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NILPOTENT ACTION BY AN AMENABLE GROUP AND EULER CHARACTERISTIC

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We prove two types of vanishing results for the Euler characteristic.

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1. Introduction

Let X be a finite connected simplicial complex, $\Gamma = \pi_1(X)$ its fundamental group, $\tilde{X}$ its universal covering space. Then Γ acts freely on $\tilde{X}$ as simplicial automorphisms and on the cohomology group $H^*(\tilde{X})$. In this note we establish the following vanishing results for the Euler characteristic $\chi(X)$ of X.

Theorem 1.1. *If $\Gamma = \pi_1(X)$ is an amenable group and Γ contains an infinite normal subgroup A which acts nilpotently on $H^*(\tilde{X})$, then the reduced ℓ_2-cohomology spaces $\overline{H}^*_{(2)}(\tilde{X}:\Gamma)$ are trivial. In particular, the Euler characteristic $\chi(X)$ of X vanishes.*

Theorem 1.2. *If $\Gamma = \pi_1(X)$ acts nilpotently on $H^*(\tilde{X})$ and contains a normal subgroup A such that the quotient group Γ/A is infinite amenable and A is Γ-nilpotent, then the Euler characteristic $\chi(X)$ of X vanishes.*

A discrete group G is called *amenable* if it admits a left invariant mean for $\ell_\infty(G)$, i.e., if there exists a functional $m : \ell_\infty(G) \to \mathbb{R}$ satisfying $m(\chi_G) = 1$ and $m(\phi x) = m(\phi)$ for all $x \in G$ and $\phi \in \ell_\infty(G)$. For example finite, Abelian, and solvable groups are amenable groups. A group containing a non-Abelian free subgroup is not amenable. A left invariant mean for a finite group G is obtained by letting $m(\phi) = \frac{1}{|G|}\sum_{x \in G} \phi(x)$. For further details on amenable groups we refer to **[9]**.

If Γ is infinite amenable and X is aspherical, then Cheeger and Gromov **[2]** and Eckmann **[5]** showed that $\chi(X) = 0$. If Γ contains a nontrivial torsion-free Abelian normal subgroup which acts nilpotently on $H^*(\tilde{X})$, then Eckmann **[4]** showed that

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$\chi(X)=0$. If Γ is a torsion-free elementary amenable group which acts nilpotently on $H^*(\tilde{X})$, then Lee and Park [8] showed that $\chi(X)=0$. If X is aspherical, then any subgroup of Γ acts nilpotently on $H^*(\tilde{X})$. Hence Theorem 1.1 generalizes the results of Cheeger and Gromov [2] and Eckmann [5]. As elementary amenable groups are amenable, Theorem 1.1 also generalizes the result of Lee and Park [8]. If Γ has finite virtual cohomological dimension and contains a nontrivial torsion-free elementary amenable normal subgroup which acts nilpotently on $H^*(\tilde{X})$, then $\chi(X)=0$. In fact, Hillman and Linnell [6] showed that any nontrivial torsion-free elementary amenable group of finite virtual cohomological dimension contains a nontrivial Abelian characteristic subgroup. Applying Eckmann's result [4] yields $\chi(X)=0$. Note that it is not known whether $X=\tilde{X}/\Gamma$ being compact implies that Γ has finite virtual cohomological dimension.

The proof of Theorem 1.1 is based on results concerning the von Neumann dimension of simplicial ℓ_2-cohomolgy spaces. Theorem 1.2 is another type of vanishing result for the Euler characteristic $\chi(X)$ of X.

2. Simplicial ℓ_2-cohomolgy

Let G be a countable group and let $\ell_2(G)$ denote the Hilbert space of real valued square summable functions on G. A pre-Hilbert space P is called a *Hilbert G-module* if:

(i) G acts on P by isometries, and

(ii) P is G-equivariantly isometric to a subspace of the tensor product $\ell_2(G)\otimes H$ of the Hilbert space $\ell_2(G)$ and some Hilbert space H with trivial G-action.

To such a P, following von Neumann and Atiyah (see [1] and [3]), one can attach a nonnegative extended real number, $0\le \dim_G P\le\infty$, called the *von Neumann dimension* of P, which is independent of the particular identification with a subspace of $\ell_2(G)\otimes H$ (See Remark 2.3). If $P\neq 0$, then $\dim_G P>0$. Moreover, the von Neumann dimension of a pre-Hilbert space is equal to that of its completion. As usual,

$$\dim_G(P_1\oplus P_2)=\dim_G P_1+\dim_G P_2.$$

For further background on Hilbert G-modules we refer the reader to [1, 2, 3].

Let G be a countable group and Y a connected simplicial complex on which G acts freely and simplicially. Denote by $Y_{(n)}$ the set of all n-simplices of Y. Define $C^n_{(2)}(Y)=\{c\in C^n(Y,\mathbb{R})\mid \sum_{s\in Y_{(n)}}c(s)^2<\infty\}$ and call it the *space of ℓ_2-cochains*. Then $C^n_{(2)}(Y)\cong \ell_2(G)\otimes H_n$ where H_n is a Hilbert space having a set S_n of representatives of $Y_{(n)}$ mod G as a basis. Hence $C^n_{(2)}(Y)$ is a *free* Hilbert G-module and $\dim_G C^n_{(2)}(Y)=$ cardinality $|S_n|$ of S_n. It is clear that the differentials $\delta^n: C^n_{(2)}(Y)\to C^{n+1}_{(2)}(Y)$ commute with the G-action. We define the simplicial *ℓ_2-cohomolgy spaces* by

$$H^n_{(2)}(Y:G)=\operatorname{Ker}\delta^n/\operatorname{Im}\delta^{n-1},$$

and we define the *(reduced) simplicial ℓ_2-cohomology spaces* by

$$\overline{H}^n_{(2)}(Y:G) = \operatorname{Ker}\delta^n / \overline{\operatorname{Im}\delta^{n-1}}.$$

Note that $C^n_{(2)}(Y) \supset \operatorname{Ker}\delta^n \cong \overline{\operatorname{Im}\delta_{n-1}} \oplus \overline{H}^n_{(2)}(Y:G)$, and hence $\operatorname{Ker}\delta^n$, $\operatorname{Im}\delta^{n-1}$, and $\operatorname{Im}\delta^{n-1}$ are Hilbert G-modules. In particular $\overline{H}^n_{(2)}(Y:G)$ acquires the structure of a Hilbert G-module and hence its von Neumann dimension is defined, denoted by $h^n(Y:G)$, and called the *nth ℓ_2-Betti number*. Moreover there is a natural G-equivariant map [2]

$$\rho : \overline{H}^*_{(2)}(Y:G) \to H^*(Y, \mathbb{R}).$$

Remark 2.1. If Y is a connected simplicial complex on which G acts freely and simplicially so that the quotient Y/G is compact, then

$$\begin{aligned}\chi(Y/G) &= \sum(-1)^n|S_n| = \sum(-1)^n \dim_G C^n_{(2)}(Y)\\ &= \sum(-1)^n \dim_G \overline{H}^n(Y:G) = \sum(-1)^n h^n(Y:G).\end{aligned}$$

The first equality follows from the fact that Y/G is a finite complex and the third equality follows from the fact that the cochain complex $\{C^*_{(2)}(Y)\}$ of Hilbert G-modules is finite.

Proposition 2.2. *For an infinite subgroup A of G, any Hilbert G-module with trivial A-action is the zero module.*

Proof. Let P be a Hilbert G-module with trivial A-action and a G-equivariant embedding $P \hookrightarrow \ell_2(G) \otimes H$. We may assume that P is a Hilbert space. Let $\{h_i\}$ be a Hilbert basis of H and let $p_i : \ell_2(G) \otimes H \to \ell_2(G)$ be the projection $1 \otimes rh_i \mapsto r \cdot 1$. With $P_0 = P$, we define inductively P_{i+1} and I_{i+1} to be the kernel and the closure of the image, respectively, of $p_{i+1} \circ j_i : P_i \hookrightarrow \ell_2(G) \otimes H \to \ell_2(G)$. I.e., $P_i = \ker p_1 \cap \ldots \cap \ker p_i \cap P$ and then I_{i+1} is the closure of the image of P_i in $\ell_2(G)$. Then $P = \sum I_i$ and I_i is a Hilbert G-module with a G-equivariant embedding $I_i \hookrightarrow \ell_2(G)$ ([3]). Since $p_{i+1} \circ j_i$ is G- and so A-equivariant, the A-action on I_i is trivial.

Note that $\ell_2(G) = \ell_2(A) \otimes \mathcal{H}$ where $\mathcal{H}$ is the Hilbert space having G/A as its Hilbert basis. By the same argument as above each Hilbert G-module I_i has a decomposition $I_i = \sum J_{i_j}$ by Hilbert A-modules such that $J_{i_j} \subset \ell_2(A)$ with trivial A-action. Now it suffices to show that each $J_{i_j} = 0$.

Let $J \subset \ell_2(A)$ with trivial A-action. Every element of J is of the form $\sum_{x\in A} a_x x$ where $\sum_{x\in A} |a_x|^2 < \infty$. If $a_x \neq 0$ for some $x \in A$, then because of the trivial action by A $a_1 = a_x \neq 0$. For any $y \in A$, $a_y = a_1 \neq 0$. Hence $\sum_{x\in A} a_x x = \sum_{x\in A} a_1 x$, so $\sum_{x\in A} |a_x|^2 = \sum_{x\in A} |a_1|^2 = \infty$. This implies $\sum_{x\in A} a_x x = 0$. Hence $J = 0$. □

Remark 2.3. As in the proof of Proposition 2.2, any Hilbert G-module P which is a Hilbert space is isomorphic to $\sum I_i$ where $I_i \hookrightarrow \ell_2(G)$. Write $1 = e_i + (1 - e_i)$ where $e_i \in I_i$ and $1 - e_i \in I_i^\perp$. Then $e_i = \sum_{x\in G}\langle e_i, x\rangle x$ where $\langle\,,\rangle$ is the inner product on $\ell_2(G)$. The trace of e_i, $\langle e_i, 1_G\rangle$, i.e., the coefficient of the identity 1_G of G, is the von Neumann dimension of I_i. The *von Neumann dimension* of P is then $\dim_G P = \sum \dim_G I_i$.

3. Nilpotent modules

Definition 3.1. Let A be a subgroup of G and let M be a $\mathbb{Z}G$-module. Then we say that A acts *nilpotently* on M if there exists a finite filtration $0 = M^{(0)} \subset M^{(1)} \subset \ldots \subset M^{(k-1)} \subset M^{(k)} = M$ by $\mathbb{Z}A$-modules such that A acts trivially on the associated graded module $\operatorname{Gr} M = \{M^{(i)}/M^{(i-1)} \mid i = 1, \ldots, k\}$.

Remark 3.2. The $M^{(i)}$ in Definition 3.1 can be chosen such that $M^{(i)}/M^{(i-1)}$ consists of all elements of $M/M^{(i-1)}$ fixed under the action of A.

Proposition 3.3 [4, Proposition 1.1]. *Let M be a $\mathbb{Z}G$-module. Suppose a subgroup A of G acts nilpotently on M so that a filtration $\{M^{(i)} \mid i = 0, \ldots, k\}$ of M is chosen as in Remark 3.2. If A is a normal subgroup of G, then the $M^{(i)}$ are $\mathbb{Z}G$-submodules of M.*

Proof. This is trivial for $i = 0$, and we assume that it holds for $i - 1$ $(i = 1, 2, \ldots, k)$. For any $h \in M^{(i)}$, $a \in A$, and $x \in G$, as A is normal in G we have $x^{-1}ax \in A$, and as A acts trivially on $M^{(i)}/M^{(i-1)}$ we have $axh = x(x^{-1}ax)h = x(h + h')$ with $h' \in M^{(i-1)}$. Since $xh' \in M^{(i-1)}$, $axh = xh + h''$ with $h'' \in M^{(i-1)}$. Thus $a \in A$ fixes the element $xh + M^{(i-1)}$ in $M/M^{(i-1)}$, and hence $xh \in M^{(i)}$. □

Theorem 3.4. *Let G be a countable group and let Y be a connected simplicial complex on which G acts freely and simplicially so that the quotient Y/G is compact. If G contains an infinite normal subgroup A which acts nilpotently on $H^*(Y)$, then the natural G-equivariant map $\rho : \overline{H}^*_{(2)}(Y : G) \to H^*(Y, \mathbb{R})$ is trivial.*

Proof. Let K be the kernel of ρ, and let $\overline{M} = \overline{H}^*_{(2)}(Y : G)$ and $M = H^*(Y, \mathbb{R})$. Take a filtration $\{M^{(i)}\}_{i=0}^k$ of M given by the nilpotent action of A on M as in Remark 3.2. By Proposition 3.3, the $M^{(i)}$ are $\mathbb{R}G$-modules. Let $\overline{M}^{(i)} = \rho^{-1}(M^{(i)})$ for $i = 0, 1, \ldots, k$. Then we have exact sequences $0 \to K \to \overline{M}^{(i)} \to M^{(i)}$, and $\overline{M}^{(i)}/\overline{M}^{(i-1)} \cong (\overline{M}^{(i)}/K)/(\overline{M}^{(i-1)}/K) \hookrightarrow M^{(i)}/M^{(i-1)}$ so A acts trivially on $\overline{M}^{(i)}/\overline{M}^{(i-1)}$; by assuming that each $\overline{M}^{(i)}$ is a Hilbert space, if it is necessary, we obtain a decomposition of $\overline{M}$ by $\ell_2(G)$-modules:

$$\overline{M} = \overline{M}^{(k)} \oplus [\overline{M}^{(k)}]^\perp = \cdots = K \oplus K^\perp \oplus [\overline{M}^{(1)}]^\perp \oplus \cdots \oplus [\overline{M}^{(k)}]^\perp,$$

where A acts trivially on the factors $K^\perp$, $[\overline{M}^{(1)}]^\perp, \cdots,$ and $[\overline{M}^{(k)}]^\perp$. By Proposition 2.2, $\overline{M} = K$. Hence ρ is a trivial map. □

Proof of Theorem 1.1. Let $Y = \tilde{X}$ and $G = \pi_1(X)$. Since G is an infinite amenable group and Y/G is a finite complex, by Lemma 3.1 of [**2**] the natural G-equivariant map $\rho : \overline{H}^*_{(2)}(Y:G) \to H^*(Y, \mathbb{R})$ is injective. On the other hand, by Theorem 3.4, ρ is the trivial map. this implies $\overline{H}^*_{(2)}(Y:G) = 0$ and in particular $\chi(X) = 0$. □

Corollary 3.5. *If* $\Gamma = \pi_1(X)$ *is an infinite amenable group and if* $\tilde{X}$ *is homotopic to an even dimensional sphere* S^{2k}, *then* $\chi(X) = 0$. *If, in addition,* Γ *has finite virtual cohomological dimension* vcd(Γ), ∞, *then the rational Euler characteristic* $\chi(\Gamma)$ *of* Γ *vanishes.*

Proof. Since $H^{2k}(\tilde{X}) = \mathbb{Z}$, the kernel Γ' of the induced action homomorphism $\Gamma \to \mathrm{Aut}(H_{2k}(\tilde{X})) = \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ has index at most 2 in Γ and acts trivially, and hence nilpotently, on $H^*(\tilde{X})$. By Theorem 1.1 $\chi(\tilde{X}/\Gamma') = 0$. Thus $\chi(X) = 0$. If $\mathrm{vcd}(\Gamma) < \infty$, then $\chi(\Gamma)$ is defined and $\chi(X) = \chi(\Gamma) \cdot \chi(\tilde{X})$ (See [**7, 8**]). Hence $\chi(\Gamma) = 0$. □

4. Proof of Theorem 1.2

Let Π be a group and let G be a Π-group, i.e., a group with ψ-action $\psi : \Pi \to \mathrm{Aut}(G)$. If G is a normal subgroup of Π we take $\psi(x)g = x \cdot g \cdot x^{-1}$. By $\Pi_2 G$ we mean the normal Π-subgroup of G generated by all elements of the form $(\psi(x)g) \cdot g^{-1}$, where $x \in \Pi$ and $g \in G$. Inductively we define $\Pi_n G = \Pi_2(\Pi_{n-1} G)$. The Π-group G is called Π-*nilpotent* if $\Pi_n G = 0$ for some n. A nilpotent group G is a G-nilpotent group.

Let $\tilde{X}_A$ denote the covering space of X corresponding to the normal subgroup A of $\Gamma = \pi_1(X)$. Then Γ/A acts on $\tilde{X}_A$ freely and simplicially with quotient X, and hence Γ acts on $\tilde{X}_A$ by composition with the quotient map $\Gamma \to \Gamma/A$. Consider the cohomology spectral sequence corresponding to the fibration $\tilde{X} \to \tilde{X}_A \to K(A, 1)$;

$$E_2^{p,q} = H^p(A; H^q(\tilde{X})) \Rightarrow H^{p+q}(\tilde{X}_A).$$

We will first show that Γ acts nilpotently on $E_2^{p,q} = H^p(A; H^q(\tilde{X}))$ and hence on $H^*(\tilde{X}_A)$.

Given an element $\alpha \in \Gamma$, let $h : \tilde{X}_A \to \tilde{X}_A$ be the associated deck transformation. This h is not necessarily base point preserving, but it can be homotoped to a map h' which preserves base point so that $h'_* : \pi_1(\tilde{X}_A) \to \pi_1(\tilde{X}_A)$ is conjugation by α. Then h' can be lifted to a map $h'' : \tilde{X} \to \tilde{X}$ which preserves base point and is freely homotopic to the deck transformation of $\tilde{X}$ associated with α. Also there is an associated based map $h' : K(A, 1) \to K(A, 1)$ so that $h'_* : \pi_1(K(A, 1)) = A \to \pi_1(K(A, 1)) = A$ is conjugation by α. This is how Γ acts on the fibration

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & \tilde{X}_A & \longrightarrow & K(A,1) \\ \downarrow h'' & & \downarrow h' & & \downarrow h' \\ \tilde{X} & \longrightarrow & \tilde{X}_A & \longrightarrow & K(A,1) \end{array}$$

each square commuting up to based homotopy, in such a way that the induced actions by Γ on $H^*(\tilde{X})$ and $H^*(\tilde{X}_A)$ are the natural actions. Hence Γ acts on the E_2 term of the spectral sequence corresponding to the fibration $\tilde{X} \to \tilde{X}_A \to K(A,1)$ and the boundary maps d_r are Γ-module maps. Since A is Γ-nilpotent, Γ acts nilpotently on $H^p(A;T)$ for any trivial A-module T. Since Γ acts nilpotently on $H^q(\tilde{X})$, we take a finite filtration $0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(k)} = H^q(\tilde{X})$ by Γ-submodules so that Γ acts trivially on $\{M^{(i)}/M^{(i-1)}\}_{i=1}^k$. In the long cohomology exact sequence of A associated with the exact sequence of coefficient modules $0 \to M^{(1)} \to M^{(2)} \to M^{(2)}/M^{(1)} \to 0$, Γ acts nilpotently on $H^p(A;M^{(1)})$ and $H^p(A;M^{(2)}/M^{(1)})$. Hence Γ acts nilpotently on $H^p(A;M^{(2)})$. By induction, Γ acts nilpotently on $E_2^{p,q} = H^p(A;H^q(\tilde{X}))$, and hence on the abutment $H^{p+q}(\tilde{X}_A)$ of the sequence.

In all, we have shown that the infinite amenable group Γ/A acts freely and simplicially on $\tilde{X}_A$ with compact quotient X and acts nilpotently on $H^*(\tilde{X}_A)$. By Theorem 3.4, the reduced ℓ_2-cohomology spaces $\overline{H}^*_{(2)}(\tilde{X}_A;\Gamma/A)$ are trivial and hence $\chi(X) = \chi(\tilde{X}_A/(\Gamma/A)) = 0$.

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