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RESEARCH ARTICLE

A note on twisted moments of Dirichlet L-functions

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Abstract
In this paper, we establish an asymptotic formula for the twisted second moments of Dirichlet L-functions with one twist when averaged over all primitive Dirichlet characters of modulus R, where R is a monic polynomial in $\mathbb{F}_q[T]$.

1. Introduction

It is well known that the study of moments of the Riemann zeta-function and L-functions is an important topic in analytic number theory. It can be even argued that a great part of research in analytic number theory in the last century has been guided and motivated by this topic.

Applications of moments of L-functions appear more notably in the Lindelöf hypothesis, but also when studying proportions of zeros satisfying the Riemann hypothesis and nonvanishing at the central point of families of L-functions. For some of these applications, it is important to understand not only the moments of L-functions but also what is known as *twisted moments*.

Let χ be a Dirichlet character modulo p, where p is a prime number. The problem is then to obtain a formula for

$$\mathcal{S}(p,h) := \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, p)} \left| L\left(\frac{1}{2},\chi\right)\right|^2 \chi(h), \tag{1.1}$$

where h is a fixed prime number and the $*$ indicates a summation over all primitive Dirichlet characters modulo p. With this notation, Conrey [3, Theorem 10] proved the following.

Theorem 1.1 (Conrey [3]). *For primes p, h with $2 \le h < p$, we have that*

$$\begin{aligned}\mathcal{S}(p,h) = {} & \frac{p^{1/2}}{h^{1/2}}\mathcal{S}(h,-p) + \frac{p}{h^{1/2}}\left(\log\frac{p}{h} + \gamma - \log(8\pi)\right) + \zeta\left(\frac{1}{2}\right)^2 p^{1/2} \\ & + O\left(h + \log p + \frac{p^{1/2}}{h^{1/2}}\log p\right),\end{aligned}$$

where γ is Euler's constant and ζ is the Riemann zeta-function.

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In [8], Young extended Conrey's result as follows.

Theorem 1.2 (Young [8]). *For primes p, h with $h < p^{1-\varepsilon}$, we have that*

$$\frac{p^{1/2}}{\varphi(p)}\mathcal{S}(p,h) - \frac{h^{1/2}}{\varphi(h)}\mathcal{S}(h,-p) = \frac{p^{1/2}}{h^{1/2}}\left(\log\frac{p}{h} + \gamma - \log(8\pi)\right) + \zeta\left(\frac{1}{2}\right)^2\left(1 - 2\frac{p^{1/2}}{\varphi(p)}(1-p^{-1/2}) + 2\frac{h^{1/2}}{\varphi(h)}(1-h^{-1/2})\right) + \mathcal{E}(p,h),$$

where $\varphi(p)$ is Euler's totient function and

$$\mathcal{E}(p,h) \ll hp^{-1-\varepsilon} + h^{-C},$$

for all fixed $\varepsilon, C > 0$.

Advancing the study of twisted moments of Dirichlet L-functions, Bettin [2] showed that the error term $\mathcal{E}(p,h)$ can be extended to a continuous function with respect to the real topology. In his work, Bettin extended the known reciprocity results for twisted moments by establishing an exact formula with shifts.

More recently, there have been some interesting developments on the study of twisted second moments of Dirichlet L-functions over rational function fields. Let q be the power of an odd prime number and $\mathbb{A} = \mathbb{F}_q[T]$ the polynomials with coefficients in the finite field $\mathbb{F}_q$. In this setting, Djanković [4] proved the following.

Theorem 1.3 (Djanković [4]). *Let P, H be irreducible polynomials in $\mathbb{F}_q[T]$ and*

$$\mathcal{S}(P,H) := \sum_{\chi \pmod P}^{*} \left|L\left(\frac{1}{2},\chi\right)\right|^2 \chi(H).$$

If $H \neq P$ and $\deg(H) \leq \deg(P)$, then

$$\frac{|P|^{1/2}}{\phi(P)}\mathcal{S}(P,H) - \frac{|H|^{1/2}}{\phi(H)}\mathcal{S}(H,-P) = \frac{|P|^{1/2}}{|H|^{1/2}}\left(\deg(P) - \deg(H) - \zeta_{\mathbb{A}}\left(\frac{1}{2}\right)^2\right) + \zeta_{\mathbb{A}}\left(\frac{1}{2}\right)^2\left(1 - 2\frac{|P|^{1/2}}{\phi(P)}\left(1-|P|^{-1/2}\right) + 2\frac{|H|^{1/2}}{\phi(H)}\left(1-|H|^{-1/2}\right)\right),$$

where $L(s,\chi)$ is the Dirichlet L-function in function fields associated with the Dirichlet character χ modulo P, with $\zeta_{\mathbb{A}}(s)$ being the zeta-function for $\mathbb{F}_q[T]$, $\phi(P)$ is the Euler's totient function for polynomials and $|P| = q^{\deg(P)}$ denotes the norm of a polynomial P in $\mathbb{F}_q[T]$.

The aim of this note is to extend the above result of Djanković. In their work, they only consider Dirichlet characters modulo a monic irreducible polynomial, that is, they only prove results for prime moduli. In this note, we establish results for general moduli. In particular, we prove the following.

Theorem 1.4. *Let H and R be monic polynomials in $\mathbb{F}_q[T]$ with $(H,R) = 1$ and $\deg(H) \ll (\frac{1}{3} - \epsilon)\deg(R)$, then*

$$\frac{1}{\phi^*(R)}\sum_{\chi \pmod R}^{*} \left|L\left(\frac{1}{2},\chi\right)\right|^2 \chi(H) = \frac{1}{|H|^{\frac{1}{2}}}\frac{\phi(R)}{|R|}(\deg(R) - \deg(H)) + O\left(\frac{1}{|H|^{\frac{1}{2}}}\log\omega(R)\right). \qquad (1.2)$$

where $\omega(R)$ is the number of distinct prime factors of R, $\phi^*(R)$ denotes the number of primitive Dirichlet characters modulo R and $*$ indicates a summation over all primitive Dirichlet characters modulo R.

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2. A short overview of Dirichlet *L*-functions over function fields

In this section, we give a short overview of Dirichlet L-functions in function fields, with most of these facts stated in [6]. Let $\mathbb{F}_q$ denote a finite field with q elements, where q is a power of an odd prime and $\mathbb{A} = \mathbb{F}_q[T]$ be its polynomial ring. Furthermore, we denote by $\mathbb{A}^+$, $\mathbb{A}^+_n$ and $\mathbb{A}^+_{\leq n}$ the set of all monic polynomials in $\mathbb{A}$, the set of all monic polynomials in $\mathbb{A}$ of degree n and the set of all monic polynomials of degree at most n in $\mathbb{A}$, respectively. For $f \in \mathbb{A}$, the norm of f, $|f|$, is defined to be equal to $q^{\deg(f)}$, and $\phi(f)$, $\mu(f)$ and $\omega(f)$ denote the Euler-Totient function for $\mathbb{A}$, the Möbius function for $\mathbb{A}$ and the number of distinct prime factors of f.

For $\Re(s) > 1$, the zeta-function for $\mathbb{A}$ is defined as

$$\zeta_{\mathbb{A}}(s) = \sum_{f\in\mathbb{A}^+} \frac{1}{|f|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1} \tag{2.1}$$

where the product is over all monic irreducible polynomials in $\mathbb{A}$. Since there are q^n monic polynomials of degree n in $\mathbb{A}$, then

$$\zeta_{\mathbb{A}}(s) = \frac{1}{1 - q^{1-s}}.$$

Definition 2.1. *Let $R \in \mathbb{A}^+$. Then, a Dirichlet character modulo R is defined to be a function $\chi : \mathbb{A} \to \mathbb{C}$, which satisfies the following properties:*

(1) $\chi(AB) = \chi(A)\chi(B), \quad \forall A, B \in \mathbb{A},$
(2) $\chi(A + BR) = \chi(A), \quad \forall A, B \in \mathbb{A},$
(3) $\chi(A) \neq 0 \iff (A, R) = 1.$

A Dirichlet character χ is said to be even if $\chi(a) = 1$ for all $a \in \mathbb{F}_q^*$. Otherwise, we say that it is odd.

Definition 2.2. *Let $R \in \mathbb{A}^+$, $S|R$ and χ be a character of modulus R. We say that S is an induced modulus of χ if there exists a character χ_1 of modulus S such that*

$$\chi(A) = \begin{cases} \chi_1(A) & \text{if } (A, R) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We say χ is primitive if there is no induced modulus of strictly smaller norm than R. Otherwise, χ is said to be non-primitive. Let $\phi^*(R)$ denote the number of primitive characters of modulus R.

Definition 2.3. *Let χ be a Dirichlet character modulo R. Then, the Dirichlet L-function corresponding to χ is defined by*

$$L(s, \chi) := \sum_{f\in\mathbb{A}^+} \frac{\chi(f)}{|f|^s} \tag{2.2}$$

which converges absolutely for $\Re(s) > 1$.

To finish this section, we will state some results about multiplicative functions in function fields, which will be used throughout this paper. Taking Euler products, we see that for all $s \in \mathbb{C}$ and all $R \in \mathbb{A}$, we have

$$\sum_{E|R} \frac{\mu(E)}{|E|^s} = \prod_{P|R} \left(1 - \frac{1}{|P|^s}\right) \tag{2.3}$$

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and differentiating (2.3), we see that for all $s \in \mathbb{C}\backslash\{0\}$, we have

$$\sum_{E|R} \frac{\mu(E)\deg(E)}{|E|^s} = -\left(\prod_{P|R} 1 - \frac{1}{|P|^s}\right)\left(\sum_{P|R} \frac{\deg(P)}{|P|^s - 1}\right). \tag{2.4}$$

Lemma 2.4 [1, Lemma 4.5]. *Let $R \in \mathbb{A}^+$. We have that*

$$\sum_{P|R} \frac{\deg(P)}{|P| - 1} \ll \log \omega(R). \tag{2.5}$$

Lemma 2.5 [7, Lemma A.2.3]. *For* $\deg(R) > 1$, *we have*

$$\omega(R) \ll \frac{\log_q |R|}{\log_q \log_q |R|} \tag{2.6}$$

where the implied constant is independent of q.

Lemma 2.6. *We have*

$$2^{\omega(R)} = \sum_{E|R} |\mu(E)|. \tag{2.7}$$

Also, for any $\epsilon > 0$, we have

$$2^{\omega(R)} \ll_\epsilon |R|^\epsilon. \tag{2.8}$$

Lemma 2.7 [7, Lemma A.2.4]. *For deg$(R) > q$, we have*

$$\phi(R) \gg \frac{|R|}{\log_q \log_q |R|}. \tag{2.9}$$

Lemma 2.8 [7, Lemma A.2.5]. *For deg$(R) > q$, we have*

$$\phi^*(R) \gg \frac{\phi(R)}{\log_q \log_q |R|}. \tag{2.10}$$

Lemma 2.9 [1, Lemma 3.7]. *Let $R \in \mathbb{A}^+$ and $A, B \in \mathbb{A}$. Then,*

$$\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, R)} \chi(A)\bar{\chi}(B) = \begin{cases} \sum_{\substack{EF=R\\ F|(A-B)}} \mu(E)\phi(F) & \text{if } (AB, R) = 1, \\ 0 & \text{otherwise} \end{cases}.$$

As a Corollary, we have the following result.

Corollary 2.10 [1, Corollary 3.8]. *For all $R \in \mathbb{A}^+$, we have that*

$$\phi^*(R) = \sum_{EF=R} \mu(E)\phi(F). \tag{2.11}$$

3. Preliminary lemmas

In this section, we state and prove results that will be needed to prove Theorem 1.4. We start by stating the approximate function equation for $\left|L\left(\frac{1}{2}, \chi\right)\right|^2$.

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Lemma 3.1 [5, Lemma 2.5]. *Let χ be a primitive Dirichlet character of modulus R. Then, we have*

$$\left|L\left(\frac{1}{2},\chi\right)\right|^2 = 2\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)}} \frac{\chi(A)\bar{\chi}(B)}{|AB|^{\frac{1}{2}}} + O\left(|R|^{-\frac{1}{2}+\epsilon}\right). \tag{3.1}$$

The next lemma will be used to obtain the main term of Theorem 1.4.

Lemma 3.2 [1, Lemma 4.12]. *Let R be a monic polynomial in $\mathbb{F}_q[T]$ and let x be a positive integer. Then,*

$$\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\ (A,R)=1}} \frac{1}{|A|} = \begin{cases} \dfrac{\phi(R)}{|R|}x + O(\log\omega(R)) & \text{if } x \geq \deg(R) \\ \dfrac{\phi(R)}{|R|}x + O(\log\omega(R)) + O\left(\dfrac{2^{\omega(R)}x}{q^x}\right) & \text{if } x < \deg(R) \end{cases}.$$

The following lemmas will be used to create a suitable bound for the error term of Theorem 1.4.

Lemma 3.3. *Let F, H and R be fixed monic polynomials in $\mathbb{F}_q[T]$ where $F|R$ and let $z<\deg(R)$. Then,*

$$\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{z}{2}}(z+1)|H|}{|F|}. \tag{3.2}$$

Proof. We consider three cases, $\deg(AH)>\deg(B)$, $\deg(AH)<\deg(B)$ and $\deg(AH)=\deg(B)$, where $AH\neq B$.

If we first consider the case $\deg(AH)>\deg(B)$ and suppose that $\deg(A)=i$, then since $AH\equiv B(\text{mod } F)$ and $AH\neq B$ we have that $AH=LF+B$ for some $L\in\mathbb{A}^+$ with $\deg(L)=i+\deg(H)-\deg(F)$ and $\deg(B)=z-\deg(A)=z-i$. Thus, combining the above, we have

$$\begin{aligned}\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ \deg(AH)>\deg(B)\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} &\leq q^{-\frac{z}{2}}\sum_{i=0}^{z}\sum_{\substack{L\in\mathbb{A}^+\\ \deg(L)=i+\deg(H)-\deg(F)}}\sum_{\substack{B\in\mathbb{A}^+\\ \deg(B)=z-i}} 1 \\ &= q^{\frac{z}{2}}\sum_{i=0}^{z} q^{-i}\sum_{\substack{L\in\mathbb{A}^+\\ \deg(L)=i+\deg(H)-\deg(F)}} 1 = \frac{q^{\frac{z}{2}}|H|}{|F|}\sum_{i=0}^{z} 1 = \frac{q^{\frac{z}{2}}(z+1)|H|}{|F|}. \end{aligned} \tag{3.3}$$

Similarly, considering the case $\deg(AH)<\deg(B)$ and using similar arguments seen previously, we have

$$\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ \deg(B)>\deg(AH)\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \leq \frac{q^{\frac{z}{2}}(z+1)}{|F|}. \tag{3.4}$$

Finally, if we consider the case where $\deg(AH)=\deg(B)=i$, then $2i=\deg(ABH)=z+\deg(H)$ and so $\deg(B)=i=\frac{z+\deg(H)}{2}$. Furthermore, since $AH\equiv B(\text{mod F})$ and $AH\neq B$, then $AH=LF+B$ where $L\in$

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$\mathbb{A}$ with $\deg(L) < i - \deg(F) = \frac{z+\deg(H)}{2} - \deg(F)$. Thus, combining the above, we have

$$\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ \deg(AH)=\deg(B)\\ AH\equiv B(\mathrm{mod}\ F)\\ AH\neq B\\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \leq q^{-\frac{z}{2}} \sum_{\substack{B\in\mathbb{A}^+\\ \deg(B)=\frac{z+\deg(H)}{2}}} \sum_{\substack{L\in\mathbb{A}\\ \deg(L)<\frac{z+\deg(H)}{2}-\deg(F)}} 1$$

$$\ll \frac{|H|^{\frac{1}{2}}}{|F|} \sum_{\substack{B\in\mathbb{A}^+\\ \deg(B)=\frac{z+\deg(H)}{2}}} 1 = \frac{q^{\frac{z}{2}}|H|}{|F|}. \tag{3.5}$$

Combining all the cases proves the result. □

Lemma 3.4. *For all $R \in \mathbb{A}^+$ and $\epsilon > 0$, we have*

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)} \ll_\epsilon |R|^{\epsilon-\frac{1}{2}}. \tag{3.6}$$

Proof. For $\deg(R) \leq q$, we know, by [7, (A.2.3)], that $\frac{\phi^*(R)}{|R|} \gg 1$. Thus, for $\deg(R) \leq q$, we have

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)} \ll \frac{2^{\omega(R)}\deg(R)}{|R|^{\frac{1}{2}}} \ll \frac{2^{\omega(R)}}{|R|^{\frac{1}{2}-\epsilon}}.$$

From Lemma 2.6, we know that $2^{\omega(R)} \ll |R|^\epsilon$, thus (3.6) holds for $\deg(R) \leq q$.

For $\deg(R) > q$, we know by Lemmas 2.7 and 2.8 that

$$\phi^*(R) \gg \frac{\phi(R)}{\log_q \log_q |R|} \gg \frac{|R|}{(\log_q \log_q |R|)^2}.$$

Thus, if $\deg(R) > q$, then

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)} \ll \frac{2^{\omega(R)}\deg(R)(\log_q \log_q |R|)^2}{|R|^{\frac{1}{2}}} \ll_\epsilon \frac{2^{\omega(R)}}{|R|^{\frac{1}{2}-\epsilon}}.$$

Finally, from Lemma 2.6, we know that $2^{\omega(R)} \ll |R|^\epsilon$, then (3.6) holds for $\deg(R) > q$ and thus completes the proof. □

4. Proof of Theorem 1.4

In this section, we use results stated previously to prove Theorem 1.4.

Proof of Theorem 1.4. Using the approximate function equation Lemma 3.1, we have

$$\frac{1}{\phi^*(R)} \sum^*_{\chi(\mathrm{mod}\ R)} \left|L\left(\frac{1}{2},\chi\right)\right|^2 \chi(H) = \frac{2}{\phi^*(R)} \sum^*_{\chi(\mathrm{mod}\ R)} \sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)}} \frac{\chi(A)\bar{\chi}(B)\chi(H)}{|AB|^{\frac{1}{2}}} + O\left(|R|^{-\frac{1}{2}+\epsilon}\right). \tag{4.1}$$

Using the orthogonality relation Lemma 2.9, we have

$$\frac{2}{\phi^*(R)} \sum^*_{\chi(\mathrm{mod}\ R)} \sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)}} \frac{\chi(A)\bar{\chi}(B)\chi(H)}{|AB|^{\frac{1}{2}}} = \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\mathrm{mod}\ F)\\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}}. \tag{4.2}$$

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For the second sum on the right-hand side of (4.2), we will consider the contribution of the diagonal, $AH = B$, and the off-diagonal, $AH \neq B$, terms separately. Thus, we write

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\bmod F)\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}} = \frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\bmod F)\\ AH=B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}$$
$$+\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\bmod F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}.$$

Considering the contribution of the diagonal, $AH = B$, the double sum over all $A, B \in \mathbb{A}^+$ with $\deg(AB) < \deg(R)$, $AH = B$ and $(ABH, R) = 1$ becomes a single sum over all $A \in \mathbb{A}^+$ with $\deg(A) < \frac{1}{2}(\deg(R) - \deg(H))$ and $(AH, R) = 1$. Therefore, using the arguments stated above and Corollary 2.10, we have

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH=B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}} = \frac{2}{|H|^{\frac{1}{2}}}\sum_{\substack{A\in\mathbb{A}^+\\ \deg(A)<\frac{\deg(R)-\deg(H)}{2}\\ (AH,R)=1}}\frac{1}{|A|}. \tag{4.3}$$

Since the condition $(AH, R) = 1$ holds if and only if $(A, R) = 1$ and $(H, R) = 1$, then since we have already assumed that $(H, R) = 1$, then, in the sum, we only need to consider the condition $(A, R) = 1$. Thus,

$$\sum_{\substack{A\in\mathbb{A}^+\\ \deg(A)<\frac{\deg(R)-\deg(H)}{2}\\ (AH,R)=1}}\frac{1}{|A|} = \sum_{\substack{A\in\mathbb{A}^+\\ \deg(A)<\frac{\deg(R)-\deg(H)}{2}\\ (A,R)=1}}\frac{1}{|A|}. \tag{4.4}$$

Using Lemma 3.2 with $x = \frac{\deg(R)-\deg(H)}{2} - 1$, we have

$$\frac{2}{|H|^{\frac{1}{2}}}\sum_{\substack{A\in\mathbb{A}^+\\ \deg(A)<\frac{\deg(R)-\deg(H)}{2}\\ (AH,R)=1}}\frac{1}{|A|} = \frac{1}{|H|^{\frac{1}{2}}}\frac{\phi(R)}{|R|}(\deg(R)-\deg(H)) + O\left(\frac{1}{|H|^{\frac{1}{2}}}\log\omega(R)\right). \tag{4.5}$$

For the contribution of the off-diagonal terms, we use Lemma 3.3 to give

$$\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\bmod F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}} = \sum_{z=0}^{\deg(R)-1}\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ AH\equiv B(\bmod F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}} \ll \sum_{z=0}^{\deg(R)-1}\frac{|H|q^{\frac{z}{2}}(z+1)}{|F|} \ll \frac{|H||R|^{\frac{1}{2}}\deg(R)}{|F|}. \tag{4.6}$$

Thus using (4.6), we have

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\bmod F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{|H||R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)}\sum_{EF=R}|\mu(E)|\frac{\phi(F)}{|F|}. \tag{4.7}$$

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Combining (4.7) with Lemmas 2.6, 3.4 and the fact that $\frac{\phi(R)}{|R|} \leq 1$, we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{2^{\omega(R)}|H||R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)} \ll |H||R|^{\epsilon-\frac{1}{2}}. \tag{4.8}$$

Since H is a monic polynomial in $\mathbb{F}_q[T]$ with $(H,R)=1$ and $\deg(H) \ll (\frac{1}{3}-\epsilon)\deg(R)$, then $|H||R|^{-\frac{1}{2}+\epsilon} \ll \frac{1}{|H|^{\frac{1}{2}}} \log\omega(R)$ as $deg(R) \to \infty$. Combining the above completes the proof of Theorem 1.4. □

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