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ON THE PROPERTY (*PU*) FOR $*$-REGULAR RANK RINGS

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Introduction. In this paper we consider an irreducible $*$-regular ring $\mathscr{R}$ with order k for some $k\geq 4$. If $\mathscr{R}$ is also a Baer ring it is a rank ring. Our first result is:

THEOREM 1.3. *Let $\mathscr{R}$ be an irreducible $*$-regular Baer ring with order k for some $k\geq 4$. The following are equivalent.*

(i) *For any $e,f\in P(\mathscr{R})$, $e\overset{a}{\sim}f\Rightarrow e\overset{*}{\sim}f$.*
(ii) *For any $e,f\in P(\mathscr{R})$, $e\overset{a}{\sim}f\Rightarrow e\overset{u}{\sim}f$.*
(iii) *For any $a\in\mathscr{R}$, $P_a\overset{*}{\sim}{}_aP$.*
(iv) *For any $e,f\in P(\mathscr{R})$, $e\cup f-f\overset{*}{\sim}e-e\cap f$.*
(v) *If $e\in P(\mathscr{R})$ and $x^*x\in e\mathscr{R}e$, then there exists $z\in e\mathscr{R}e$ with $x^*x=z^*z$.*

We give the name "property (*PU*)" to (i) of Theorem 1.3, and express our second result as:

THEOREM 3.1. *Let $\mathscr{R}$ be an irreducible $*$-regular Baer ring with order k, $k\geq 4$, which satisfies property (PU). Then property (PU) lifts from $\mathscr{R}$ to $\mathscr{R}_n$.*

The ring $\mathscr{R}$ can be a rank ring without being a Baer ring. In this case, the completion $\mathscr{R}^\wedge$ of $\mathscr{R}$ in rank metric is a Baer ring. Our third result is:

THEOREM 6.1. *Let $\mathscr{R}$ be an irreducible $*$-regular rank ring with order k, $k\geq 4$, in which comparability holds and which satisfies property (PU). Then property (PU) extends from $\mathscr{R}$ to $\mathscr{R}^\wedge$.*

We conclude with an application to rank metric completions of certain inductive limits.

1. **Preliminaries.** A ring $\mathscr{R}$ is *regular* if the equation $axa=a$ is soluble in $\mathscr{R}$ for any a in $\mathscr{R}$. A $*$-*regular* ring is a regular ring $\mathscr{R}$ which admits an involution with the property that for any $a\in\mathscr{R}$, $a^*a=0$ implies $a=0$. We say that an irreducible regular ring is discrete if its projection lattice is atomic; otherwise, it is continuous. The rings with which we will be concerned will have a unit and will either be continuous rings or discrete rings with order k for some $k\geq 4$ [7, Definition 3.6, p. 100]. A ring is a *Baer* ring if the left and right annihilators of every subset are generated by idempotents. A Baer ring has a unit and a $*$-regular ring is a

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Baer ring if and only if it is complete (that is, its lattice of projections is a complete lattice). If a ring is regular but not Baer it may fail to have a unit element; in this case, we will always specify the existence of a unit element. A *rank ring* is a regular ring $\mathscr{R}$ which admits a rank function. That is, there exists a real-valued function $R(a)$, $a \in \mathscr{R}$, such that:

(i) $0 < R(a) < \infty$ for all $a \neq 0$;

(ii) $R(ab) \leq R(a), R(b)$;

(iii) If e and f are orthogonal idempotents then $R(e+f) = R(e) + R(f)$.

If $\mathscr{R}$ has a unit element and $R(1)=1$ the rank function is said to be normalized. An irreducible $*$-regular Baer ring admits a normalized rank function. In fact:

THEOREM 1.1. *An irreducible $*$-regular ring $\mathscr{R}$ with order k, $k \geq 4$, is a Baer ring if and only if it admits a normalized rank function with range contained in the closed unit interval and is complete in rank metric.*

Proof. Suppose that $\mathscr{R}$ is Baer. Then the lattice of projections (which we identify with the lattice of principal right ideals) of $\mathscr{R}$ is a continuous geometry (see [4]). A continuous geometry admits a dimension function D with range contained in the closed unit interval [7, Theorem 6.9, p. 52]. For $a \in \mathscr{R}$ define $R(a)$ by $R(a) = D(e)$ where e is the unique projection which generates the principal right ideal of a. This function $R: \mathscr{R} \to [0, 1]$ is a rank function and $\mathscr{R}$ is complete in rank metric [7, Theorem 17.4, p. 230].

Suppose that $\mathscr{R}$ admits a rank function with range contained in the closed unit interval and is complete in rank metric. Then $\mathscr{R}$ has a unit [1, Theorem 3.7 (iv), p. 716]. Thus, [7, Theorem 18.1, p. 237] applies to yield the result claimed.

This completes the proof.

We call an element x of a $*$-regular ring $\mathscr{R}$ *positive* if x has the form a^*a for some $a \in \mathscr{R}$. We denote the lattice of projections of $\mathscr{R}$ by $P(\mathscr{R})$. If $e, f \in P(\mathscr{R})$ and there exists $x \in e\mathscr{R}f$ and $y \in f\mathscr{R}e$ with $e = xy$ and $f = yx$, e and f are *algebraically equivalent*; in case e and f are algebraically equivalent, we write $e \overset{a}{\sim} f$. If there exists $w \in e\mathscr{R}f$ (to be called *partial unitary*) with $e = ww^*$ and $f = w^*w$, e and f are *$*$-equivalent*; in this case we write $e \overset{*}{\sim} f$. If e and f are exchangeable by a unitary, that is, if there exists $u \in \mathscr{R}$ with $uu^* = u^*u = 1$ and $ueu^* = f$, we say that e and f are *unitarily equivalent*; in this case we write $e \overset{u}{\sim} f$. If e and f have a common complement in $P(\mathscr{R})$ they are *perspective*; in this case we write simply $e \sim f$.

We denote by $a\mathscr{R}$ the right ideal generated by the element a of a $*$-regular ring $\mathscr{R}$ and we denote by p_a the unique projection satisfying $a\mathscr{R} = p_a\mathscr{R}$; we call p_a the *left projection* of a. Left ideal is denoted and *right projection* is defined analogously; we denote by ${}_ap$ the right projection of $a \in \mathscr{R}$. If $e, f \in P(\mathscr{R})$ we denote by $e \cup f$ and $e \cap f$ their lattice join and their lattice meet, respectively, in $P(\mathscr{R})$. If $e, f \in P(\mathscr{R})$ are orthogonal, $e \cup f = e + f$; if $e, f \in P(\mathscr{R})$ and $e \leq f$, then $e = f - g$ where $g \in P(\mathscr{R})$ is the orthogonal complement of e in f.

DEFINITION. We say that a $*$-regular ring $\mathscr{R}$ has property (PU) if any two algebraically equivalent projections in $\mathscr{R}$ are $*$-equivalent.

It is well-known that not every $*$-regular ring has property (PU); we give an example below of a $*$-regular ring which does not.

EXAMPLE. Let Q denote the field of rational numbers and Q_4 the ring of 4×4 matrices over Q. This ring is an irreducible $*$-regular Baer ring which has order 4. Moreover, the ring has infinitely many elements and the centre (namely, Q) has characteristic zero. It does not have property (PU). Let the matrices $E=(e_{ij})$, $F=(f_{ij})$, $X=(x_{ij})$, $Y=(y_{ij})\in Q_4$ be defined by:

$$e_{ij}=\begin{cases}\frac{1}{5}, & i=j=1\\ \frac{2}{5}, & i=1, j=2\\ \frac{2}{5}, & i=2, j=1\\ \frac{4}{5}, & i=j=2\\ 0, & \text{otherwise};\end{cases}\qquad f_{ij}=\begin{cases}1, & i=j=1\\ 0, & \text{otherwise};\end{cases}$$

$$x_{ij}=\begin{cases}3, & i=j=1\\ 6, & i=2, j=1\\ 0, & \text{otherwise};\end{cases}\qquad y_{ij}=\begin{cases}\frac{1}{15}, & i=j=1\\ \frac{2}{15}, & i=1, j=2\\ 0, & \text{otherwise}.\end{cases}$$

Then E and F are algebraically equivalent projections (via X and Y). However, it is easy to see that E and F are not $*$-equivalent (since there is no rational number whose square is $1/5$).

LEMMA 1.1. [10, Lemma 2, p. 74]. *Let $\mathscr{R}$ be a $*$-regular ring with unit which satisfies the condition*

$$x_1^*x_1+\cdots+x_n^*x_n=0 \quad\text{implies}\quad x_1=\cdots=x_n=0$$

$(n=1, 2, \ldots)$, $x_i\in\mathscr{R}$ $(i=1, 2, \ldots, n)$. *Then the centre of $\mathscr{R}$ contains a subfield isomorphic to the field of rational numbers.*

In particular, if $\mathscr{R}$ is an irreducible $*$-regular Baer ring which has order k for some $k\geq 4$ and possesses property (PU), then the following conditions are satisfied (this was proved by von Neumann in [6]; for a proof, see [9, Theorem 4, p. 220]):

(a) If $x, y\in\mathscr{R}$ are both positive, then $x+y$ is positive.
(b) If $x, y\in\mathscr{R}$ are both positive and $x+y=0$, then $x=y=0$.

It is easily seen that these two conditions imply the condition in the statement of Lemma 1.1.

We shall be concerned mainly with irreducible $*$-regular Baer rings with order k, $k\geq 4$. These arise naturally as the coordinatizing rings of those continuous geometries which admit an orthocomplementation. Conversely, the projection lattice of an irreducible $*$-regular Baer ring is a continuous geometry with this property.

Our attention will be focused on the property (PU). This property in an irreducible $*$-regular Baer ring with order k for some $k \geq 4$ has several equivalent formulations.

THEOREM 1.3. *Let $\mathscr{R}$ be an irreducible $*$-regular Baer ring with order k for some $k \geq 4$. The following are equivalent.*

(i) *For any $e, f \in P(\mathscr{R})$, $e \overset{a}{\sim} f \Rightarrow e \overset{*}{\sim} f$.*
(ii) *For any $e, f \in P(\mathscr{R})$, $e \overset{a}{\sim} f \Rightarrow e \overset{u}{\sim} f$.*
(iii) *For any $a \in \mathscr{R}$, $p_a \overset{*}{\sim} {}_a p$.*
(iv) *For any $e, f \in P(\mathscr{R})$, $e \cup f - f \overset{*}{\sim} e - e \cap f$.*
(v) *If $e \in P(\mathscr{R})$ and $x^*x \in e\mathscr{R}e$, then there exists $z \in e\mathscr{R}e$ with $x^*x = z^*z$.*

Proof. Let R denote the normalized rank function of $\mathscr{R}$.

(i)$\Rightarrow$(ii) we have $e \overset{a}{\sim} f$. Hence $1-e \overset{a}{\sim} 1-f$ [3, Exercise 2, p. 88]. So for some $w \in \mathscr{R}f$, $v \in (1-e)\mathscr{R}(1-f)$, we have $ww^* = e$, $w^*w = f$, $vv^* = 1-e$, $v^*v = 1-f$. Put $u = w^* + v^*$. Then

$$\begin{aligned} uu^* &= (w^*+v^*)(w+v) \\ &= w^*w + w^*v + v^*w + v^*v \\ &= f + w^*e(1-e)v + v^*(1-e)ew + 1 - f \\ &= 1 \\ &= u^*u. \end{aligned}$$

And

$$\begin{aligned} ueu^* &= (w^*+v^*)e(w+v) \\ &= w^*ew + w^*ev + v^*ew + v^*ev \\ &= f + w^*e(1-e)v + v^*(1-e)ew + v^*(1-e)ev \\ &= f. \end{aligned}$$

(ii)$\Rightarrow$(iii) For any $a \in \mathscr{R}$, $p_a \overset{a}{\sim} {}_a p$ [5, Exercise 7, p. 38]. Hence there exists unitary $u \in \mathscr{R}$ with $up_a u^* = {}_a p$. Put $w = up_a$. Then $ww^* = {}_a p$, $w^*w = p_a$.

(iii)$\Rightarrow$(iv) Let $e, f \in P(\mathscr{R})$ and let $g = e \cup f - f$, $h = e - e \cap f$. Then $g \overset{a}{\sim} h$ [4, Lemma 1, p. 525]. It follows that $1-g \overset{a}{\sim} 1-h$ and that there exists $x \in g\mathscr{R}h$, $y \in h\mathscr{R}g$, $u \in (1-g)\mathscr{R}(1-h)$, $v \in (1-h)\mathscr{R}(1-g)$ with $xy = g$, $yx = h$, $uv = 1-g$, $vu = 1-h$.

Let $s = x+u$, $t = y+v$. Then

$$\begin{aligned} st &= (x+u)(y+v) \\ &= xy + xv + uy + uv \\ &= g + xh(1-h)v + u(1-h)hy + 1 - g \\ &= 1 \\ &= ts. \end{aligned}$$

Hence $t=s^{-1}$ and

$$\begin{aligned} s^{-1}gs &= (y+v)g(x+u) \\ &= ygx+ygu+vgx+vgu \\ &= h+yg(1-g)u+v(1-g)gx+v(1-g)gu \\ &= h. \end{aligned}$$

We therefore have $g\cdot gs=gs$, $gs\cdot s^{-1}=g$. So $p_{gs}=g$. Also, $gs\cdot h=gs$, $s^{-1}\cdot gs=h$. So ${}_{gs}p=h$. It follows that $g\overset{*}{\sim}h$.

(iv)⇒(i) Suppose that $e,f\in P(\mathscr{R})$ with $e\overset{a}{\sim}f$, $e\perp f$. There exists $g\in P(\mathscr{R})$ with $e+f=e\cup g=f\cup g$ and $e\cap g=f\cap g=0$ [7, Theorem 15.3(c), p. 215]. We will show that $e\overset{*}{\sim}g$. Since $e\perp f$, we may write $e^{\perp}=f+f'$ where f' is the orthogonal complement of f in $e^{\perp}$.

$$\begin{aligned} g\cap e^{\perp} &= g\cap(f+f') \\ &= g\cap(f\cup(f'\cap(g\cup f))) \\ &= g\cap(f\cup(f'\cap(e+f))) \\ &= g\cap(f\cup 0) \\ &= g\cap f \\ &= 0. \end{aligned}$$

So $g-g\cap e^{\perp}=g$. Also,

$$\begin{aligned} g\cup e^{\perp} &= g\cup(f+f') \\ &= g\cup(f\cup f') \\ &= (g\cup f)\cup f' \\ &= (e\cup f)\cup f' \\ &= e\cup(f\cup f') \\ &= e\cup e^{\perp}=1. \end{aligned}$$

So $g\cup e^{\perp}-e^{\perp}=e$. Hence $e\overset{*}{\sim}g$. Similarly, $g\overset{*}{\sim}f$; so $e\overset{*}{\sim}f$. Suppose now that $e\overset{a}{\sim}f$ and $R(e)=R(f)\leq 1/4$. There exists $h\leq 1-e\cup f$ with $e\overset{a}{\sim}h\overset{a}{\sim}f$. From the above (since $h\perp e,f$), $e\overset{*}{\sim}h\overset{*}{\sim}f$; so $e\overset{*}{\sim}f$. Suppose finally that $e\overset{a}{\sim}f$ without restriction. We may write

$$\begin{aligned} e &= e_1+\cdots+e_k, \\ f &= f_1+\cdots+f_k, \end{aligned}$$

with $e_i\perp e_j$, $f_i\perp f_j$ $(i\neq j)$, and $R(e_i)=R(f_i)\leq 1/4$ $(i=1,2,\ldots,k)$. It follows that $e_i\overset{*}{\sim}f_i$ and that $e\overset{*}{\sim}f$ since $*$-equivalence is finitely additive [5, Theorem 25, p. 33].

(i)⇔(v). This follows from [9, Theorem 1, p. 215].

This completes the proof.

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Property (PU) may be regarded as that property which an irreducible $*$-regular Baer ring with order k, $k \geq 4$, must have in order for the notions of equality in rank, perspectivity, algebraic equivalence, $*$-equivalence, and unitary equivalence of projections to coincide.

THEOREM 1.4. *Let $\mathscr{R}$ be an irreducible $*$-regular Baer ring with order k, $k \geq 4$. Let R denote the normalized rank of $\mathscr{R}$. Then $\mathscr{R}$ has property (PU) if and only if the following are equivalent for $e, f \in P(\mathscr{R})$.*

(i) $R(e) = R(f)$.
(ii) $e \sim f$.
(iii) $e \overset{a}{\sim} f$.
(iv) $e \overset{*}{\sim} f$.
(v) $e \overset{u}{\sim} f$.

Proof. If these condition are equivalent, in particular, (iii)$\Rightarrow$(iv); that is, $\mathscr{R}$ has property (PU).

Conversely, suppose that $\mathscr{R}$ has property (PU). Then (iii)$\Rightarrow$(iv).

(iv)$\Rightarrow$(v) We have $e \overset{*}{\sim} f$ and (using property (PU)) $1-e \overset{*}{\sim} 1-f$. It follows as in the proof of (i)$\Rightarrow$(ii) of Theorem 1.3 and from property (PU) that $e \overset{u}{\sim} f$.

(v)$\Rightarrow$(i) This follows from [7, Theorem 17.1(d), p. 224].

(i)$\Rightarrow$(ii) This follows from [7, Theorem 6.9, p. 52].

(ii)$\Rightarrow$(iii) This follows from [7, Theorem 15.3(a), p. 215].

This completes the proof.

2. **The matrix ring over a regular ring.** We denote by $\mathscr{R}_n$ the ring of $n \times n$ matrices over a ring $\mathscr{R}$. Von Neumann showed that $\mathscr{R}_n$ is regular if and only if $\mathscr{R}$ is regular. If $\mathscr{R}$ is a regular ring with unit and with normalized rank function R, then $\mathscr{R}_n$ admits a unique normalized rank function R'_n [2]. If $\mathscr{R}$ is complete with respect to the metric of R, $\mathscr{R}_n$ is complete with respect to the metric of R'_n. We will find it convenient to work with the rank function $R_n = nR'_n$ which has the property:

$$R_n(E(e)) = nR(e)$$

whenever e is idempotent in $\mathscr{R}$ and $E(e)$ is the matrix in $\mathscr{R}_n$ which has e for all diagonal entries and zeros elsewhere. We denote by $E_i(a)$ the matrix in $\mathscr{R}_n$ which has a for ith diagonal entry and zeros elsewhere and by $E(a)$ the matrix $\sum_{i=1}^n E_i(a)$. We note that if e is idempotent in $\mathscr{R}$, then

$$R_n(E_i(e)) = \frac{1}{n} R_n(E(e)) = R(e).$$

It is easy to see that $\mathscr{R}_n$ is irreducible if $\mathscr{R}$ is irreducible.

If $\mathscr{R}$ is $*$-regular, $\mathscr{R}_n$ may fail to be $*$-regular. In fact, $\mathscr{R}_n$ is $*$-regular if and only if $\mathscr{R}$ has the property

$$x_1^* x_1 + \cdots + x_n^* x_n = 0 \quad \text{implies} \quad x_1 = \cdots = x_n = 0$$

$x_i \in \mathscr{R}$ $(i=1, 2, \ldots, n)$ (see [5, Exercise 8, p. 38]). In particular, if $\mathscr{R}$ is an irreducible $*$-regular Baer ring which has order k for some $k \geq 4$ and possesses property (PU), $\mathscr{R}_n$ is $*$-regular.

3. **Lifting property (PU) from $\mathscr{R}$ to $\mathscr{R}_n$.** Throughout this section, $\mathscr{R}$ is an irreducible $*$-regular Baer ring. We suppose that $\mathscr{R}$ has order k for some $k \geq 4$. The object of this section is to raise property (PU) from $\mathscr{R}$ to $\mathscr{R}_2$ and thence to $\mathscr{R}_n$ $(n=3, 4, \ldots)$. $\mathscr{R}$ admits a normalized rank function and our analysis is carried out in terms of the rank function. The completeness of $\mathscr{R}$ is assumed only to ensure that comparability of projections is at hand: for $e, f \in P(\mathscr{R})$, either there exists $e_1 \in P(\mathscr{R})$ with $e_1 \leq e$ and $e_1 \sim f$ or there exists $f_1 \in P(\mathscr{R})$ with $f_1 \leq f$ and $e \sim f_1$. Completeness is not a necessary condition for comparability, and comparability could be assumed outright. In this case, $\mathscr{R}$ may fail to admit a rank function; nevertheless, the analysis can be carried out by comparing projections with a fixed reference projection.

LEMMA 3.1. *Let $E=(a_{ij}) \in P(\mathscr{R}_2)$ and suppose that $0 < R_2(E) \leq 1/4$. Then there exists unitary $U \in \mathscr{R}_2$ and $e \in P(\mathscr{R})$ with $UEU^* = E_1(e)$.*

Proof. From [2, §7.2, p. 332] it follows that $R(a_{ij}) \leq 1/4$ $(i, j=1, 2)$. Let

$$p_{a_{11}} \cup p_{a_{12}} = f, \qquad p_{a_{21}} \cup p_{a_{22}} = g.$$

If $f=0$, then $a_{11}=a_{12}=a_{21}^*=a_{21}=0$ and $a_{22}=g$; if $g=0$, then $a_{22}=a_{21}=a_{12}^*=a_{12}=0$ and $a_{11}=f$; in either case, the problem is trivial. Assume, then, that $f, g \neq 0$. We have $a_{11}=fa_{11}$, $a_{12}=fa_{12}$, $a_{21}=ga_{21}$, $a_{22}=ga_{22}$. Also, $R(f)$, $R(g) \leq 1/2$, so there exists unitary $u \in \mathscr{R}$ with $ufu^* = f_1 \leq (1-g)$. It follows that $guf=0$. Define

$$U = \begin{bmatrix} (1-g)u & g \\ gu & 1-g \end{bmatrix}.$$

Easy calculations show that U is unitary in $\mathscr{R}_2$ and that

$$UE = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$$

for some $x, y \in \mathscr{R}$. Thus

$$\begin{aligned} UEU^* &= (UE)(UE)^* \\ &= \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^* & 0 \\ y^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} xx^*+yy^* & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

It is obvious that $e=xx^*+yy^* \in P(\mathscr{R})$. This completes the proof of the lemma.

LEMMA 3.2. *Let E, $F \in P(\mathscr{R}_2)$ and suppose that $R_2(E)=R_2(F) \leq 1/4$. Then there exists unitary $U \in \mathscr{R}_2$ with $UEU^*=F$.*

Proof. There exists unitaries U_1, $U_2 \in \mathscr{R}$ and $e, f \in P(\mathscr{R})$ with $U_1 E U_1^* = E_1(e)$, $U_2 F U_2^* = E_1(f)$. Now $R(e) = R_2(E_1(e)) = R_2(E) = R_2(F) = R_2(E_1(f)) = R(f)$, so there exists unitary $u \in \mathscr{R}$ with $ueu^* = f$. Define $U = U_2^* \cdot E(u) \cdot U_1$. Then U is unitary in $\mathscr{R}_2$ and $UEU^* = F$. This completes the proof of the lemma.

LEMMA 3.3. *Let $E, F \in P(\mathscr{R}_2)$ with $R_2(E) = R_2(F)$. Then there exists $W \in E\mathscr{R}_2 F$ with $E = WW^*$, $F = W^*W$.*

Proof. We may write

$$E = E_1 + \cdots + E_k,$$
$$F = F_1 + \cdots + F_k,$$

with $E_i \perp E_j$, $F_i \perp F_j$ $(i \neq j)$, and $R_2(E_i) = R_2(F_i) \leq 1/4$ $(i = 1, 2, \ldots, k)$. By Lemma 3.2, there exists unitary $U_i \in \mathscr{R}_2$ with $U_i E_i U_i^* = F_i$ $(i = 1, 2, \ldots, k)$. So $E_i \overset{*}{\sim} F_i$ (via $W_i = E_i U_i^*$). Now $*$-equivalence is finitely additive [5, Theorem 25, p. 33]. This completes the proof of the lemma.

LEMMA 3.4. *Let $m = 2^k$ for some $k \geq 1$, and let $E, F \in P(\mathscr{R}_m)$ with $R_m(E) = R_m(F)$. Then there exists $W \in E\mathscr{R}_m F$ with $E = WW^*$, $F = W^*W$.*

Proof. We observe that $\mathscr{R}_{2^k} \overset{*}{\cong} (\mathscr{R}_{2^{k-1}})_2$. The lemma now follows by a simple induction from Lemma 3.3.

COROLLARY. *Property (PU) lifts from $\mathscr{R}$ to $\mathscr{R}_m$, $m = 2^k$.*

Proof. Let $E, F \in P(\mathscr{R}_m)$ with $E \overset{a}{\sim} F$. Then $E = XY$, $F = YX$ for some $X \in E\mathscr{R}_m F$, $Y \in F\mathscr{R}_m E$. So

$$R_m(E) = R_m(XY) \leq R_m(Y) = R_m(FY) \leq R_m(F)$$
$$= R_m(YX) \leq R_m(X) = R_m(EX) \leq R_m(E).$$

Hence $R_m(E) = R_m(F)$ and the corollary follows from the lemma.

It follows readily from condition (v) in the statement of Theorem 1.3 that if $e \in P(\mathscr{R})$, then $e\mathscr{R}e$ has property (PU). We prove the following lemma.

LEMMA 3.5. *Let $\mathscr{R}$ be a $*$-regular ring with normalized rank function R and the property: projections with equal rank are $*$-equivalent. Let $e, f \in P(\mathscr{R})$ with $R(e) = R(f)$ and let $e \cup f = g$. Then there exists unitary $u \in \mathscr{R}$ with $ueu^* = f$ and $uh = h$ for all $h \leq (1 - g)$.*

Proof. Let $e_1 = g - e$, $f_1 = g - f$. Then

$$R(e_1) = R(g - e) = R(g) - R(e)$$
$$= R(f) - R(e) = R(f - e) = R(f_1).$$

There exist unitaries v, $v_1 \in \mathscr{R}$ with $v_1 e_1 v_1^* = f_1$, $vev^* = f$. Define $w = ve + v_1 e_1$. Then

$$ww^* = (ve + v_1 e_1)(ev^* + e_1 v_1^*)$$
$$= vev^* + vee_1 v_1^* + v_1 e_1 ev^* + v_1 e_1 v_1^*$$
$$= f + f_1 = g.$$

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Similarly, $w^*w=g$ and $wew^*=f$. Let $u=w+1-g$. Then clearly u is unitary and $ueu^*=wew^*=f$.

Moreover, for $h\leq(1-g)$,

$$\begin{aligned} uh &= u(1-g)h \\ &= (w+1-g)(1-g)h \\ &= (w(1-g)+1-g)h \\ &= (wg(1-g)+1-g)h \\ &= (1-g)h = h. \end{aligned}$$

This completes the proof.

THEOREM 3.1. *Let $\mathscr{R}$ be an irreducible $*$-regular Baer ring with order k, $k\geq 4$, which satisfies property (PU). Then property (PU) lifts from $\mathscr{R}$ to $\mathscr{R}_n$ $(n=1, 2, \ldots)$.*

Proof. Choose k so that $2^k=m\geq n$. Let $P=(p_{ij})\in P(\mathscr{R}_m)$ be defined by

$$p_{ij}=\begin{cases}1, & i=j\leq n\\ 0, & \text{otherwise.}\end{cases}$$

Then $P\mathscr{R}_m P\overset{*}{\simeq}\mathscr{R}_n$. Now $\mathscr{R}_m$ has property (PU) by Lemma 3.4. Hence $\mathscr{R}_n$ has property (PU) by the remark preceding Lemma 3.5. This completes the proof of the theorem.

4. **The inductive limit of a system of rings.** Let $(I, \leq)$ be a partially ordered set which is directed up (i.e., for $i, j\in I$ there is a $k\in I$ with $i\leq k, j\leq k$). Suppose that for each $i\in I$, $\mathscr{R}_i$ is a ring and that for each i, j with $i\leq j$ there is a ring homomorphism $\Psi_{ji}:\mathscr{R}_i\to\mathscr{R}_j$ such that whenever $i\leq j\leq k$, we have

$$\Psi_{kj}\Psi_{ji}=\Psi_{ki}.$$

Let S be the subset of $(\bigcup_{i\in I}\mathscr{R}_i)\times I$ described by $\{(a, i): i\in I \text{ and } a\in\mathscr{R}_i\}$. We define a relation $\rho\subset S\times S$ by $(a, i)\rho(b, j)$ if there is a $(c, k)\in S$ with $i\leq k$, $j\leq k$ and $\Psi_{ki}a=\Psi_{kj}b=c$. The relation ρ is clearly an equivalence relation on S. The equivalence classes of S form a ring, which we denote by $\mathscr{R}$ and call the *inductive limit* of the system $(I, \leq, \mathscr{R}_i, \Psi_{ji})$ with respect to the following operations. If $\rho_{(a,i)}$ denotes the equivalence class of $(a, i)\in S$, addition and multiplication in $\mathscr{R}$ are defined by the rules:

(1) $\rho_{(a,i)}+\rho_{(b,j)}=\rho_{(\Psi_{ki}a+\Psi_{kj}b,k)}$

(2) $\rho_{(a,i)}\cdot\rho_{(b,j)}=\rho_{(\Psi_{ki}a\cdot\Psi_{kj}b,k)}$,

where $i\leq k, j\leq k$.

We note the following (see [2, §9]).

(i) If each $\mathscr{R}_i$ is regular, $\mathscr{R}$ is regular.

(ii) If each Ψ_{ji} is injective, then the mapping $a\mapsto\rho_{(a,i)}$ is an injective ring embedding of $\mathscr{R}_i$ in $\mathscr{R}$.

(iii) If each $\mathscr{R}_i$ is a regular ring with rank function $R_i:\mathscr{R}_i\to[0, 1]$† and each Ψ_{ji} preserves the rank, then the function $R:\mathscr{R}\to[0, 1]$ given by

$$R(\rho_{(a,i)}) = R_i(a)$$

is a rank function on $\mathscr{R}$.

(iv) If each $\mathscr{R}_i$ is irreducible and the Ψ_{ji}'s are injective, $\mathscr{R}$ is irreducible.

(v) If each $\mathscr{R}_i$ is $*$-regular (we denote the involution in each $\mathscr{R}_i$ by $*$) and each Ψ_{ji} preserves the involution, then the mapping $*:\mathscr{R}\to\mathscr{R}$ defined by

$$\rho^*_{(a,i)} = \rho_{(a^*,i)}$$

is an involution with respect to which $\mathscr{R}$ is $*$-regular.

(vi) If I has infinitely many elements and each $\mathscr{R}_i$ is a rank ring, $\mathscr{R}$ need not be complete in rank metric even if each $\mathscr{R}_i$ is complete.

It follows that if each $\mathscr{R}_i$ is an irreducible $*$-regular Baer ring, $\mathscr{R}$ is an irreducible $*$-regular rank ring with unit. We will see in the next section that in this case the completion $\mathscr{R}^\wedge$ of $\mathscr{R}$ in rank metric is again an irreducible $*$-regular Baer ring.

5. **The completion of a regular rank ring.** Let $\mathscr{R}$ be a regular ring with unit and rank function R. We denote by N the set of positive integers and by $X_{n\in N}\mathscr{R}$ the collection of mappings

$$\alpha: N \to \mathscr{R}$$

$$n \mapsto \alpha_n.$$

Let T be the subset of $X_{n\in N}\mathscr{R}$ whose elements α satisfy $R(\alpha_n-\alpha_m)\to 0$ as $n, m\to\infty$. We define a relation $\equiv\subset T\times T$ by $\alpha\equiv\beta$ if $R(\alpha_n-\beta_n)\to 0$ as $n\to\infty$. The relation $\equiv$ is clearly an equivalence relation on T. The equivalence classes of $\equiv$ form a ring, which we denote by $\mathscr{R}^\wedge$ and call the *completion* of $\mathscr{R}$, with respect to the usual pointwise operations.

We extend the rank function on $\mathscr{R}$ to a function (again denoted by R) on $\mathscr{R}^\wedge$ in the following way. Let $a \in \mathscr{R}^\wedge$ and suppose that $\alpha \in a$; then

$$R(a) = \lim_{n\to\infty} R(\alpha_n).$$

Let $a \in \mathscr{R}$; we denote the equivalence class of the mapping $\alpha:N\to\mathscr{R}$ given by $\alpha_n=a$ $(n=1, 2, \ldots)$ by $\hat{a}$, and we define $\wedge:\mathscr{R}\to\mathscr{R}^\wedge$ by $a\mapsto\hat{a}$.

THEOREM 5.1. [1, 3.6 and 3.7, p. 716].

(i) *$\mathscr{R}^\wedge$ is a regular ring.*

(ii) *R is a rank function on $\mathscr{R}^\wedge$.*

(iii) *$\mathscr{R}^\wedge$ is complete with respect to its rank metric.*

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† In the case of matrix rings this notation conflicts with the notation $R_i=iR_i'$ introduced on p. 11.

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(iv) *The mapping* $\wedge:\mathscr{R}\to\mathscr{R}^\wedge$ *is a ring isomorphic embedding of* $\mathscr{R}$ *into* $\mathscr{R}^\wedge$ *preserving rank.*

(v) $\wedge(\mathscr{R})=\mathscr{R}^\wedge$ *if and only if* $\mathscr{R}$ *is complete in rank metric.*

We note the following.

(i) If $\mathscr{R}$ is a $*$-regular ring, we extend the involution (again denoted by $*$) to $\mathscr{R}^\wedge$ as follows. Let $a\in\mathscr{R}^\wedge$ and let $\alpha\in a$. Define $\alpha':N\to\mathscr{R}$ by

$$\alpha'_n=\alpha^*_n \qquad (n=1,2,\ldots).$$

Then $\alpha'\in T$. If a^* denotes the equivalence class of α', then the mapping $a\mapsto a^*$ is the required extension of the involution to $\mathscr{R}^\wedge$. Moreover, $\mathscr{R}^\wedge$ is $*$-regular.

(ii) If $\mathscr{R}$ is irreducible, so is $\mathscr{R}^\wedge$ [3, Lemma 3(ii), p. 479].

We have, then, that if $\mathscr{R}$ is an irreducible $*$-regular rank ring with unit, $\mathscr{R}^\wedge$ is an irreducible $*$-regular Baer ring. Combining this with the results of the previous section, we have that if $(I,\leq,\mathscr{R}_i,\Psi_{ji})$ is a system of irreducible $*$-regular Baer rings, then the completion $\mathscr{R}^\wedge$ of

$$\mathscr{R}=\lim_{\to}(I,\leq,\mathscr{R}_i,\Psi_{ji})$$

is an irreducible $*$-regular Baer ring. If in addition each $\mathscr{R}_i$ has property (PU), it follows readily that $\mathscr{R}$ has property (PU). In the next section we prove, under the additional assumption of comparability in $\mathscr{R}$, that property (PU) persists from $\mathscr{R}$ to $\mathscr{R}^\wedge$.

6. **Extension of property** (PU) **from incomplete** $\mathscr{R}$ **to complete** $\mathscr{R}^\wedge$**.** Throughout this section, $\mathscr{R}$ is an irreducible $*$-regular ring with unit and normalized rank function R. We suppose that $\mathscr{R}$ has order k for some $k\geq 4$ and possesses property (PU). Since we are assuming $\mathscr{R}$ to be not complete, Theorem 1.4 does not apply. However,

$$e\overset{a}{\sim}f\Rightarrow R(e)=R(f)$$

does (as always in a rank ring) hold. If the rank function is to be useful as a means of analysing $\mathscr{R}$ some kind of converse of this is needed. The rank function does not itself give an idea of relative size; if $e,f\in P(\mathscr{R})$ and $R(e)<R(f)$, e cannot usefully be thought of as being smaller than f unless there is an image of e (having the same size as e in terms of the appropriate notion of size) inside f. To ensure that $\mathscr{R}$ is tractible and that no pathology arises we will assume in all that follows that comparability holds in $\mathscr{R}$:

If $e,f\in P(\mathscr{R})$ and $R(e)\leq R(f)$, then there exists $f_1\in P(\mathscr{R})$ with $f_1\leq f$ and $e\sim f_1$.

A formally weaker approach is to assume in place of property (PU) the condition:

(1) $R(e)=R(f)\Rightarrow e\overset{*}{\sim}f$.

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Then the condition:

(2) If $e, f \in P(\mathscr{R})$ and $R(e) \leq R(f)$, then there exists $f_1 \in P(\mathscr{R})$ with $f_1 \leq f$ and $R(e) = R(f_1)$

ensures that $\mathscr{R}$ is amenable to analysis. We observe that condition (1) is equivalent here (and in any $*$-regular rank ring) to

$$R(e) = R(f) \Rightarrow e \overset{u}{\sim} f.$$

LEMMA 6.1. *$\mathscr{R}$ has comparability if and only if $\mathscr{R}$ satisfies conditions* (1) *and* (2) *above.*

Proof. Suppose that $\mathscr{R}$ has comparability. Let $e, f \in P(\mathscr{R})$ with $R(e) = R(f)$. There exists $f_1 \leq f$ with $e \sim f_1$. It follows that $e \overset{a}{\sim} f_1$ [7, Theorem 15.3(a), p. 215] and that $R(e) = R(f_1)$. Since $R(f - f_1) = R(f) - R(f_1) = R(e) - R(e) = 0$ we have $f = f_1$. That is, $e \overset{a}{\sim} f$. Since $\mathscr{R}$ has property (PU), $e \overset{*}{\sim} f$. This establishes that $\mathscr{R}$ satisfies condition (1). Now let $e, f \in P(\mathscr{R})$ with $R(e) \leq R(f)$. There exists $f_1 \leq f$ with $e \sim f_1$. Again we have $e \overset{a}{\sim} f_1$ and $R(e) = R(f_1)$. This establishes that $\mathscr{R}$ satisfies condition (2).

Conversely, suppose that $\mathscr{R}$ satisfies conditions (1) and (2). Let $e, f \in P(\mathscr{R})$ with $R(e) \leq R(f)$. There exists $f_1 \leq f$ with $R(e) = R(f_1)$. Write

$$e = e \cap f_1 + e',$$
$$f_1 = e \cap f_1 + f_1'.$$

Then

$$\begin{aligned} R(e') &= R(e - e \cap f_1) = R(e) - R(e \cap f_1) \\ &= R(f_1) - R(e \cap f_1) = R(f_1 - e \cap f_1) = R(f_1'). \end{aligned}$$

So $e' \overset{*}{\sim} f_1'$. Moreover, $e' \cap f_1' = 0$. We may apply [7, Theorem 15.3(c), p. 215] to obtain $e' \sim f_1'$. Now [7, Theorem 3.5, p. 20] applies to yield $e \sim f_1$. This completes the proof.

LEMMA 6.2. *Let $e \in P(\mathscr{R}^\wedge)$. Then there exists $\alpha \in a$ with $\alpha_n \in P(\mathscr{R})$ $(n = 1, 2, \ldots)$.*

Proof. Let $\beta \in e$. Define $\gamma_n = (\beta_n + \beta_n^*)/2$ (for the invertibility of 2, cf. Lemma 1.1). Denote by $\bar{\gamma}_n$ the relative inverse of γ_n [4, p. 525] and let $e_n = P_{\gamma_n}$. Then $\gamma_n \bar{\gamma}_n = e_n$, $e_n \gamma_n = \gamma_n$. We have

$$\begin{aligned} R(e_n - \beta_n) &= R(e_n - \gamma_n + \gamma_n - \beta_n) \\ &\leq R(e_n - \gamma_n) + R(\gamma_n - \beta_n) \\ &= R((e_n - \gamma_n)^2) + R(\beta_n^* - \beta_n) \end{aligned}$$

since $e_n - \gamma_n = (e_n - \gamma_n)^*$ and $\gamma_n - \beta_n = (\beta_n^* - \beta_n)/2$. Now easy calculations verify

$$(e_n - \gamma_n)^2 = (\gamma_n - \gamma_n^2)(\bar{\gamma}_n - 1)$$

and

$$\gamma_n - \gamma_n^2 = \{2(\beta_n - \beta_n^2) + 2(\beta_n - \beta_n^2)^* + (\beta_n - \beta_n^*)\}/4.$$

Hence,

$$R((e_n-\gamma_n)^2) \leq R(\gamma_n-\gamma_n^2)$$
$$\leq 2R(\beta_n-\beta_n^2)+R(\beta_n-\beta_n^*).$$

So

$$R(e_n-\beta_n) \leq 2\{R(\beta_n-\beta_n^2)+R(\beta_n-\beta_n^*)\}.$$

Since e is a projection, both $R(\beta_n-\beta_n^2)$ and $R(\beta_n-\beta_n^*)$ approach zero as n becomes large. Let $\alpha:N\to\mathscr{R}$ be given by

$$\alpha_n = e_n \qquad (n = 1, 2, \ldots).$$

Then $\alpha\equiv\beta$, so $\alpha \in e$ and $\alpha_n \in P(\mathscr{R})$. This completes the proof.

Lemma 6.3. *Suppose that $e\mathscr{R}=p\mathscr{R}$ and $f\mathscr{R}=q\mathscr{R}$, where $e,f \in P(\mathscr{R})$ and p, q are orthogonal idempotents. Then $R(p-q)\leq 2R(e-f)$.*

Proof. We have

$$ep = p, \qquad pe = e,$$
$$fq = q, \qquad qf = f.$$

Also,

$$pq = pf = 0 = qp = qe.$$

Hence

$$(p+q)(e-f)(p+q)+(p-q)(e-f)(p-q) = 2(p-q).$$

So

$$R(p-q) = R(2(p-q)) \leq 2R(e-f).$$

This completes the proof.

Lemma 6.4. *Let $e,f \in P(\mathscr{R})$. We have*

$$R(e \cap f)+R(e-f) \geq (R(e)+R(f))/2.$$

Proof. Write

$$e = e'+e \cap f,$$
$$f = f'+e \cap f.$$

Then $e-f=e'-f'$ and $e' \cap f'=0$. There exist orthogonal idempotents $p, q \in \mathscr{R}$ with

$$e'\mathscr{R} = p\mathscr{R}, \qquad f'\mathscr{R} = q\mathscr{R}$$

[1, Lemma 2.1 (2.12), p. 711]. By Lemma 6.3, $R(p-q)\leq 2R(e'-f')=2R(e-f)$. Now $(p-q)^2=p^2-pq-qp+q^2=p+q$ and $p-q=p^2-q^2=(p-q)(p+q)$. Hence

$$R(p)+R(q) = R(p+q) = R((p-q)^2) \leq R(p-q),$$

and

$$R(p-q) = R(p-q)(p+q)) \leq R(p+q) = R(p)+R(q).$$

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Combining these, we obtain $R(p-q)=R(p)+R(q)$. Since $R(e')+R(f')=R(p)+R(q)$, we have $R(e')+R(f')\leq 2R(e-f)$. Also, $R(e\cap f)=R(e-e')=R(f-f')=R(e)-R(e')=R(f)-R(f')$. So $2R(e\cap f)+R(e')+R(f')=R(e)+R(f)$. Hence $2R(e\cap f)+2R(e-f)\geq R(e)+R(f)$.

This completes the proof.

LEMMA 6.5. *Let $e_1, e_2, f_1, f_2\in P(\mathscr{R})$ with $R(e_1)=R(f_1)$, $R(e_2)=R(f_2)$, and let unitary $u_1\in\mathscr{R}$ be such that $u_1e_1u_1^*=f_1$. Then there exists unitary $u_2\in\mathscr{R}$ with $u_2e_2u_2^*=f_2$ and*

$$R(u_2-u_1)\leq 2(R(e_2-e_1)+R(f_2-f_1)).$$

Proof. Let $R(e_2-e_1)=\xi$, $R(f_2-f_1)=\eta$. Also let $f_3=u_1e_2u_1^*$. Then $f_3-f_1=u_1e_2u_1^*-u_1e_1u_1^*=u_1(e_2-e_1)u_1^*$, and $R(f_3-f_1)=R(e_2-e_1)=\xi$. Hence $R(f_3-f_2)=R(f_3-f_1+f_1-f_2)\leq R(f_3-f_1)+R(f_2-f_1)=\xi+\eta$. Write

$$f_2=f_2\cap f_3+f_2',$$
$$f_3=f_2\cap f_3+f_3'.$$

Then $R(f_3'-f_2')=R(f_3-f_2)\leq\xi+\eta$. Also, $R(f_3)=R(e_2)=R(f_2)$. So $R(f_3')=R(f_2')$. Put $f_2'\cup f_3'=g$. Then by Lemma 3.5, there exists unitary $u_3\in\mathscr{R}$ with $u_3f_2'u_3^*=f_3'$ and $u_3h=h$ for all $h\leq 1-g$. Now $f_2'\leq 1-f_2\cap f_3$ and $f_3'\leq 1-f_2\cap f_3$, so $f_2'\cup f_3'=g\leq 1-f_2\cap f_3$. Hence $1-g\geq f_2\cap f_3$ and $u_3(f_2\cap f_3)=f_2\cap f_3$. Therefore,

$$\begin{aligned}u_3f_2u_3^* &= u_3(f_2'+f_2\cap f_3)u_3^*\\ &= u_3f_2'u_3^*+u_3(f_2\cap f_3)u_3^*\\ &= f_3'+f_2\cap f_3\\ &= f_3.\end{aligned}$$

Moreover, $(1-u_3)(1-g)=1-g-(1-g)=0$. So $R(1-u_3)=R((1-u_3)g)\leq R(g)$. Now $R(g)=R(f_2'\cup f_3')=R(f_2')+R(f_3')$, and by Lemma 6.4, $R(f_3')+R(f_2')\leq 2R(f_3'-f_2')$. Hence $R(g)\leq 2R(f_3'-f_2')\leq 2(\xi+\eta)$. So $R(1-u_3)\leq 2(\xi+\eta)$. We have $u_3f_2u_3^*=f_3=u_1e_2u_1^*$, or $(u_3^*u_1)e_2(u_3^*u_1)^*=f_2$. Put $u_3^*u_1=u_2$. Then u_2 is unitary, $u_2e_2u_2^*=f_2$, and

$$\begin{aligned}R(u_2-u_1) &= R(u_3^*u_1-u_1)\\ &= R((1-u_3)^*u_1)\\ &\leq R(1-u_3)\\ &\leq 2(\xi+\eta)\\ &= 2(R(e_2-e_1)+R(f_2-f_1)).\end{aligned}$$

This completes the proof.

LEMMA 6.6. *Let $e, f\in P(\mathscr{R}^\wedge)$ with $R(e)=R(f)$. Then there exists unitary $u\in\mathscr{R}^\wedge$ with $ueu^*=f$.*

Proof. Let $\lambda=R(e)=R(f)$. If both e and f are images under the mapping $\wedge:\mathscr{R}\to\mathscr{R}^\wedge$ of projections in $\mathscr{R}$, there is nothing to prove. We will suppose that e is not the image under $\wedge$ of a projection in $\mathscr{R}$. Let $\alpha\in e$, $\beta\in f$ with $\alpha_n, \beta_n\in P(\mathscr{R})$ $(n=1, 2, \ldots)$. Since

$$\lim_{n\to\infty} R(\alpha_n)=\lambda=R(e),$$

there is a subsequence (α_{k_n}) of (α_n) such that $R(\alpha_{k_n})$ either increases monotonously to λ or decreases monotonously to λ. We will suppose that $R(\alpha_{k_n})$ decreases monotonously to λ. Put

$$\alpha_{k_n}=\gamma_n \qquad (n=1, 2, \ldots).$$

Then the mapping

$$\gamma: N\to\mathscr{R}$$

$$n\mapsto\gamma_n$$

satisfies $\gamma\equiv\alpha\in e$. Now $R(e-\hat{\gamma}_n)\to 0$ [1, Theorem 3.7(iii), p. 716], so we may assume that

$$R(e-\hat{\gamma}_n)\le 2^{-n}\lambda \qquad (n=1, 2, \ldots).$$

(If this is not the case, we may again select a subsequence of (γ_n) for which this is true.) We then have

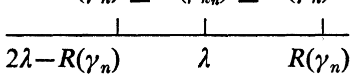
$$\begin{aligned} R(\gamma_{n+1}-\gamma_n) &= R(\hat{\gamma}_{n+1}-\hat{\gamma}_n)\\ &= R(\hat{\gamma}_{n+1}-e+e-\hat{\gamma}_n)\\ &\le R(\hat{\gamma}_{n+1}-e)+R(\hat{\gamma}_n-e)\\ &< 2^{-(n+1)}\lambda+2^{-n}\lambda\\ &= 3.2^{-(n+1)}\lambda. \end{aligned}$$

Also, $R(e-\hat{\gamma}_n)\ge|R(e)-R(\gamma_n)|=R(\gamma_n)-R(e)$. So $R(\gamma_n)\le R(e)+R(e-\hat{\gamma}_n)<\lambda(1+2^{-n})$. We may suppose that $R(\beta_{n+1}-\beta_n)<2^{-(n+1)}\lambda$. Then if (β_{k_n}) is any subsequence of (β_n), we have $k_{n+1}=k_n+l$ for some $l\ge 1$ and since $k_n\ge n$, $2^{-(k_n+1)}\le 2^{-(n+1)}$. Thus

$$\begin{aligned} R(\beta_{k_{n+1}}-\beta_{k_n}) &= R(\beta_{k_n+l}-\beta_{k_n+l-1}+\cdots+\beta_{k_n+1}-\beta_{k_n})\\ &\le R(\beta_{k_n+l}-\beta_{k_n+l-1})+\cdots+R(\beta_{k_n+1}-\beta_{k_n})\\ &< 2^{-(k_n+l)}\lambda+\cdots+2^{-(k_n+1)}\lambda\\ &= 2^{-(k_n+1)}\lambda(1+2^{-1}+\cdots+2^{-(l-1)})\\ &< 2^{-(n+1)}\lambda\cdot 2=2^{-n}\lambda. \end{aligned}$$

Now since $R(\gamma_n)\downarrow\lambda$ and $\lim_{n\to\infty}R(\beta_n)=\lambda$, for each $n\in N$ there is a $k_n\in N$ with

$$2\lambda-R(\gamma_n)\le R(\beta_{k_n})\le R(\gamma_n).$$



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There exists $\beta'_{k_n} \in P(\mathscr{R})$ with $\beta'_{k_n} \leq \beta_{k_n}$ and $R(\beta'_{k_n}) = R(\gamma_n)$. Put

$$\beta'_{k_n} = \delta_n \qquad (n = 1, 2, \ldots).$$

Then the mapping

$$\delta : N \to \mathscr{R}$$
$$n \mapsto \delta_n$$

satisfies $\delta \equiv \beta \in f$. Now $-R(\beta_{k_n}) \leq R(\gamma_n) - 2\lambda$. So $R(\gamma_n) - R(\beta_{k_n}) \leq \lambda(1+2^{-n}) + \lambda(1+2^{-n}) - 2\lambda = 2^{-n+1}\lambda$. Hence

$$\begin{aligned} R(\beta'_{k_n} - \beta_{k_n}) &= R(\beta'_{k_n}) - R(\beta_{k_n}) \\ &= R(\gamma_n) - R(\beta_{k_n}) \\ &\leq 2^{-n+1}\lambda. \end{aligned}$$

We therefore have

$$\begin{aligned} R(\delta_{n+1} - \delta_n) &= R(\beta'_{k_{n+1}} - \beta'_{k_n}) \\ &\leq R(\beta'_{k_{n+1}} - \beta_{k_{n+1}}) + R(\beta_{k_{n+1}} - \beta_{k_n}) + R(\beta'_{k_n} - \beta_{k_n}) \\ &\leq 2^{-(n+1)+1}\lambda + 2^{-n}\lambda + 2^{-n+1}\lambda \\ &= 2^{-n}\lambda(1+1+2) = 2^{-n+2}\lambda. \end{aligned}$$

Summarizing, we have $\gamma \in e$, $\delta \in f$ with $\gamma_n, \delta_n \in P(\mathscr{R})$, $R(\gamma_n) = R(\delta_n)$, and

$$\left.\begin{aligned} R(\gamma_{n+1} - \gamma_n) &\leq 3.2^{-(n+1)}\lambda \\ R(\delta_{n+1} - \delta_n) &\leq 8.2^{-(n+1)}\lambda \end{aligned}\right\} \qquad (n = 1, 2, \ldots).$$

By induction from Lemma 7.5, there exists a sequence of unitaries $v_n \in \mathscr{R}$ with $v_n \gamma_n v_n^* = \delta_n$ and

$$\begin{aligned} R(v_{n+1} - v_n) &\leq 2(R(\gamma_{n+1} - \gamma_n) + R(\delta_{n+1} - \delta_n)) \\ &\leq 11.2^{-n}\lambda. \end{aligned}$$

Therefore, $R(v_m - v_n) \to 0$ as $m, n \to \infty$ and the mapping

$$v : N \to \mathscr{R}$$
$$n \mapsto v_n$$

belongs to T. Let u be the equivalence class of v. Then, clearly, u is unitary in $\mathscr{R}^\wedge$ and $ueu^* = f$. This completes the proof.

THEOREM 6.1. *Let $\mathscr{R}$ be an irreducible $*$-regular rank ring with order k, $k \geq 4$, in which comparability holds and which satisfies property* (*PU*). *Then property* (*PU*) *extends from $\mathscr{R}$ to $\mathscr{R}^\wedge$.*

Proof. Let $e, f \in P(\mathscr{R})$ with $e \overset{a}{\sim} f$. Then $R(e) = R(f)$ and there exists unitary $u \in \mathscr{R}^\wedge$ with $ueu^* = f$. Put $w = eu^*$. Then $w \in e\mathscr{R}^\wedge f$ and

$$ww^* = e, \qquad w^*w = f.$$

This completes the proof of the theorem.

7. **An application to inductive limits.** Suppose in this section that $\mathscr{R}$ is an irreducible $*$-regular Baer ring which has order k for some $k \geq 4$ and possesses property (PU). We denote as usual by $\mathscr{R}_n$ the ring of $n \times n$ matrices over $\mathscr{R}$. We define a relation $/ \subset N \times N$ by: m/n if $n=km$ for some $k \in N$. Then $(N, /)$ is a partially ordered set which is directed up. If m/n, we define an injective ring isomorphism

$$\Psi_{nm} : \mathscr{R}_m \to \mathscr{R}_n$$

as follows: if $A=(a_{ij}) \in \mathscr{R}_m$, then $\Psi_{nm}(A)$ shall be the matrix $B=(b_{pq}) \in \mathscr{R}_n$ such that

$$b_{(s-1)k+i,(s-1)k+j} = \begin{cases} a_{ij}, & 1 \leq i, j \leq m, 1 \leq s \leq k \\ 0, & \text{otherwise,} \end{cases}$$

where $n=km$ (i.e., B has copies of A down the principal diagonal and zeros elsewhere).

Suppose that $I \subset N$ and that for $m, n \in I$ there exists $k \in I$ such that m/k, n/k. We write

$$\mathscr{R}_I = \varinjlim(I, /, \mathscr{R}_n, \Psi_{nm})$$

for the inductive limit of the system $(I, /, \mathscr{R}_n, \Psi_{nm})$. In §5, we observed that $\mathscr{R}_I$ is an irreducible $*$-regular ring with unit and normalized rank function. That $\mathscr{R}_I$ has comparability and possesses property (PU) are consequences of the following theorem.

THEOREM 7.1. *Let $e, f \in P(\mathscr{R}_I)$. Then:*

(1) *$R(e)=R(f) \Rightarrow e \overset{*}{\sim} f$.*

(2) *$R(e) \leq R(f)$ implies that there exists $f_1 \in P(\mathscr{R}_I)$ with $f_1 \leq f$ and $R(e)=R(f_1)$.*

Proof. (1) For some $n, m \in I$ and projections $E \in \mathscr{R}_n$, $F \in \mathscr{R}_m$ we have $e=\rho_{(E,n)}$, $f=\rho_{(F,m)}$. Let $k \in I$ with n/k, m/k and let $\Psi_{kn}E=E' \in P(\mathscr{R}_k)$, $\Psi_{km}F=F' \in P(\mathscr{R}_k)$. We have

$$\rho_{(E,n)} = \rho_{(E',k)} = e,$$
$$\rho_{(F,m)} = \rho_{(F',k)} = f,$$

and

$$R(e) = R(\rho_{(E',k)}) = R_k(E'),$$
$$R(f) = R(\rho_{(F',k)}) = R_k(F').$$

Hence $R_k(E')=R_k(F')$. Since $\mathscr{R}_k$ is an irreducible $*$-regular Baer ring, Theorems 1.4 and 3.1 imply that there exists $W \in E'\mathscr{R}_k F'$ with

$$WW^* = E', \qquad W^*W = F'.$$

Let $\rho_{(W,k)}=w \in \mathscr{R}_I$; then $w^*=\rho_{(W^*,k)}$ and

$$ww^* = \rho_{(W,k)}\rho_{(W^*,k)} = \rho_{(WW^*,k)} = \rho_{(E',k)} = e,$$
$$w^*w = \rho_{(W^*,k)}\rho_{(W,k)} = \rho_{(WW^*,k)} = \rho_{(F',k)} = f.$$

This completes the proof of (1).

(2) Again, we obtain $k \in I$ and projections $E', F' \in \mathscr{R}_k$ with $\rho_{(E',k)}=e$, $\rho_{(F',k)}=f$. Since $R(e)\le R(f)$, $R_k(E')\le R_k(F')$. Since $\mathscr{R}_k$ is an irreducible $*$-regular Baer ring, $\mathscr{R}_k$ has comparability: there exists $F_1' \in P(\mathscr{R}_k)$ with $F_1' \le F'$ and $R_k(E')=R_k(F_1')$. Let $\rho_{(F',k)}=f_1 \in \mathscr{R}_I$. Then $ff_1=\rho_{(F',k)}\rho_{(F_1',k)}=\rho_{(F'F_1',k)}=\rho_{(F_1',k)}=f_1$, so $f_1\le f$. Moreover, $R(e)=R_k(E')=R_k(F_1')=R(f_1)$.

This completes the proof.

We may now employ Theorem 6.1 to obtain that $\mathscr{R}_I\hat{}$ has property (PU). Under suitable conditions $\mathscr{R}_I\hat{}$ will be a continuous ring even if $\mathscr{R}$ is discrete.

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