

THE SOLUTION OF DUAL COSINE SERIES BY THE USE OF ORTHOGONALITY RELATIONS

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1. Introduction

In this paper expressions are derived for the unknown coefficients in the dual cosine series:

$$\frac{1}{2}a_0 \cos \frac{x}{2} + \sum_{n=1}^{\infty} a_n \cos (n + \frac{1}{2})x = f_1(x), \quad 0 < x < c, \quad (1)$$

$$\frac{1}{2}\alpha a_0 \cos \frac{x}{2} + \sum_{n=1}^{\infty} a_n \frac{n(n+1)}{2n+1} \cos (n + \frac{1}{2})x = f_2(x), \quad c < x < \pi, \quad (2)$$

where α is a given constant, and $f_1(x)$ and $f_2(x)$ are prescribed functions.

The method is an extension of that of a previous paper, see Noble and Whiteman (5), and is a generalization of the method of multiplying-factors described by Noble (4), Section 3, or the equivalent fractional integration technique of Erdélyi and Sneddon (2). As far as we know, the solution of the series (1) and (2) is completely new. In the analysis that follows, which is purely formal, two orthogonality relations for Legendre polynomials are needed. These and other required results are stated in Section 3.

2. Technique for solution

The method outlined here will be used directly in Section 3 to find the coefficients a_n in (1) and (2). As it can also be used with slight modification to solve some well-known dual series, the details are given in general terms. The dual series (1) and (2) have the form:

$$\sum_{n=0}^{\infty} a_n \alpha(n) \phi_n(x) = f_1(x), \quad 0 < x < c, \quad (3)$$

$$\sum_{n=0}^{\infty} a_n \beta(n) \phi_n(x) = f_2(x), \quad c < x < \pi. \quad (4)$$

In order to find the a_n we multiply (3) by $p(\theta, x)$ and (4) by $q(\theta, x)$, where the forms of p and q will be specified later, and integrate the respective results from 0 to θ ($0 \leq \theta \leq c$) and θ to π ($c \leq \theta \leq \pi$) with respect to x . After change in the order of summation and integration, the resulting expressions can be

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written as

$$\sum_{n=0}^{\infty} a_n \alpha(n) \xi_n(\theta) = \int_0^\theta p(\theta, x) f_1(x) dx, \tag{5}$$

$$\sum_{n=0}^{\infty} a_n \beta(n) \eta_n(\theta) = \int_\theta^\pi q(\theta, x) f_2(x) dx, \tag{6}$$

where

$$\xi_n(\theta) = \int_0^\theta p(\theta, x) \phi_n(x) dx$$

and

$$\eta_n(\theta) = \int_\theta^\pi q(\theta, x) \phi_n(x) dx$$

and the functions $\xi_n(\theta)$ and $\eta_n(\theta)$ are connected by the relations

$$\frac{d\xi_n(\theta)}{d\theta} = \frac{n(n+1)}{2n+1} \frac{\eta_n(\theta)}{\sin \theta} \tag{7}$$

and

$$\frac{d\eta_n(\theta)}{d\theta} = -(2n+1) \sin \theta \xi_n(\theta). \tag{8}$$

Equations (3) and (4) contain a_0 terms which do not have the form of the remainder of the series. These are assimilated in the right-hand sides of the equations, and we write

$$\check{f}_1(x) = f_1(x) - \frac{1}{2} a_0 \cos \frac{1}{2} x.$$

and

$$\check{f}_2(x) = f_2(x) - \frac{1}{2} a_0 \cos \frac{1}{2} x.$$

Equation (5) is now multiplied by $\xi_m(\theta) \sin \theta$ and the result is integrated from 0 to c with respect to θ . Assuming that $\xi_n(0) = 0$, so that the end contributions vanish, and that (5) holds for $\theta = c$, we have

$$\sum_{n=1}^{\infty} a_n \beta(n) \int_0^c \eta_n(\theta) \eta_m(\theta) \frac{d\theta}{\sin \theta} = \int_0^c \eta_m(\theta) d\theta \frac{d}{d\theta} \int_0^\theta p(\theta, x) \check{f}_1(x) dx. \tag{9}$$

In the right-hand side of (9) the inner integral involving the a_0 term is a constant, so that the $\check{f}_1(x)$ here may be written as $f_1(x)$. It follows immediately from (6) that

$$\sum_{n=1}^{\infty} a_n \beta(n) \int_c^\pi \eta_n(\theta) \eta_m(\theta) \frac{d\theta}{\sin \theta} = \int_c^\pi \eta_m(\theta) \frac{d\theta}{\sin \theta} \int_\theta^\pi q(\theta, x) \check{f}_2(x) dx. \tag{10}$$

Equations (9) and (10) are combined, and the orthogonality relation

$$\int_0^\pi \eta_n(\theta) \eta_m(\theta) \frac{d\theta}{\sin \theta} = \frac{2\delta_{mn}}{\beta(n)}, \tag{11}$$

where δ_{mn} is the Kronecker delta, is used so that

$$a_n = \frac{1}{2} \int_0^c \eta_n(\theta) d\theta \frac{d}{d\theta} \int_0^\theta p(\theta, x) f_1(x) dx + \frac{1}{2} \int_c^\pi \eta_n(\theta) \frac{d\theta}{\sin \theta} \int_\theta^\pi q(\theta, x) \check{f}_2(x) dx. \tag{12}$$

3. Mathematical formulae

Two well-known relations for Legendre polynomials and their derivatives are

$$P_n(\cos \theta) = - \frac{1}{(2n+1) \sin \theta} \frac{d}{d\theta} [P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta)] \tag{13}$$

and

$$P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta) = \frac{2n+1}{n(n+1)} \sin \theta \frac{d}{d\theta} P_n(\cos \theta), \tag{14}$$

(see Magnus and Oberhettinger (3)). When the orthogonality relation

$$\int_0^\pi P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2\delta_{mn}}{2n+1} \tag{15}$$

is integrated by parts, the use of (13) and (14) gives

$$\begin{aligned} \int_0^\pi [P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta)][P_{m+1}(\cos \theta) - P_{m-1}(\cos \theta)] \frac{d\theta}{\sin \theta} \\ = \frac{2(2n+1)}{n(n+1)} \delta_{mn}. \end{aligned} \tag{16}$$

Mehler's integrals for Legendre polynomials are

$$P_n(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos(n+\frac{1}{2})x dx}{(\cos x - \cos \theta)^{\frac{1}{2}}} = \frac{\sqrt{2}}{\pi} \int_\theta^\pi \frac{\sin(n+\frac{1}{2})x dx}{(\cos \theta - \cos x)^{\frac{1}{2}}} \tag{17}$$

(Magnus and Oberhettinger (3)), and from the second of these we obtain

$$P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta) = \frac{2\sqrt{2}}{\pi} \int_\theta^\pi \frac{\cos(n+\frac{1}{2})x \sin x dx}{(\cos \theta - \cos x)^{\frac{1}{2}}}. \tag{18}$$

4. Solution of the series

The coefficients a_n in the dual series

$$\frac{1}{2}a_0 \cos \frac{x}{2} + \sum_{n=1}^\infty a_n \cos(n+\frac{1}{2})x = f_1(x), \quad 0 < x < c, \tag{19}$$

$$\frac{1}{2}\alpha a_0 \cos \frac{x}{2} + \sum_{n=1}^\infty a_n \frac{n(n+1)}{2n+1} \cos(n+\frac{1}{2})x = f_2(x), \quad c < x < \pi, \tag{20}$$

are found using the technique outlined in Section 2. Referring back to that method, we choose

$$p(\theta, x) = \frac{\sqrt{2}}{\pi(\cos x - \cos \theta)^{\frac{1}{2}}}, \tag{21}$$

and

$$q(\theta, x) = \frac{2\sqrt{2} \sin x}{\pi(\cos \theta - \cos x)^{\frac{1}{2}}}, \tag{22}$$

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so that from (17) and (18)

$$\xi_n(\theta) = P_n(\cos \theta) \tag{23}$$

and

$$\eta_n(\theta) = [P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta)]. \tag{24}$$

The orthogonality relation (11) is thus identical with (16), so that from (12)

$$a_n = \frac{1}{\sqrt{2\pi}} \int_0^c [P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta)] d\theta \cdot \frac{d}{d\theta} \int_0^\theta \frac{f_1(x) dx}{(\cos x - \cos \theta)^{\frac{1}{2}}} + \frac{\sqrt{2}}{\pi} \int_c^\pi [P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta)] \frac{d\theta}{\sin \theta} \int_\theta^\pi \frac{(f_2(x) - \frac{1}{2}\alpha a_0 \cos \frac{1}{2}x) \sin x dx}{(\cos \theta - \cos x)^{\frac{1}{2}}} \tag{25}$$

for $n = 1, 2, \dots$

The remaining coefficient α_0 is found by substituting (25) for a_n in (19), and then using the relation

$$\sum_{n=1}^\infty [P_{n+1}(\cos \theta) - P_{n-1}(\cos \theta)] \cos(n + \frac{1}{2})x = \frac{\sqrt{2}H(x-\theta) \sin x}{(\cos \theta - \cos x)^{\frac{1}{2}}} - (1 + \cos \theta) \cos \frac{1}{2}x,$$

$H(x-\theta)$ being the Heaviside unit function (see Sneddon (6)). After some simplification we obtain

$$a_0 = \frac{1}{K} \left\{ \frac{1}{\sqrt{2\pi}} \int_0^c (1 + \cos \theta) d\theta \cdot \frac{d}{d\theta} \int_0^\theta \frac{f_1(x) dx}{(\cos x - \cos \theta)^{\frac{1}{2}}} + \frac{\sqrt{2}}{\pi} \int_c^\pi \cot \frac{1}{2}\theta d\theta \int_\theta^\pi \frac{f_2(x) dx}{(\cos \theta - \cos x)^{\frac{1}{2}}} \right\},$$

where $K = \frac{1}{2}(1 - \alpha \cos^2 \frac{1}{2}c - 2\alpha \log \sin \frac{1}{2}c)$.

The method of Section 2 can be used to solve the much-studied series

$$\sum_{n=0}^\infty a_n(2n+1)^p \cos(n + \frac{1}{2})x = f_1(x), \quad 0 < x < c, \tag{26}$$

$$\sum_{n=0}^\infty a_n \cos(n + \frac{1}{2})x = f_2(x), \quad c < x < \pi, \tag{27}$$

where $p = \pm 1$. These series are those of equations (3) and (4) when

$$\alpha(n) = (2n+1)^p$$

and

$$\beta(n) = 1.$$

For the case $p = 1$ our method gives the result of Srivastav (7). When in addition $f_2(x) = 0$, it gives that of Tranter (8). For the case $p = -1$ we obtain an expression for a_n , which for $f_2(x) = 0$ is that of Tranter (8). However, for general $f_1(x)$ and $f_2(x)$ we have been unable to transform this into that given by Srivastav (7), which was later corrected by Cooke (1).

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