K^e - 1) = \varepsilon(J^l - 1) = 0, \tag{7.5}$$

$$S(E^e) = S(F^e) = S(K^e) - 1 = S(J^l) - 1 = 0. \tag{7.6}$$

The only non-trivial computations concern the vanishing $\Delta(E^e) = \Delta(F^e) = 0$. Following Proposition 2.3,

$$\Delta(E^e) = \sum_{u=0}^{e} q^{u(e-u)} \begin{bmatrix} e \\ u \end{bmatrix} (J^{-rtu}E^{e-u} \otimes J^{rt(e-u)}E^u).$$

Since $\left[{e \atop u}\right] = 0$ for $0 < u < e$, $\Delta(E^e) = E^e \otimes J^{rte} + J^{-rte} \otimes E^e$. Thus $\Delta(E^e) = 0$ as $E^e = 0$. One can prove that $\Delta(F^e) = 0$ in a similar way.

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By Lemma 7.3, we obtain a Hopf algebra $U_{r,t,l}$ with dimension le^3. □

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REFERENCES

1. V. G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, *Sov. Math. Dokl.* **32** (1985), 254–258.

2. I. Frankel, M. Khovanov and C. Stroppel, A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products, *Selecta Math.* **12**(3–4) (2006), 379–431.

3. M. Jimbo, A q-difference analogue of $U(\mathcal{G})$ and the Yang–Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.

4. V. G. Kac, *Infinite dimensional Lie algebras*, 3rd edition (Cambridge University Press, Cambridge, 1990).

5. G. Lusztig, *Introduction to quantum group* (Birkhäuser, Boston, 1993).

6. M. Rosso, Finite-dimensional representations of the quantum analog of enveloping algebra of a complex simple Lie algebra, *Comm. Math. Phys.* **117** (1998), 581–593.

7. A. Savage, The tensor product of representations of $U_q(\mathfrak{sl}_2)$ via quivers, *Adv. Math.* **177**(2) (2003), 297–340.

8. Ng. Siu-Hung, Hopf algebras of dimension *pq*, *J. Algebra* **319**(7) (2008), 2772–2788.

9. T. Tanisaki, Harish-Chandra isomorphisms for quantum algebras, *Comm. Math. Phys.* **127** (1990), 555–571.

10. Y. Vivek and Y. Savasvati, On the models of certain p, q-algebra representations: The p, q-oscillator algebra, *J. Math. Phys.* **49**(5) (2008), 053504, 1–12.

11. Zhixiang Wu, A class of weak Hopf algebras related to a Borcherds-Cartan Matrix, *J. Phys. A Math. gen.* **39** (2006), 14611–14626.