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Semi-classical Integrability, Hyperbolic Flows and the Birkhoff Normal Form

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Abstract. We prove that a Hamiltonian $p \in C^\infty(T^*\mathbf{R}^n)$ is locally integrable near a non-degenerate critical point ρ_0 of the energy, provided that the fundamental matrix at ρ_0 has rationally independent eigenvalues, none purely imaginary. This is done by using Birkhoff normal forms, which turn out to be convergent in the C^∞ sense. We also give versions of the Lewis-Sternberg normal form near a hyperbolic fixed point of a canonical transformation. Then we investigate the complex case, showing that when p is holomorphic near $\rho_0 \in T^*\mathbf{C}^n$, then Re p becomes integrable in the complex domain for real times, while the Birkhoff series and the Birkhoff transforms may not converge, *i.e.*, p may not be integrable. These normal forms also hold in the semi-classical frame.

0 Introduction

Birkhoff's theorem reduces Hamiltonians near an elliptic equilibrium to quasi-integrable systems. More precisely, let $p \in C^\infty(T^*\mathbf{R}^n)$ have a local non degenerate minimum at $\rho_0 = (x_0, \xi_0) = 0$ with non resonant frequencies $\lambda_1, \dots, \lambda_n$, *i.e.*, the fundamental matrix F_{ρ_0} defined by

$$p''_{\rho_0}(t, s) = \frac{1}{2}\sigma(t, F_{\rho_0}(s)) \tag{0.1}$$

(here the hessian p'' and the symplectic 2-form are considered as quadratic forms on $\mathbf{R}^{2n}$) has eigenvalues $\pm i\lambda_1, \dots, \pm i\lambda_1$ linearly independent over $\mathbf{Z}$, $\lambda_j > 0$. Then there is (locally near ρ_0,) a canonical transform $\kappa \in C^\infty$ preserving the origin $\rho_0 = 0$, formally defined through its Taylor series, such that

$$q(y, \eta) = p \circ \kappa(y, \eta) \sim \sum_{\alpha \in \mathbf{N}^n \setminus 0} a_\alpha \iota^\alpha, \quad \iota_j = \frac{1}{2}(\eta_j^2 + y_j^2) \tag{0.2}$$

near 0 (in the sense of Taylor series) with linear part $\sum_{j=1}^n \lambda_j \iota_j$. The function q is known as the Birkhoff normal form of p (see [BamGraPa, Bi, Gal, GiDeFoGaSim, Sj3, Vi], etc.) A theorem of C. Siegel [Si1, Si2] says that Birkhoff series are in general divergent (because of small denominators) and there is no hope to reduce p to a completely integrable system. A gigantic literature has been devoted to integrability of Hamiltonian systems; we have listed below some of the most famous references ([Ar, ArNo, CuB, Mo, Si1, Si2, SiMo], etc.) but this work has been in part inspired by [El, It and IaSj]. See also [Au] for a somewhat less conventional and more algebraic approach.

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Classification of quadratic Hamiltonians was made by Williamson [Ar, App. 6]. We know that eigenvalues of F_{ρ_0} are of the form $\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda}$. These Hamiltonians have a particularly simple normal form when the eigenvalues are all distinct and non vanishing. Assuming that F_{ρ_0} is semi-simple (diagonalizable) in suitable symplectic coordinates $(x, \xi) \in \mathbf{R}^{2n}$, the normal form is given as follows:

$$p(x,\xi) = \sum_{j=1}^{\ell} a_j x_j \xi_j + \sum_{j=1}^{m} \Big(c_j \big(x_{\ell+2j-1}\xi_{\ell+2j-1} + x_{\ell+2j}\xi_{\ell+2j} \big) + d_j \big(x_{\ell+2j-1}\xi_{\ell+2j} - x_{\ell+2j}\xi_{\ell+2j-1} \big) \Big) + \frac{1}{2} \sum_{j=\ell+2m+1}^{n} b_j (\xi_j^2 + x_j^2) \tag{0.3}$$

We call "action variables" the elementary polynomials that enter the expression (0.3). The eigenvalues λ_j of F_{ρ_0} are of the form $\pm a_j$, $\pm(c_j \pm i d_j)$, and $\pm i b_j$, with the convention $a_j,\ b_j, c_j > 0$. Here we consider the case where none of the eigenvalues λ_j is purely imaginary, *i.e.,* no b_j occur in the decomposition. We say then that p, or H_p (the Hamiltonian vector field), is hyperbolic, or of complex hyperbolic type, if we want to stress that some λ_j's are complex. Since the construction of Birkhoff series is a purely algebraic algorithm, it extends trivially to the hyperbolic, or complex hyperbolic case (provided, of course, the eigenvalues are rationally independent.)

In the analytic category, H. Ito has proved [It] that Birkhoff series and Birkhoff transforms are convergent iff the Hamiltonian is integrable, *i.e.,* the corresponding dynamical system has, locally, n Poisson commuting, analytic integrals of motion.

Complex eigenvalues occur in small oscillations around an unstable equilibrium. As a first example we consider a top spinning around its apex O, with inertial momenta $I_1 \leq I_2 < I_3$, where the principal axis of inertia corresponding to eigenvalue I_3 goes through O. For $I_1 = I_2$ (the so-called Lagrange top), the Hamiltonian is integrable at all energies, but in general there are only 2 integrals of motion. See *e.g.,* [Au] for details. When the top is spinning fast enough, the total energy is close to a minimum, and the Hamiltonian orbits (expressed in suitable Euler angles) are confined within compact energy surfaces, on quasi-invariant torii. Then the motion can be described by means of the Birkhoff normal form (0.2). Some of these torii are invariant (the KAM torii), but most of them will be eventually destroyed. When kinetic energy decreases however, we approach a critical value of the Hamiltonian, and the motion becomes unstable.

As a second example, we may consider a satellite, with inertial momenta $I_1 < I_2 < I_3$, spinning around the principal axis of inertia corresponding to the intermediate eigenvalue I_2. Again, within certain regimes, such a motion is unstable.

Then we may ask whether the Hamiltonian becomes integrable near critical energies.

In the smooth case (or in case of finite regularity), G. Belitskii, I. Bronstein and A. Kopanskii [BeKo1, BeKo2, BrKo] used recently an idea of A. Banyaga, R. de la Llave and C. Wayne [BaLlWa] to prove that, under somewhat more general conditions of non resonance, such hyperbolic (or complex hyperbolic) flows are locally integrable.

From the point of view of classical mechanics, this matter may look rather futile, since the system will leave the unstable position long before the effects of non integrablity become relevant. Divergence from equilibrium grows in general exponentially fast with time, with exception however of the trajectories sufficiently close to the stable manifold. Thus, such an improvement may be of "microlocal" nature.

In (semi-classical) quantum mechanics however, particles are reputed to tunnel in classically forbidden regions. A local minimum of the classical Hamiltonian becomes a saddle point "seen from the complex side". Consider for instance a semiclassical Schrödinger operator $P = -h^2\Delta + V(x)$ for energies E close to a non-degenerate minimum of V, $V(x_0) = 0$. The classical Hamiltonian reads $p(x,\xi) = \xi^2 + V(x)$. When extending quasi-invariant tori in $V(x) > E$, we replace p by $\widetilde{p}(x,\xi) = \xi^2 - V(x)$, which becomes hyperbolic, and it is very convenient to know, in tunneling problems (as in [MaSo, KaRo, Ro1]) that the resulting Hamiltonian, written in (hyperbolic) action-angle coordinates is completely integrable.

Our main result for integrability and Birkhoff transformations in the real C^∞ sense is to give a self-contained proof of the following :

Theorem 0.1 *Assume $p \in C^\infty$ is real, with a non-degenerate critical point at ρ_0, such that the eigenvalues $\lambda_1, \dots, \lambda_n$ of F_{ρ_0} are rationally independent, and none of them is purely imaginary. Then, in a neighborhood of $(0,0)$, there is a C^∞ canonical map κ, $\kappa(0,0) = (0,0)$, $d\kappa(0,0) = \mathrm{Id}$, and a C^∞ function q of the elementary action variables ι as in (0.3) such that $p \circ \kappa(y,\eta) = q(\iota)$.*

(Then we shall say that p has exact Birkhoff normal form, while the term "resonant" means that the relation $p \circ \kappa(y,\eta) = q(\iota)$ holds modulo flat terms at ρ_0.)

A related problem concerns conjugation of a real canonical transformation to a time-one Hamiltonian flow ; this is the so-called Lewis-Sternberg normal form [St]. A typical situation is this of the Poincaré map, and a lot of work has been devoted to the subject [Bru, Fr, BaLlWa, It, IaSj], etc.

As for the Birkhoff normal form, a central question is convergence of the process of reduction. The Lewis-Sternberg theorem was stated at the level of formal series, and a proof of convergence in the symplectic, hyperbolic case was recently given in [BaLlWa].

So let $\Phi\colon T^*\mathbf{R}^n \to T^*\mathbf{R}^n$ be a local diffeomorphism preserving the symplectic structure, $\Phi(0,0) = (0,0)$. Assume that $d\Phi(0,0)$ has eigenvalues $\lambda_1, \dots, \lambda_n$, and none of them is negative or of modulus 1. We say then that Φ is hyperbolic at $(0,0)$.

Assume also the frequencies $\lambda_1, \dots, \lambda_n$ are non resonant in the strong sense, *i.e.*,

$$\lambda_1^{m_1} \cdots \lambda_n^{m_n} = 1 \text{ for } m_j \in \mathbf{Z} \quad \Longrightarrow \quad m_j = 0 \text{ for all } j. \tag{0.4}$$

Note that if H_p is a Hamiltonian vector field, then H_p is hyperbolic in the sense above iff the time-one map $\exp H_p$ is hyperbolic, because of the formula $\kappa \circ \exp H_p \circ \kappa^{-1} = \exp H_{p\circ\kappa^{-1}}$. By a slight abuse of notations, if Φ is a map in $T^*\mathbf{R}^n$ and κ a local diffeomorphism, we denote again $\kappa \circ \Phi \circ \kappa^{-1}$ by Φ, since the conjugation is simply a change of variables, both in source and target space. Loosely speaking, a Birkhoff normal form for p gives a Sternberg normal form for $\exp H_p$. This is the main idea in the following:

Theorem 0.2 *Let Φ be as above, satisfying* (0.4)*. Then there is a smooth function $q(\iota)$ defined in a neighborhood $(0,0)$, depending on the action variables ι alone such that $\Phi(\rho) = \exp H_q(\iota)$.*

This theorem has the following semi-classical counterpart. Let U be an elliptic h-Fourier Integral Operator (FIO for short) of order 0, defined microlocally near ρ_0, and associated with the canonical transformation Φ as in Theorem 0.2. Let $I = (I_1, \cdots, I_n)$ be the semi-classical Weyl quantization of the action variables ι_j.

Theorem 0.3 *Let U be as above, whose canonical transformation verifies* (0.4)*. Then there exists a classical symbol $F(\iota, h) = F_0(\iota) + hF_1(\iota) + h^2F_2(\iota) + \cdots$, $F_0(\iota) = q(\iota)$, such that $U = e^{iF(I,h)/h}$ microlocally near ρ_0.*

(For terminology and basic results on FIO's see Appendix A.2.) Thus we specialize the result of [IaSj] in the hyperbolic case. Such a normal form may be useful when studying the quantization of some billiard maps as in [SjZw].

Next we turn to the holomorphic case, and focus on the reduction of Hamiltonians. (See [It] for a discussion on necessary and sufficient conditions ensuring that such Hamiltonians are integrable.)

Again the problem arises naturally in semi-classical quantum mechanics. As an example, consider $p(x, \xi)$ real analytic near $\rho_0 = (0,0) \in \mathbf{R}^{2n}$, with a non-degenerate minimum at ρ_0, and let $\pm i\lambda_1, \ldots, \pm i\lambda_n$ be the purely imaginary eigenvalues of F_{ρ_0}, $\lambda_j > 0$, which we assume again rationally independent. When trying to construct the solution of some eikonal equation, one introduces $\widetilde{p}(z, \zeta) = -p(z - \zeta, i\zeta)$ as an holomorphic function on a neighborhood of 0 in $T^*\mathbf{C}^n$. Then $\widetilde{p}$ verifies the hypotheses above, namely if $\widetilde{p}_2$ denotes the quadratic part of $\widetilde{p}$, then $\langle d\widetilde{p}_2(0,0), (z,\zeta)\rangle = 2\sum_{j=1}^n \lambda_j z_j(\zeta_j - \frac{1}{2}z_j)$. This situation is met when studying microlocal properties of eigenfunctions for certain PDO's (see [MaSo]).

As usual in complex symplectic geometry, it is convenient to distinguish between several symplectic structures ; we send the reader to [Sj1], [MeSj] for the theory, and recall here simply the following fact: $\mathbf{C}^{2n}$ is endowed with the complex canonical 2-form $\sigma_{\mathbf{C}} = \sum_{j=1}^n d\zeta_j \wedge dz_j$, $z_j = x_j + iy_j$, $\zeta_j = \xi_j + i\eta_j$, which makes it a symplectic space, and two real symplectic 2-forms: $\mathrm{Re}\sigma_{\mathbf{C}} = \sum_{j=1}^n d\xi_j \wedge dx_j - d\eta_j \wedge dy_j$, and $\mathrm{Im}\sigma_{\mathbf{C}} = \sum_{j=1}^n d\xi_j \wedge dy_j + d\eta_j \wedge dx_j$. Concerning integrability in the complex domain, we are naturally led to introduce the following:

Definition 0.4 Let $p(z, \zeta)$ be a complex Hamiltonian near ρ_0 and have a non degenerate critical point at ρ_0. We say that p is R-integrable iff there is a $\mathrm{Re}\,\sigma_{\mathbf{C}}$-canonical map $\kappa \in C^\infty$ around ρ_0 and a C^∞ function $q(\iota')$ such that $\mathrm{Re}\, p \circ \kappa(z, \zeta) = q(\iota')$. (Here ι' stand for the real and imaginary part of the complex action variables as in (0.3), and Poisson commute for the real symplectic structure.)

Equivalently, there exists an $\mathrm{Im}\,\sigma_{\mathbf{C}}$-canonical map $\widetilde{\kappa} \in C^\infty$, and a C^∞ function $\widetilde{q}(\iota')$, such that $\mathrm{Im}\, p \circ \widetilde{\kappa}(z,\zeta) = \widetilde{q}(\iota')$. We could define analogously an I-integrable Hamiltonian by requiring that $\mathrm{Im}\, p \circ \kappa(z, \zeta) = q(\iota')$ for some $\mathrm{Re}\,\sigma_{\mathbf{C}}$-canonical map κ. Roughly speaking, an R- (resp. I-) integrable Hamiltonian is integrable for real

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(resp. imaginary) times. If p is holomorphic and $\mathbf{C}$-integrable, (*i.e.*, with respect to $\sigma_{\mathbf{C}}$), then it is both R and I-integrable, but there are not so many Hamiltonians because of Siegel's result. We have:

Theorem 0.5 *Let $p(z,\zeta)$ be a complex Hamiltonian near ρ_0 and have a non degenerate critical point at ρ_0. Assume that $\overline{\partial}_{(z,\zeta)}p = \mathcal{O}(|z,\zeta|^\infty)$, and that the fundamental matrix F_{ρ_0} (in the holomorphic sense) has no purely imaginary eigenvalues. Then p is R-integrable in a complex neighborhood of ρ_0. Moreover, if κ denotes the $\operatorname{Re}\sigma_{\mathbf{C}}$-canonical map as in Definition* (0.4)*, we have $\overline{\partial}_{(z,\zeta)}\kappa = \mathcal{O}(|z,\zeta|^\infty)$, and $\kappa^*(\sigma_{\mathbf{C}}) = \sigma_{\mathbf{C}} + \mathcal{O}(|z,\zeta|^\infty)$.*

Our result still looks quite poor, in the sense that we lose on the way almost every track of analyticity; reduction to the normal Birkhoff form holds only modulo functions with $\overline{\partial}$ of rapid decrease near ρ_0. Of course again, we cannot expect convergence of Birkhoff series or Birkhoff transforms in a full complex neighborhood of ρ_0, except in the one dimensional case, see [It] and [HeSj2, App. B]. A more thorough approach should rely on resurgence theory for functions of several complex variables as in [Ec]; this would of course help to understand better how the system switches from integrability to non-integrability when moving around the origin in complex directions. (See also [Ro2] for another type of result, where we study integrability and monodromy of κ, as a map defined on the covering in $T^*\mathbf{C}^n$, of the complement of the stable and unstable manifolds.) The paper is organized as follows:

In Section 1 we prove Theorem 0.1 for Hamiltonians and discuss briefly the case of a closed hyperbolic orbit. Then we treat the semi-classical case.

Section 2 is devoted to the Lewis-Sternberg normal form for canonical transforms and Fourier Integral Operators quantizing a Poincaré map. We sketch a slightly different proof for the normal form of canonical maps given in [BaLlWa].

In Section 3, we extend the Birkhoff normal form of Theorem 0.1 and the Sternberg normal form of Theorem 0.2 to the parameter dependent case, in the spirit of [IaSj].

In Section 4, we recall some well known facts about complex symplectic geometry and prove first the center stable/unstable manifold theorem in the almost holomorphic case. Then we turn to the proof of Theorem 0.5, which is very similar to that of Theorem 0.1. We conclude with some remarks on monodromy.

In the Appendix, we first recall a simple way of constructing Birkhoff series, including parameters. We conclude with some review on FIO's.

We close this Introduction by listing some open problems:

(1) What can be said about integrability when $\operatorname{Spec} F_{\rho_0} \cap i\mathbf{R} = \{i\lambda, -i\lambda\}$, $\lambda > 0$, *i.e.*, when the center-manifold associated with purely imaginary eigenvalues is of dimension 2? For higher dimensions, it is known that KAM torii can occur (see [Gr]).

(2) What can be said about integrability in the (complex-) hyperbolic case, when some of the frequencies are resonant, or more precisely when the equilibrium point ρ_0 is "simply resonant" in the sense of [It]?

(3) Do our results extend to time-dependent, (or non autonomous) Hamiltonians? (See [Sie] and again [It].)

1 Birkhoff Normal Form and Integrability: The Real Case

We discuss here "convergence" of Birkhoff normal forms for smooth, real valued Hamiltonians near a fixed point ρ_0.

1.1 Classical Integrability

Let p be a real valued Hamiltonian with a nondegenerate critical point $\rho_0 \in T^*\mathbf{R}^n$ of complex hyperbolic type. First we recall some well-known facts about the geometry of bicharacteristics of p near ρ_0 (see [Ch2, Sj2], though there seems to be some confusion in [Ch2, p. 707] between the invariant manifolds for the vector field X and its linear part X_0, the main arguments show up already in that paper.) Then we discuss a solvability problem for H_p in the class of smooth, flat functions at ρ_0. At last we prove Theorem 0.1 by the method of homotopy. Let F_{ρ_0} denote the fundamental matrix of p at $\rho_0 = (0, 0)$,

$$2F_{\rho_0} = \begin{pmatrix} \dfrac{\partial^2 p}{\partial x \partial \xi} & \dfrac{\partial^2 p}{\partial \xi^2} \\ -\dfrac{\partial^2 p}{\partial x^2} & -\dfrac{\partial^2 p}{\partial x \partial \xi} \end{pmatrix} (\rho_0) = J \,\mathrm{Hess}(p)(\rho_0) \tag{1.1}$$

(where J is the symplectic matrix), verifying

$$\mathrm{Hess}(p)(\rho_0)(t, s) = p''_{\rho_0}(t, s) = \frac{1}{2}\sigma(t, F_{\rho_0}(s)).$$

The factor $\frac{1}{2}$ is for convenience of notations. Since p''_{ρ_0} is non degenerate, F_{ρ_0} has no zero eigenvalues. As we are interested in the Birkhoff normal form, we readily assume that F_{ρ_0} is diagonalizable. Let $\Lambda_\pm \subset T_{\rho_0}\mathbf{R}^{2n}$ be the sum of all eigenspaces corresponding to eigenvalues with positive (resp. negative) real parts.

Assuming that F_{ρ_0} has no purely imaginary eigenvalues, in suitable symplectic coordinates $(x, \xi) \in \mathbf{R}^{2n}$, the normal form for the quadratic part p_2 of p at ρ_0 is given as in (0.3) with no elliptic terms, i.e. $\ell + 2m = n$. So Λ_+ is the sum of eigenspaces associated with $\lambda_j = a_j$, $(j = 1, \dots, \ell\,)$ $\lambda_j = c_{j-\ell} \pm i d_{j-\ell}$, $(j = \ell + 1, \dots, \ell + m)$, and Λ_- is the sum of eigenspaces associated with the corresponding $-\lambda_j$, and $\Lambda_+ \oplus \Lambda_- = T_{\rho_0}\mathbf{R}^{2n}$. In these symplectic coordinates $\Lambda_+ = \{\xi = 0\}$, $\Lambda_- = \{x = 0\}$, and F_{ρ_0} has block diagonal form, the diagonal terms $(\lambda_1, \dots, \lambda_\ell)$, the 2×2 matrices $\left(\begin{smallmatrix} c_j & d_j \\ -d_j & c_j \end{smallmatrix}\right)$ $(j = \ell + 1, \dots, \ell + m)$, the diagonal terms $(-\lambda_1, \dots, -\lambda_\ell)$, and the 2×2 matrices $\left(\begin{smallmatrix} -c_j & d_j \\ -d_j & -c_j \end{smallmatrix}\right)$ $(j = \ell + 1, \dots, \ell + m)$ respectively, which is the so-called Cartan decomposition. Note that Λ_+ and Λ_- are dual spaces for the symplectic form on $\mathbf{R}^{2n}$.

To simplify notations, we shall sometimes introduce complex symplectic coordinates

$$
(1.3)\qquad
\begin{aligned}
z_{\ell+2j} &= \frac{1}{\sqrt{2}}(x_{\ell+2j} + i x_{\ell+2j-1}), &\qquad \zeta_{\ell+2j} &= \frac{1}{\sqrt{2}}(\xi_{\ell+2j} - i\xi_{\ell+2j-1}),\\
z_{\ell+2j-1} &= \frac{1}{\sqrt{2}}(x_{\ell+2j} - i x_{\ell+2j-1}), &\qquad \zeta_{\ell+2j-1} &= \frac{1}{\sqrt{2}}(\xi_{\ell+2j} + i\xi_{\ell+2j-1}),
\end{aligned}
$$

$j = 1, \dots, m$. (the variables x_j and ξ_j being as in (0.3)). Further we denote x_j for z_j, ξ_j for the dual coordinate ζ_j, and eventually label the collection of these symplectic coordinates, so that

$$
(1.4)\qquad H_{p_2} = \sum_{j=1}^{n} \lambda_j (x_j \frac{\partial}{\partial x_j} - \xi_j \frac{\partial}{\partial \xi_j})
$$

or

$$
p''_{\rho_0}(t,s) = \sum_{j=1}^{n} \lambda_j \big(t_{x_j} s_{\xi_j} + t_{\xi_j} s_{x_j} \big) .
$$

Of course, we shall keep in mind that the complexification here is only formal, since no analyticity is assumed; this is no more than the usual identification consisting for instance in taking complex coordinates which diagonalize a rotation in the plane.

Now we turn to the non-linear case and recall the stable-unstable manifold theorem. This theorem has a long history, see *e.g.*, [Ha] in the differentiable case, [Ch2] or [Ne] for a proof based on Sternberg's linearization theorem, [AbMar, AbRob, HiPuSh] and references therein for more general statements. Note that these results are generally stated without symplectic structure, but most of them easily extend to this setting. See however [Sj2, App.] in the analytic category, and Theorem 4.2 below for the almost holomorphic case.

Theorem 1.1 *With notations above, in a neighborhood of ρ_0, there are H_p-invariant Lagrangian manifolds $\mathcal{J}_\pm$ passing through ρ_0, such that $T_{\rho_0}(\mathcal{J}_\pm) = \Lambda_\pm$. Within $\mathcal{J}_+$ (resp. $\mathcal{J}_-$), ρ_0 is repulsive (resp. attractive) for H_p, and $p|_{\mathcal{J}_\pm} = 0$. We can also find real symplectic coordinates, denoted again by (x,ξ), such that their differential at ρ_0 verifies $d(x,\xi)(\rho_0) = \mathrm{Id}$, and $\mathcal{J}_+ = \{\xi = 0\}$, $\mathcal{J}_- = \{x = 0\}$. In these coordinates*

$$
(1.5)\qquad p(x,\xi) = \langle A(x,\xi)x, \xi\rangle
$$

where $A(x,\xi)$ is a real, $n \times n$ matrix with C^∞ coefficients, $A_0 = dA(\rho_0) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with the convention that if λ_j is complex, diag$(\lambda_j, \overline{\lambda}_j)$ denotes $\begin{pmatrix} c_j & d_j \\ -d_j & c_j \end{pmatrix}$.

It follows that

$$
(1.6)\qquad H_p = A_1(x,\xi)x \cdot \frac{\partial}{\partial x} - A_2(x,\xi)\xi \cdot \frac{\partial}{\partial \xi}
$$

with $A_j(x,\xi) = A_0 + \mathcal{O}\big(x,\xi\big)$, $A_0 = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$, $A_1(x,\xi) = A(x,\xi) + {}^t\partial_\xi A(x,\xi)\cdot \xi$, $A_2(x,\xi) = {}^tA(x,\xi) + \partial_x A(x,\xi)\cdot x$, and $\mathrm{Spec}\, A(x,\xi) = \mathrm{Spec}\, {}^tA(x,\xi) \subset \mathbf{R}^+$. Possibly after relabeling the coordinates, we may assume $0 < \mathrm{Re}\,\lambda_1 \le \cdots \le \mathrm{Re}\,\lambda_n$.

Now we describe the flow of H_p, using Theorem 1.1. Let $\|\cdot\|$ denote the usual euclidean norm on $\mathbf{R}^n$. We put

$$B_0 = \int_0^\infty e^{-s({}^tA_0)} e^{-sA_0}\, ds.$$

which is a positive definite symmetric matrix with the property ${}^tA_0B_0 + B_0A_0 = \mathrm{Id}$. In the present case where A_0 is diagonalizable,

$$B_0 = \mathrm{diag}\big(\lambda_1, \cdots, \lambda_l, \frac{1}{2}c_{\ell+1}, \frac{1}{2}c_{\ell+1}, \cdots, \frac{1}{2}c_{\ell+m}, \frac{1}{2}c_{\ell+m}\big).$$

If $\|x\|_0^2 = \langle B_0x, x\rangle$ is the corresponding norm, then

$$A_0x \cdot \partial_x\|x\|_0^2 = \|x\|^2, \quad A_0\xi \cdot \partial_\xi\|\xi\|_0^2 = \|\xi\|^2. \tag{1.7}$$

It follows from this and (1.6) that if $\|x\|_0^2 + \|\xi\|_0^2 \le \delta^2$, for some $\delta > 0$ small enough, then

$$\frac{d}{dt}\|x\|_0^2 = H_p\|x\|_0^2 \ge C\|x\|^2, \quad -H_p\|\xi\|_0^2 \ge C\|\xi\|^2, \quad C > 0.$$

For $\delta > 0$, we define the outgoing region

$$\Omega_\delta^{\mathrm{out}} = \{(x,\xi) : \|\xi\|_0 < 2\|x\|_0,\ \|x\|_0^2 + \|\xi\|_0^2 < \delta^2\}$$

and let $\partial\Omega_\delta^{\mathrm{out}}$ denote its boundary. Let $t \mapsto (x(t), \xi(t)) = \exp tH_p\big(x(0), \xi(0)\big)$ be an integral curve of H_p with $\rho = \big(x(0), \xi(0)\big) \in \Omega_\delta^{\mathrm{out}}$. We have

$$\dot{x}(t) = A_1(x(t), \xi(t))x(t),\ \dot{\xi}(t) = -A_2(x(t), \xi(t))\xi(t).$$

So when $\rho \in \Omega_\delta^{\mathrm{out}}$, $\|x(t)\|_0$ is increasing and $\|\xi(t)\|_0$ decreasing as long as $\big(x(t), \xi(t)\big) \in \Omega_\delta^{\mathrm{out}}$, and moreover there is $C > 0$ such that for $\delta > 0$ sufficiently small and all $t \in \mathbf{R}$:

$$e^{-\operatorname{Re}\lambda_+(t)t}e^{-C\delta|t|}\|\xi(0)\|_0 \le \|\xi(t)\|_0 \le e^{-\operatorname{Re}\lambda_-(t)t}e^{C\delta|t|}\|\xi(0)\|_0 \tag{1.8}$$

$$e^{\operatorname{Re}\lambda_-(t)t}e^{-C\delta|t|}\|x(0)\|_0 \le \|x(t)\|_0 \le e^{\operatorname{Re}\lambda_+(t)t}e^{C\delta|t|}\|x(0)\|_0 \tag{1.9}$$

with the convention $\lambda_+(t) = \lambda_n$ and $\lambda_-(t) = \lambda_1$ for $t > 0$, $\lambda_+(t) = \lambda_1$ and $\lambda_-(t) = \lambda_n$ for $t < 0$. It follows that for any $\delta_0 > 0$, there is $\delta_1 > 0$ (say $\delta_1 = \delta_0/2$), such that if $\rho \in \Omega_{\delta_1}^{\mathrm{out}}$, then $\exp(-tH_p)(\rho) \in \Omega_{\delta_0}^{\mathrm{out}}$, $t \ge 0$, until the path meets $\partial\Omega_{\delta_0}^{\mathrm{out}} \cap \{\|\xi\|_0 = 2\|x\|_0\}$. For each $\rho \in \Omega_{\delta_1}^{\mathrm{out}}$, we define the hitting time

$$T_-^{\mathrm{out}}(\rho) = \inf\{t > 0 : \|\xi(-t)\|_0 \ge 2\|x(-t)\|_0\}, \tag{1.10}$$

i.e., the time for the path $\exp(-tH_p)(\rho)$ to reach the cone $\|\xi\|_0 = 2\|x\|_0$. Since $\exp(-tH_p)(\rho)$ is a C^∞ function of ρ and t, it follows from the implicit function theorem that $T_-^{\rm out}(\rho)$ is a C^∞ function of $\rho \in \Omega_{\delta_1}^{\rm out} \setminus \mathcal{J}_+$. For $\rho = (x, 0) \in \mathcal{J}_+$, we set $T_-^{\rm out}(\rho) = +\infty$, and we leave it undefined for $\rho = 0$. Similarly, for $\rho \in \Omega_{\delta_1}^{\rm out}$ we define

$$T_+^{\rm out}(\rho) = \inf\{t > 0 : \|x(t)\|_0^2 + \|\xi(t)\|_0^2 \geq \delta_0^2\}, \tag{1.11}$$

to be the time for the path $\exp(tH_p)(\rho)$ to leave the ball $\|x\|_0^2 + \|\xi\|_0^2 < \delta_0^2$. Again, $T_+^{\rm out}(\rho)$ is a C^∞ function of $\rho \in \Omega_{\delta_1}^{\rm out}$. Moreover, there is $\tau > 0$ such that for all $\rho \in \Omega_{\delta_1}^{\rm out}$, $\exp(tH_p)(\rho) \notin \Omega_{\delta_0}^{\rm out}$ for $T_-^{\rm out}(\rho) \leq t \leq T_-^{\rm out}(\rho) + \tau$. Since we are interested in local properties of the flow near ρ_0, we can modify, without loss of generality, $p(x, \xi)$ outside a small neighborhood of ρ_0 such that the path $\exp(tH_p)(\rho)$, $\rho \in \Omega_{\delta_1}^{\rm out}$, will never enter $\Omega_{\delta_0}^{\rm out}$ again after time $T_+^{\rm out}(\rho)$, *i.e.*, we may assume $\tau = +\infty$. From now on, we change notation δ_0 and δ_1 to δ for simplicity, keeping in mind that δ is a sufficiently small, but fixed positive number.

We define in a similar way the incoming region

$$\Omega_\delta^{\rm in} = \big\{(x, \xi) : \|x\|_0 < 2\|\xi\|_0,\ \|x\|_0^2 + \|\xi\|_0^2 < \delta^2\big\} \tag{1.12}$$

and the hitting times $T_\pm^{\rm in}(\rho)$. More precisely,

$$T_-^{\rm in}(\rho) = \inf\{t > 0 : \|x(-t)\|_0^2 + \|\xi(-t)\|_0^2 \geq \delta^2\} \tag{1.13}$$

$$T_+^{\rm in}(\rho) = \inf\{t > 0 : \|x(t)\|_0 \geq 2\|\xi(t)\|_0\} \tag{1.14}$$

As above, we may assume that the flow starting from any point $\rho \in \mathbf{R}^{2n}$ crosses at most once the region $\Omega_\delta = \Omega_\delta^{\rm in} \cup \Omega_\delta^{\rm out}$. Then estimates (1.8) and (1.9) hold for all $(x, \xi) \in \Omega_\delta$, and all $t \in \mathbf{R}$ provided $\big(x(t), \xi(t)\big) \in \Omega_\delta$.

Now let I denote the ideal of $C^\infty(\mathbf{R}^{2n})$ consisting in all smooth functions vanishing at ρ_0. We want to solve the homological equation $H_p f = g$ in I^∞. This is of course essentially well-known (see *e.g.*, [GuSc, p. 175] for analogous results). So let $\chi^{\rm out} + \chi^{\rm in} = 1$ be a smooth partition of unity in the unit sphere $\mathbf{S}^{2n-1}$ such that supp $\chi^{\rm out} \subset \{\|\xi\|_0 < 2\|x\|_0\}$, supp $\chi^{\rm in} \subset \{\|x\|_0 < 2\|\xi\|_0\}$. We extend $\chi^{\rm out}$, $\chi^{\rm in}$ as homogeneous functions of degree 0 on $T^*\mathbf{R}^n \setminus \rho_0$.

Proposition 1.2 *Let ρ_0 be an hyperbolic fixed point for p as above, and $g \in I^\infty$. Let*

$$f^{\rm out}(\rho) = \int_{-\infty}^0 \big(\chi^{\rm out} g\big) \circ \exp(tH_p)(\rho)\, dt,\ \ f^{\rm in}(\rho) = -\int_0^\infty \big(\chi^{\rm in} g\big) \circ \exp(tH_p)(\rho)\, dt$$

Then $f = f^{\rm out} + f^{\rm in} \in I^\infty$ solves $H_p f = g$.

Proof We treat the case of $f^{\rm out}$, this of $f^{\rm in}$ is similar. Let $\delta_0 > 0$ small enough, and $\Omega_{\delta_1}^{\rm out/in}$ be as above. Without loss of generality, we may assume supp $g \subset \Omega_{\delta_0} = \Omega_{\delta_0}^{\rm out} \cup \Omega_{\delta_0}^{\rm in}$, so $\operatorname{supp}(\chi^{\rm out} g) \subset \Omega_{\delta_0}^{\rm out}$. Then it is easy to see that

$$(\operatorname{supp} f^{\rm out}) \cap \Omega_{\delta_1} \subset \Omega_{\delta_1}^{\rm out},$$

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so we will assume $\rho \in \Omega^{\mathrm{out}}_{\delta_1}$, and as above write δ for δ_0 or δ_1. If $\rho \in \Omega^{\mathrm{out}}_{\delta} \setminus \mathcal{J}_+$, we have $f^{\mathrm{out}}(\rho) = \int_{-T^{\mathrm{out}}_-(\rho)}^{0} \big(\chi^{\mathrm{out}} g\big) \circ \exp(tH_p)(\rho)dt$, since $\exp(tH_p)(\rho) \notin \operatorname{supp} \chi^{\mathrm{out}}$ for $t < -T^{\mathrm{out}}_-(\rho)$. Furthermore,

$$H_p f^{\mathrm{out}}(\rho) = \int_{-\infty}^{0} \frac{d}{dt}\Big(\big(\chi^{\mathrm{out}} g\big) \circ \exp(tH_p)(\rho)\Big)\, dt = (\chi^{\mathrm{out}} g)(\rho).$$

When $\rho \in \mathcal{J}_+$, $\exp(tH_p)(\rho) \to 0$ when $t \to -\infty$ and the integral makes sense because of (1.9) and the fact that $g(\rho) = \mathcal{O}(\rho)$, as $\rho \to 0$. Again $H_p f^{\mathrm{out}}(\rho) = \chi^{\mathrm{out}} g(\rho)$. We are left to show that $f^{\mathrm{out}} \in I^{\infty}$. Because of (1.9) and $\|\xi(t)\|_0 \leq 2\|x(t)\|_0$ in supp χ^{out}, f^{out} is continuous and vanishes at $\rho = 0$. To show that $f^{\mathrm{out}} \in C^1$, we write, following [IaSj]:

$$d\Big((\chi^{\mathrm{out}} g) \circ \exp(tH_p)(\rho)\Big) = \Big(d(\chi^{\mathrm{out}} g)(\exp(tH_p)(\rho)\Big) \circ d\exp(tH_p)(\rho) \tag{1.15}$$

so we need to examine the evolution of $d\kappa_t(\rho) = d\exp(tH_p)(\rho)$ along the integral curve κ_t of H_p starting at ρ. Differentiating $\partial_t \kappa_t(\rho) = H_p\big(\kappa_t(\rho)\big)$ we find

$$\partial_t d\kappa_t(\rho) = \frac{\partial H_p}{\partial \rho}(\kappa_t(\rho)) \circ \big(d\kappa_t(\rho)\big), \quad d\kappa_0(\rho) = \mathrm{Id} \tag{1.16}$$

with $\frac{\partial H_p}{\partial \rho}(\rho) = 2F_{\rho_0} + \mathcal{O}(\rho)$, and the Gronwall lemma applied to (1.16), as in (1.8) and (1.9), gives, for $\kappa_t(\rho) \in \Omega^{\mathrm{out}}_{\delta}$ and all $t \leq 0$:

$$e^{-(\operatorname{Re}\lambda_1 - C\delta)t} \leq \|d\xi_t(\rho)\| \leq e^{-(\operatorname{Re}\lambda_n + C\delta)t} \tag{1.17}$$

$$e^{(\operatorname{Re}\lambda_n + C\delta)t} \leq \|dx_t(\rho)\| \leq e^{(\operatorname{Re}\lambda_1 - C\delta)t} \tag{1.18}$$

so $d\kappa_t(\rho) = \mathcal{O}\big(e^{-(\operatorname{Re}\lambda_n + C\delta)t}\big)$.

On the other hand, g being flat at 0, $d(\chi^{\mathrm{out}} g)(\exp(tH_p)(\rho) = \mathcal{O}(\|x_t(\rho)\|^N)$ for any N, so taking N large enough, we see that $d\big((\chi^{\mathrm{out}} g) \circ \exp(tH_p)(\rho)\big)$ is integrable, so $f^{\mathrm{out}} \in C^1$, and vanishes at 0. To continue, we take the partial derivative of (1.16) with respect to ρ_j, $j = 1, \cdots, 2n$ and write

$$\partial_t \frac{\partial}{\partial \rho_j} d\kappa_t(\rho) - \frac{\partial H_p}{\partial \rho}(\kappa_t(\rho)) \circ \Big(\frac{\partial}{\partial \rho_j}(d\kappa_t(\rho)\Big) = F_j(t,\rho)$$

with

$$F_j(t,\rho) = \sum_{k=1}^{2n} \frac{\partial^2 H_p}{\partial \rho_k \partial \rho}(\kappa_t(\rho)) \frac{\partial}{\partial \rho_j} \kappa_{t,k}(\rho) \circ d\kappa_t(\rho).$$

Using the group property, we write (1.16) as

$$\partial_t d\kappa_{t-\tilde{t}}\big(\kappa_{\tilde{t}}(\rho)\big) \circ d\kappa_{\tilde{t}}(\rho) = \frac{\partial H_p}{\partial \rho}\Big(\kappa_{t-\tilde{t}}\big(\kappa_{\tilde{t}}(\rho)\big)\Big) \circ d\kappa_{t-\tilde{t}}\big(\kappa_{\tilde{t}}(\rho)\big) \circ d\kappa_{\tilde{t}}(\rho), \tag{1.19}$$

$$d\kappa_0(\rho) = \mathrm{Id}\,. \tag{1.20}$$

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Since $\kappa_{\tilde t}$ is a canonical map, $d\kappa_{\tilde t}$ is invertible, so

$$\partial_t d\kappa_{t-\tilde t}\big(\kappa_{\tilde t}(\rho)\big) = \frac{\partial H_p}{\partial\rho}\Big(\kappa_{t-\tilde t}\big(\kappa_{\tilde t}(\rho)\big)\Big)\circ d\kappa_{t-\tilde t}\big(\kappa_{\tilde t}(\rho)\big), \quad d\kappa_0(\rho)=\mathrm{Id}\,.$$

So we recognize $d\kappa_{t-\tilde t}\big(\kappa_{\tilde t}(\rho)\big)$, $d\kappa_{\tilde t-\tilde t}(\rho)=\mathrm{Id}$ as the fundamental matrix of our $2n\times 2n$ system of ordinary differential equations, and since $\frac{\partial}{\partial\rho_j}d\kappa_t(\rho)|_{t=0}=0$, Duhamel's principle gives:

$$\frac{\partial}{\partial\rho_j}d\kappa_t(\rho)=\int_0^t d\kappa_{t-\tilde t}\big(\kappa_{\tilde t}(\rho)\big)\circ F_j(\tilde t,\rho)\,d\tilde t.$$

From (1.17) and (1.18) we find the estimate $F_j(\tilde t,\rho)=\mathcal{O}\big(e^{-2(\mathrm{Re}\,\lambda_n+C\delta)\tilde t}\big)$, and by integration

$$\frac{\partial}{\partial\rho_j}d\kappa_t(\rho)=\mathcal{O}\big(e^{-2(\mathrm{Re}\,\lambda_n+C\delta)t}\big). \tag{1.21}$$

On the other hand, differentiating (1.15) with respect to ρ_j we get

$$\begin{aligned}\frac{\partial}{\partial\rho_j}d\big((\chi^{\mathrm{out}}g)\circ\kappa_t(\rho)\big) &= d(\chi^{\mathrm{out}}g)\big(\kappa_t(\rho)\big)\circ\frac{\partial}{\partial\rho_j}d\kappa_t(\rho)\\ &\quad+\sum_{k=1}^{2n}\frac{\partial}{\partial\rho_k}d\big(\chi^{\mathrm{out}}g\big)\big(\kappa_t(\rho)\big)\frac{\partial}{\partial\rho_j}\kappa_{t,k}(\rho)\circ d\kappa_t(\rho)\end{aligned}$$

Using (1.21), and again (1.17), (1.18), the estimates

$$d(\chi^{\mathrm{out}}g)\circ\big(\kappa_t(\rho)\big),\quad \frac{\partial}{\partial\rho_k}d\big(\chi^{\mathrm{out}}g\big)\big(\kappa_t(\rho)\big)=\mathcal{O}\big(\|x_t(\rho)\|^N\big)$$

ensure once more the integrability of $\frac{\partial}{\partial\rho_j}d\big((\chi^{\mathrm{out}}g)\circ\kappa_t(\rho)\big)$, so $f^{\mathrm{out}}\in C^2$ and we can see that its second derivatives vanish at 0. The argument carries over easily by induction, so the Proposition is proved. ∎

Now we are ready to prove Theorem 0.1, by combining the Birkhoff normal form (see *e.g.,* Appendix for a simple proof) and a deformation argument. When p has a non-degenerate critical point with non-resonant frequencies, we know that there is a smooth canonical transform κ between neighborhoods of 0, leaving fixed the origin, such that $p\circ\kappa(x,\xi)=q_0(\iota)+r(x,\xi)$, where $\iota=(\iota_1,\cdots,\iota_n)$ are the action variables as in (0.3), and $r\in I^\infty$ depends also on the corresponding dual (angle) variables. The Hamiltonian $q_0(\iota)$ satisfies the same hypotheses as p, and is constructed from the formal Taylor series by a Borel sum of the type $q_0(\iota)=\sum_{k=1}^\infty\tilde q_k(\iota)\chi(\iota/\varepsilon_k)$, $\chi\in C_0^\infty(\mathbf{R}^n)$ equal to 1 near 0, $\varepsilon_k\to 0$ fast enough as $k\to\infty$, and $\tilde q_k(\iota)$ is homogeneous of degree k. The canonical transformation is of the form $\kappa=\exp H_{\tilde f}$ for some smooth $\tilde f$. We

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shall try to construct a family κ_s of canonical transformations, $0 \le s \le 1$, tangent to identity at infinite order, such that $\kappa_0 = \mathrm{Id}$ and κ_1 solves $p \circ \kappa \circ \kappa_1 = q_0$. The deformation (or homotopy) method consists in finding a C^∞ one-parameter family of vector fields $s \mapsto X_s$ along which some property is conserved, in that case the property for Hamiltonians, interpolating between p and q_0, of being integrable. It reduces here essentially to solving a homological equation as in Proposition 1.2. (See [ArVaGo] for an introduction, and also [GuSc, p. 168, HeSj2, App. A, MeSj, BaLlWa, BrKo, IaSj, *etc.*], for other applications more directly relevant to our problem.) So let $q_s = q_0 + sr$, $0 \le s \le 1$, and look for κ_s such that

$$q_s \circ \kappa_s = q_0. \tag{1.22}$$

Then $\kappa_s|_{s=1}$ will solve our problem. The deformation field

$$X_s(\rho) = \sum_{j=1}^{2n} v_{s,j}(\rho) \frac{\partial}{\partial \rho_j} \in I^\infty(T\mathbf{R}^{2n})$$

is such that

$$\partial_s \kappa_s = X_s \circ \kappa_s. \tag{1.23}$$

Differentiating (1.22) gives $r \circ \kappa_s + \frac{\partial q_s}{\partial \rho}(\kappa_s) \circ \partial_s \kappa_s = 0$, or

$$r \circ \kappa_s + \big\langle X_s(\kappa_s(\rho)), q_s(\kappa_s(\rho)) \big\rangle = 0.$$

Furthermore, we require X_s to be Hamiltonian, *i.e.*, $X_s = H_{f_s}$, $f_s \in I^\infty$, so we get

$$\langle H_{f_s}, q_s \rangle = -\langle H_{q_s}, f_s \rangle = -r, \tag{1.24}$$

all quantities being evaluated at $\kappa_s(\rho)$. We want to apply Proposition 1.2 to $p = q_s$, $g = r$, so we move to the new symplectic coordinates (adapted to the outgoing/incoming manifolds) by composing with smooth canonical transformations Φ_s, *i.e.*, replace H_{q_s} by $(\Phi_s)^* H_{q_s}$, f_s by $(\Phi_s)^* f_s$, etc., so omitting for brevity these coordinate transformations when no confusion might occur, Proposition 1.2 gives $f_s \in I^\infty$ solving (1.24). So we are led to show that, given $H_{f_s} \in I^\infty$, (1.23) has a solution of the form $\kappa_s = \mathrm{Id} + \kappa_s'$, $\kappa_s' \in I^\infty$. Existence for $0 \le s \le 1$ follows *e.g.*, from Gronwall's lemma, truncating q_s outside a neighborhood of 0, and the condition $\kappa_0 = \mathrm{Id}$ gives

$$\|\kappa_s(\rho)\| \le C\|\rho\|, \quad C > 0 \tag{1.25}$$

for $\|\rho\| < \delta$. We want to show $\kappa_s'(\rho) = \mathcal{O}(\rho^\infty)$. Recall from the proof of Proposition 1.2 that, by the group property, $d\kappa_s(\rho)$ is the fundamental solution for the system $\partial_s Y(\rho, s) = \frac{\partial H_{f_s}}{\partial \rho}\big(\kappa_s(\rho)\big) Y(\rho, s)$. Since $d\kappa_s'(\rho)$ solves

$$\partial_s d\kappa_s'(\rho) - \frac{\partial H_{f_s}}{\partial \rho}(\kappa_s(\rho)) \circ \big(d\kappa_s'(\rho)\big) = \frac{\partial H_{f_s}}{\partial \rho}(\kappa_s(\rho)), \quad d\kappa_s'(0) = 0. \tag{1.26}$$

Duhamel's principle gives

$$d\kappa_s'(\rho) = \int_0^s d\kappa_{s-\widetilde{s}}\big(\kappa_{\widetilde{s}}(\rho)\big) \circ \frac{\partial H_{f_s}}{\partial \rho}(\kappa_{\widetilde{s}}(\rho))d\widetilde{s}.$$

Since $\frac{\partial H_{f_s}}{\partial \rho}(\kappa_{\widetilde{s}}(\rho)) = \mathcal{O}(\|(\kappa_{\widetilde{s}}(\rho))\|^N)$, (1.25) gives $\frac{\partial H_{f_s}}{\partial \rho}(\kappa_{\widetilde{s}}(\rho)) = \mathcal{O}(\|\rho\|^N)$, and $d\kappa_{s-\widetilde{s}}\big(\kappa_{\widetilde{s}}(\rho)\big) = \mathcal{O}(1)$, so, choosing N large enough, we get $d\kappa_s'(\rho) = \mathcal{O}(\|\rho\|^2)$. Integrating this relation, we get again $\kappa_s'(\rho) = \mathcal{O}(\|\rho\|)$. Taking partial derivative of (1.26) with respect to ρ_j as in the proof of Proposition 1.2 yields also $\frac{\partial}{\partial \rho_j} d\kappa_s'(\rho) = \mathcal{O}(\|\rho\|)$, and a straightforward induction argument shows $\kappa_s' \in I^\infty$, uniformly for s on compact sets. Taking $s = 1$ and undoing the transformation $\Phi_s|_{s=1}$ give eventually the result. ■

1.2 Two Simple Applications

As a first application, we present a different statement of theorem 0.1. It is sometimes convenient to perform the Birkhoff transform in action-angle coordinates (see [Gal, p. 473] for the elliptic case.) We restrict for simplicity to the usual case of a (real-) hyperbolic fixed point, where

$$p(x,\xi) = \xi^2 - \sum_{j=1}^n \lambda_j^2 x_j^2 + \mathcal{O}(\|x\|^3)$$

The corresponding Williamson coordinates are then given by the linear symplectic transformation $\kappa_1(x,\xi) = (y,\eta)$, $\sqrt{2}\lambda_j y_j = \lambda_j x_j + \xi_j$, $\sqrt{2}\eta_j = -\lambda_j x_j + \xi_j$. Outside the hyperplanes $x_i = 0$, we can construct smooth hyperbolic action-angle coordinates (ι,φ). Restricting for simplicity to $x_j > 0$, all j, they are defined for $\iota_j > 0$, $\varphi_j \in \mathbf{R}$, by the formulas $\lambda_j x_j = \sqrt{2\lambda_j \iota_j}\cosh\varphi_j$, $\xi_j = \sqrt{2\lambda_j\iota_j}\sinh\varphi_j$. We set $\kappa_0(\iota,\varphi) = (x,\xi)$.

Let κ be the canonical transform of Theorem 0.1, and define $\widetilde{\kappa} = \kappa_0^{-1}\circ\kappa_1^{-1}\circ\kappa\circ\kappa_1\circ\kappa_0$. Then, with $\kappa(y,\eta) = (y',\eta') = (y,\eta) + \mathcal{O}(|y,\eta|^2)$, we have $\widetilde{\kappa}(\iota,\varphi) = (\iota',\varphi')$, $2\lambda_j\iota_j' = -2\lambda_j y_j'\eta_j' = -\xi_j'^2 + \lambda_j{}^2 x_j'^2$, where $\kappa_1(x',\xi') = (y',\eta')$. Actually we can check that we can choose κ such that $\kappa_1^{-1}\circ\kappa\circ\kappa_1$ preserves the hyperplanes $\xi_j = 0$. (This is done at the level of Birkhoff series as in [KaRo, App], and an inspection of the proof of theorem 0.1 shows that this carries out to the corrections mod I^∞.)

Moreover, there exists a smooth generating function $S(\iota',\varphi)$ such that $\iota = \partial_\varphi S(\iota',\varphi)$, $\varphi' = \partial_{\iota'}S(\iota',\varphi)$, and of the form $S(\iota',\varphi) = \langle\iota',\varphi\rangle + \Phi(\iota',\varphi)$. Here $\partial_{\iota'}\Phi(\iota',\varphi) = \mathcal{O}(\iota')$, $\partial_\varphi\Phi(\iota',\varphi) = \mathcal{O}(\iota'^2)$, uniformly for ι' small enough. Finally, $p = q(\iota')$.

As for the second application, we consider an Hamiltonian flow with a non trivial center manifold. More precisely, let $p \in C^\infty(T^*\mathbf{R}^n)$ such that $dp \neq 0$ on the characteristic variety $p(\rho) = 0$, and p has a closed trajectory γ_0 of hyperbolic type at energy 0 (see *e.g.*, [Ar, App. 7, GeSj, SjZw]). A basic example is $p(x,\xi) = \xi^2 + \lambda_1^2 x_1^2 - \sum_{j=2}^n \lambda_j^2 x_j^2$, near an energy level $E > 0$. Another example is given by a smooth family $p = p_E$ of Hamiltonians depending on $2(n-1)$ phase

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variables $(x', \xi') \in T^*\mathbf{R}^{n-1}$, periodic with respect to $\theta \in \mathbf{S}^1$; parameter E then stands for the dual variable.

Since near every point of γ_0 there are symplectic coordinates (y, η), such that $p = \eta_1$, Hamiltonian p is locally integrable, but because of topological obstructions, there is no such global coordinate patch in a neighborhood of γ_0. So we may address the problem of "semi-global" integrability.

Let K be the set of trapped trajectories near energy 0:

$$K = \{\rho \in p^{-1}(E),\ E \in [-E_0, E_0],\ \exp(tH_p)(\rho) \not\to \infty, \text{ as } t \to \pm\infty\}$$

Let $K_E = K \cap p^{-1}(E)$, E small, and assume we are in the situation where $K_0 = \gamma_0$ is a closed trajectory of hyperbolic type.

Then in a neighborhood of K, there is a smooth, symplectic, closed submanifold $\Sigma \subset T^*\mathbf{R}^n$ of dimension 2, containing K_0 and such that H_p is tangent to Σ everywhere. We call Σ the center manifold of γ_0, and it is nothing but the one-parameter family of closed trajectories $\gamma_E \subset p^{-1}(E)$, E small. The restriction σ_Σ of σ to $T\Sigma^\perp$ (where $(\cdot)^\perp$ stands for " symplectic orthogonal") is clearly invariant under H_p. Hyperbolicity means that p vanishes of second order on Σ, and for all $\rho \in \Sigma$, the fundamental matrix $F_\rho|_{\Sigma^\perp}$ as in (1.1) is of rank $2n-2$, and has no purely imaginary eigenvalues. In the case at hand, we will assume that these eigenvalues are rationally independent. For $\rho \in \Sigma$, let $\Lambda_\pm(\rho) \subset T_\rho(\mathbf{R}^{2n})$ be as above the $(n-1)$-dimensional isotropic subspaces whose complexifications are the sum of all complex eigenspaces corresponding to eigenvalues with positive/negative real parts. We have the splitting $(T_\rho\Sigma)^\perp = \Lambda_+(\rho) \oplus \Lambda_-(\rho)$. We can also find real symplectic coordinates, denoted again by $(x, \xi) = ((x', x''), (\xi', \xi''))$, such that their differential verifies $d(x, \xi)|_\Sigma = \mathrm{Id}$, Σ is given by $(x', \xi') = 0$, and $\mathcal{J}_+ = \{\xi' = 0\}$ and $\mathcal{J}_- = \{x' = 0\}$ are the stable/unstable manifolds, tangent to $\Lambda_\pm(\rho)$, $\rho \in \Sigma$.

Let $\rho_0 \in \Sigma$ be such that the non resonance condition holds on eigenvalues $\lambda_1(\rho_0)$, $\dots, \lambda_{n-1}(\rho_0)$, and apply the Birkhoff normal form to p. Then there exists a smooth canonical transform κ for the symplectic 2-form σ_Σ, and a smooth Hamiltonian $q_0(\iota'; x'', \xi'')$, where $\iota' = (\iota_1, \dots, \iota_{n-1})$ are action variables as in (0.3) built from the (x', ξ')-coordinates, such that

$$p \circ \kappa(x, \xi) = q_0(\iota'; x'', \xi''),\ (x, \xi) \in \text{neigh}\,(\rho_0, \mathbf{R}^{2n})$$

To formulate a semi-global result we assume that the fundamental matrix of p (for the 2-form σ_Σ) is constant on Σ, with non resonant frequencies as above. Since the coordinates above can be defined globally on a neighborhood of γ_0 (see *e.g.*, [GeSj]), and the constructions above depend smoothly on $\rho_0 \in \Sigma$, we have found a smooth fibre bundle over Σ whose sections are action-angle coordinates in $T\Sigma^\perp$ adapted to p. Of course such a result is of mere academic interest, since κ a priori does not preserve the full symplectic structure, but it makes sense for the family p_E as above (non autonomous case.) See [CuB, Vu1, Vu2] for other (semi-)global aspects of integrability.

1.3 Semi-Classical Quantization and the Exact Birkhoff Normal Form

Let $P = P^w(x, hD, h)$ be a h-PDO with principal symbol p as above, so that $P(\rho, h) = p(\rho) + hp_1(\rho) + \cdots$ (in the sense of asymptotic sums) is real valued. Let κ be as in theorem 0.1, $\varphi(x, \eta)$ a generating function and let U be an elliptic FIO associated with the phase function φ and an amplitude we can choose so that U is microlocally unitary near ρ_0. Then the principal symbol $\widetilde{p}_0 = p \circ \kappa$ of $\widetilde{P} = U^{-1}PU$ is in the exact Birkhoff normal form given by theorem 0.1 (see Appendix B for a short review on pseudo differential calculus.) We try to correct U by a h-PDO of the form $B = e^{iW}$, where $W = W^w(x, hD, h)$, $W(\rho, h) = w_0(\rho) + hw_1(\rho) + \cdots$, and proceed as in [KaRo, IaSj] to show that we can choose W such that the Weyl symbol of

$$Q = B^{-1}PB = e^{-iW}Pe^{iW} = \sum_{j \geq 0} \frac{1}{j!}[iW, [iW, \ldots, [iW, P] \cdots]] \tag{1.27}$$

(we have dropped the tilde for convenience), is in the exact Birkhoff normal form. Let $p(x, \xi, h) = p_0(x, \xi) + hp_1(x, \xi) + \cdots$ be the Weyl symbol of P, where p_0 is in the exact Birkhoff normal form by construction. The coefficient of h in (1.27) is given by

$$q_1 = p_1 + \{w_0, p_0\}. \tag{1.28}$$

Working first at the level of formal Taylor series we can find q_1 resonant, and w_0 such that $q_1 = p_1 + \{w_0, p_0\}$ modulo I^∞, then we correct w_0 by changing w_0 in $w_0 + w_0'$ where w_0' solves an equation of the form $H_{p_0} w_0' = g \in I^\infty$. This can be achieved because of Proposition 1.2, so the principal symbol w_0 of $W(\rho, h)$ can be chosen such that $q_1 = q_1(\iota)$, and the two first terms in (1.27) are in the exact Birkhoff form. The choice of w_0 will influence the h^2 term in the symbol of $e^{-iW}Pe^{iW}$ only through the term $[iW, P]$ and to make the h^2 term in the exact Birkhoff normal form leads to a new equation of the same type as (1.28). It is clear that this construction can be iterated and we have found W such that the Weyl symbol $q(x, \xi, h) = p_0(\iota) + hq_1(\iota) + h^2 q_2(\iota) + \cdots$ is real and in the exact Birkhoff normal form. At last we set $A = UB$. If $I = (I_1, \ldots, I_n)$ denote the Weyl quantization of the action variables ι (I_j are commuting operators), our computations so far can be summarized in the following:

Proposition 1.3 *Let $P(x, hD, h)$ be the Weyl quantization of the symbol $p(x, \xi, h) = p_0(x, \xi) + hp_1(x, \xi) + \cdots$, real valued, and such that p_0 verifies the hypothesis of theorem 0.1. Then there is a (formally) unitary FIO A, and a smooth symbol $F(\iota_1, \ldots, \iota_n)$ defined near ρ_0, such that $A^{-1}PA = F(I_1, \ldots, I_n, h)$ (microlocally near ρ_0.)*

2 The Lewis-Sternberg Normal Form for the Poincaré Map

In this section we prove Theorems 0.2 and 0.3. First we recall the following version of a theorem of Lewis-Sternberg (see [St, Theorem 1, Corollary 1.1; Fr, Theorem V.1] and [IaSj] for a detailed proof). For simplicity we content to a particular case relevant to our problem. So assume A is a real $2n \times 2n$ symplectic matrix and has eigenvalues $\lambda_1, \ldots, \lambda_n$, $1/\lambda_1, \ldots, 1/\lambda_n$, $\overline{\lambda}_1, \ldots, \overline{\lambda}_n$, $1/\overline{\lambda}_1, \ldots, 1/\overline{\lambda}_n$, where none of them is

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negative. Then there is a natural choice of the logarithm $B = \log A$, and B is antisymmetric for the canonical 2-form on $T^*\mathbf{R}^n$. Let $\mu_j = \log \lambda_j$, in such a way that $\overline{\lambda}_j$ corresponds to $\overline{\mu}_j$, and $p_0(\rho) = b(\rho) = \frac{1}{2}\sigma(\rho, B\rho)$. Assume that for $k_j \in \mathbf{Z}$,

$$\sum k_j\mu_j \in 2i\pi\mathbf{Z} \Longrightarrow \sum k_j\mu_j = 0. \tag{2.1}$$

We have the following:

Theorem 2.1 *Let $\Phi\colon \operatorname{neigh}(0, \mathbf{R}^{2n}) \to \operatorname{neigh}(0, \mathbf{R}^{2n})$ be a smooth canonical transformation, leaving fixed $\rho_0 = 0$, and $A = d\Phi(\rho_0)$ as above. Then there is $p \in C^\infty$ defined near ρ_0, uniquely determined modulo I^∞, (for a given choice of p_0) such that $p(\rho) = p_0(\rho) + \mathcal{O}(\rho^3)$ and*

$$\Phi(\rho) = \exp H_p(\rho) + \mathcal{O}(\rho^\infty).$$

We state now the counterpart of Theorem 1.1 for canonical maps involving a discrete dynamical system (see *e.g.*, [BaLlWa] and references therein.)

Theorem 2.2 *Let $f\colon \operatorname{neigh}(0, \mathbf{R}^{2n}) \to \operatorname{neigh}(0, \mathbf{R}^{2n})$ be a smooth canonical transformation, leaving fixed $\rho_0 = 0$, and assume $A = df(\rho_0)$ is non degenerate and has no eigenvalues of modulus 1. Let L_+ (resp. L_-) be the sum of eigenspaces associated with eigenvalues λ_j of modulus > 1 (resp. < 1). So $L_\pm$ are Lagrangian subspaces. Then there exist smooth Lagrangian manifolds $\mathcal{L}_\pm$ passing through ρ_0, tangent to $L_\pm$ at ρ_0, invariant by f, and such that within $\mathcal{L}_+$ (resp. $\mathcal{L}_-$), ρ_0 is repulsive (resp. attractive) for f.*

For $\rho = (x, \xi)$, we denote by $\rho^{(N)} = (x^{(N)}(\rho), \xi^{(N)}(\rho)) = f^N(\rho)$, $N \in \mathbf{Z}$, the N-th iterate of ρ under f. If $\mathcal{L}_+$ (resp. $\mathcal{L}_-$) is given by $\xi = 0$ (resp. $x = 0$), it is again possible to define the outgoing region

$$\Omega_\delta^{\text{out}} = \left\{ (x, \xi) : \|\xi\|_0 < 2\|x\|_0,\ \|x\|_0^2 + \|\xi\|_0^2 < \delta^2 \right\}$$

for some suitable euclidean norm $\|\cdot\|_0$, and express the expansion and contraction properties of our discrete dynamical system in term of Lyapunov exponents as in (1.8–1.9). The same holds of course for the incoming region. Now we recall the following result, which is the symplectic version of the Lewis-Sternberg theorem. At least to prepare for Theorem 0.3, it could be useful to sketch a simple proof based on the previous arguments.

Theorem 2.3 [BaLlWa] *Let $f, f_0\colon \operatorname{neigh}(0, \mathbf{R}^{2n}) \to \operatorname{neigh}(0, \mathbf{R}^{2n})$ be smooth canonical transformations, leaving fixed $\rho_0 = 0$, and assume they are tangent to infinite order at ρ_0. Let $A = df(\rho_0)$ have its spectrum outside the unit circle as above. Then there is a smooth canonical transform g leaving fixed ρ_0, $dg(\rho_0) = \mathrm{Id}$, such that $g^{-1} \circ f \circ g = f_0$.*

Outline of Proof It relies again on the stable/unstable manifolds theorem above and a deformation argument, which goes as follows. Let f_s, $0 \leq s \leq 1$ be a smooth family of canonical transformations interpolating between f_0 and $f_1 = f$. We can take $f_s = \widehat{f}_s \circ f_0$, with $\widehat{f}_s(\rho) = \frac{1}{s}\widehat{f}(s\rho)$ for $s > 0$, and $\widehat{f}_0(\rho) = \rho$, for $s = 0$, where $\widehat{f} = f \circ f_0^{-1}$. We look for a family of canonical transformations g_s with $g_0 = \mathrm{Id}$, satisfying

$$g_s^{-1} \circ f_s \circ g_s = f_0. \tag{2.2}$$

The deformation fields are of the form

$$\partial_s f_s = \mathcal{F}_s \circ f_s, \quad \partial_s g_s = \mathcal{G}_s \circ g_s \tag{2.3}$$

with $\mathcal{F}_s$ and $\mathcal{G}_s$ Hamiltonian, *i.e.*, $\mathcal{F}_s = H_{F_s}$, $\mathcal{G}_s = H_{G_s}$. Since f and f_0 are tangent to infinite order at ρ_0, we have $F_s \in I^\infty$ and we look for $\mathcal{G}_s$ in the same class. The crucial observation in [BaLlWa] is the following. Taking derivative with respect to s in (2.2) we obtain the homological equation:

$$\partial_s\big(g_s^{-1} \circ f_s \circ g_s\big) = \big(g_s^{-1}\big)_*\big(\mathcal{F}_s - \mathcal{G}_s + (f_s)_*\mathcal{G}_s\big) \circ \big(g_s^{-1} \circ f_s \circ g_s\big) = 0$$

and it is clear that (2.2) can be solved iff we can find a C^1 family of vector fields $\mathcal{G}_s$ satisfying $\mathcal{F}_s - \mathcal{G}_s + (f_s)_*\mathcal{G}_s = 0$. At the level of Hamiltonians this relation takes the form

$$G_s - G_s \circ f_s^{-1} = F_s. \tag{2.4}$$

This equation will be solved as in Proposition 1.2, changing the continuous dynamical system $t \mapsto \exp(tH_p)(\rho)$, $t \in \mathbf{R}$, to $N \mapsto f^N(\rho)$, $N \in \mathbf{Z}$. So let $\chi^{\mathrm{out}} + \chi^{\mathrm{in}} = 1$ be a smooth partition of unity such that supp $\chi^{\mathrm{out}} \subset \{\|\xi\|_0 < 2\|x\|_0\}$, supp $\chi^{\mathrm{in}} \subset \{\|x\|_0 < 2\|\xi\|_0\}$, where we have chosen symplectic coordinates adapted to f_s as in theorem 2.2. After modifying suitably the functions outside a fixed neighborhood of ρ_0, define

$$G_s^{\mathrm{out}}(\rho) = \sum_{N\geq 0}(\chi^{\mathrm{out}}F_s) \circ f_s^{-N}(\rho), \quad G_s^{\mathrm{in}}(\rho) = -\sum_{N\geq 1}(\chi^{\mathrm{in}}F_s) \circ f_s^{N}(\rho). \tag{2.5}$$

Then $G_s = G_s^{\mathrm{out}} + G_s^{\mathrm{in}}$ formally solves (2.4) and using exponential estimates on the discrete flow f_s^N shows that G_s is C^1 and vanishes at ρ_0. For higher derivatives we use the "tangent functor trick" of [BaLlWa], which is the discrete analogue of (1.16), and differentiate (2.4) to obtain

$$dG_s - dG_s(f_s^{-1}) \circ df_s^{-1} = dF_s. \tag{2.6}$$

This is a linear equation in dG_s similar to (2.4), whose solution is again given (formally) by $dG_s = G_s^{\mathrm{out}} + G_s^{\mathrm{in}}$,

$$\begin{aligned} dG_s^{\mathrm{out}}(\rho) &= \sum_{N\geq 0}(\chi^{\mathrm{out}}dF_s) \circ f_s^{-N}(\rho) \prod_{j=0}^{N} df_s^{-1} \circ f_s^{-N+j}(\rho) \\ dG_s^{\mathrm{in}}(\rho) &= -\sum_{N\geq 1}(\chi^{\mathrm{in}}dF_s) \circ f_s^{N}(\rho) \prod_{j=0}^{N} df_s^{-1} \circ f_s^{N-j}(\rho) \end{aligned} \tag{2.7}$$

(the product being understood as a product of matrices). As above, it is easy to see that the two series converge uniformly in $\Omega_\delta = \Omega_\delta^{\mathrm{out}} \cup \Omega_\delta^{\mathrm{in}}$ to continuous functions vanishing at ρ_0, and the same holds for the first derivative. A uniqueness argument further shows that dG_s as defined in (2.7) is actually the derivative of (2.5), so G_s is C^2, and vanishes to second order at ρ_0. To continue, we take partial derivative in (2.6) with respect to ρ_j, $j = 1, \cdots, 2n$, which gives an equation analogous to (2.6), and argue again as in Proposition 1.2 (precise estimates can be found in [BaLlWa]). So by induction we proved $G_s \in I^\infty$, and (2.3) gives $g_s = \mathrm{Id} + \mathcal{O}(\rho^\infty)$ as in the argument after (1.25). So we have proved the theorem. ∎

Applying Theorem 2.3 to $f = \Phi$, $f_0 = \exp H_p$, where p is given in Theorem 2.1, we get:

Proposition 2.4 *Let Φ be as in Theorem 2.1,* i.e., *none of the eigenvalues λ_j of $A = d\Phi(\rho_0)$ is negative, and $\mu_j = \log \lambda_j, |\lambda_j| \neq 1$ satisfy* (2.1)*. Then there exists a smooth Hamiltonian q defined near ρ_0, $q(\rho) = p_0(\rho) + \mathcal{O}(\rho^3)$, such that $\Phi(\rho) = \exp H_q(\rho)$.*

If the μ_j's are rationally independent, we can write q in the exact Birkhoff normal form, so Theorem 0.2 immediately follows from Proposition 2.4 and Theorem 0.1.

2.1 Semiclassical Integrability

Here we prove theorem 0.3. From Proposition 2.4 we may already assume that U is associated with a canonical transformation of the form $\kappa = \exp H_p$ (for the moment we have no need on the rational independence of the μ_j's.) We could follow [IaSj] but we prefer a similar proof based on the argument of Section 1.3. So consider the family of FIO's $U_s = sU + (1-s)U_0$, $U_0 = e^{iP_0/h}$, $0 \le s \le 1$, $P_0 = p^w(x, hD) + h\alpha$ (where α is a constant subprincipal symbol we choose so that U_s is elliptic for all s,) all associated with κ. See *e.g.*, again [Iv, Section 2] for a proof of the fact that $e^{iP_0/h}$ is a FIO, and related properties. We look for a smooth family $W_s(x, hD, h)$ of h-PDO's of order 0 such that

$$e^{-iW_s} U_s e^{iW_s} = e^{iP_0/h}. \tag{2.8}$$

Taking derivative with respect to s we get

$$U_0^{-1} U_s \partial_s W_s - U_0^{-1} \partial_s W_s U_s - i(U_0^{-1} U - \mathrm{Id}) = 0. \tag{2.9}$$

Since all FIO's are associated with the same canonical relation, $U_0^{-1} U_s \partial_s W_s$, $U_0^{-1} \partial_s W_s U_s$ and $U_0^{-1} U$ are h-PDO's of order 0. Denoting the Weyl symbol of W_s by the same letter, $W_s(\rho, h) = w_0(\rho, s) + h w_1(\rho, s) + h^2 w_2(\rho, s) + \cdots$, by $a^0(\kappa(\rho), \rho, s)$, the principal symbol of U_s, $b^0(\rho, \kappa(\rho))$ this of U_0^{-1}, we first identify the principal symbol of (2.9). From the well-known calculus on FIO's that we recall in Appendix B, we get from (2.9)

$$\begin{aligned} &b^0(\rho,\kappa(\rho)) a^0(\kappa(\rho),\rho,s) \partial_s w_0(\rho,s) - b^0\big(\rho,\kappa(\rho)\big)\, \partial_s w_0(\kappa(\rho),s) a^0\big(\kappa(\rho),\rho,s\big) \\ &\qquad - i\Big(b^0\big(\rho,\kappa(\rho)\big)\, a^0\big(\kappa(\rho),\rho,s\big) - 1\Big) = 0. \end{aligned} \tag{2.10}$$

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Dividing this equation by $b^0\big(\rho, \kappa(\rho)\big)\, a^0\big(\kappa(\rho), \rho, s\big) \neq 0$, we get

$$\partial_s w_0(\rho, s) - \partial_s w_0(\kappa(\rho), s) = -\int_0^1 (\partial_s w_0) \circ \exp(tH_p)(\rho)\, dt = c_0(\rho, s). \tag{2.11}$$

As in [IaSj, Theorem 3.2], this can be solved mod I^∞ by successive approximations, so we are left, changing w_0 to $w_0 + w_0'$, with

$$\partial_s w_0'(\rho, s) - \partial_s w_0'(\kappa(\rho), s) = c_0'(\rho, s) \in I^\infty. \tag{2.12}$$

This is exactly equation (2.4) with κ replacing f_s^{-1}, so as in the proof of Theorem 2.3 we can find a smooth family $\partial_s w_0(\rho, s) \in I^\infty$ solving (2.12). Integrating for $0 \leq s \leq 1$ with the initial value $w_0(\rho, 0) = 0$, we are done with the principal symbol $w_0(\rho, s)$, which is unique mod I^∞ according to the uniqueness part of [IaSj, Theorem 3.2]. Of course, it is essential to notice that (2.11) and (2.12) can be solved in the whole neighborhood of ρ_0 where c_0' is defined.

Let us consider now the coefficient of h in (2.9). Using (A.4) and (A.5) and the usual calculus on h-PDO's, we see that $\partial_s w_1(\rho, s)$ verifies again an equation of the form of (2.11), where the right hand side also depends on $w_0(\rho, s)$. This can again be solved in the same neighborhood of ρ_0. So an easy inductive argument shows that (2.8) holds microlocally near ρ_0. For $s = 1$ we get, by usual estimates (see *e.g.*, [Iv, Section 1]) microlocally near ρ_0: $U = e^{iW_1} e^{iP_0/h} e^{-iW_1} = \exp(ie^{iW_1} P_0 e^{-iW_1}/h)$ and so we have proved:

Proposition 2.5 *Let U be an elliptic FIO microlocally defined near ρ_0, associated with a canonical transform Φ as in Proposition 2.4. Then there is an h-PDO, $P = P(x, hD, h)$, with principal symbol p given by Proposition 2.4, such that $\Phi = \exp H_p$ and $U = e^{iP/h}$ microlocally near ρ_0.*

Combining Propositions 1.3, 2.4 and 2.5 eventually gives Theorem 0.3.

3 Parameter Dependent Case

We extend some of the previous results, taking advantage of the fact observed in [IaSj], that the Birkhoff normal form can be carried out nearby critical points with non resonant frequencies, modulo small error terms. Thinking of the Poincaré map, which depends smoothly on energy E, if the frequencies are non resonant for some $E = E_0$, they may become resonant for values of E arbitrarily close to E_0. So it is interesting to investigate some weak form of integrability. We content here to classical Hamiltonians, but quantization could be easily treated as above.

3.1 The Birkhoff Normal Form

As in the Appendix, let $p^s \in C^\infty$ depend smoothly on $s \in \text{neigh}(0, \mathbf{R}^k)$, $p^s(\rho_0) = 0$, and have a non-degenerate critical point of hyperbolic type at ρ_0. (In some applications, the critical point depends on s, but choosing suitable linear symplectic coordinates and changing p^s by a constant we are in this situation.) After possibly performing another linear symplectic transformation, we may assume that its quadratic part is of the form

$$p_2^s(x,\xi) = \sum_{j=1}^{n} \mu_j^s x_j \xi_j \tag{3.1}$$

with coordinates independent of s. For $s = 0$, we suppose the $\mu_j = \mu_j^0$ rationally independent. Then Proposition A.1 below shows there is a smooth family of canonical transforms, κ^s, $\kappa^s(\rho_0) = \rho_0$, such that

$$p^s \circ \kappa^s(\rho) = q^s(\iota) + r^s(\rho),\ r^s(\rho) = \mathcal{O}(\rho^\infty) + \rho^3\mathcal{O}(s^\infty)$$

with the principal part of q^s as in (3.1). Looking at the deformation procedure, we see that we can apply the stable/unstable manifold theorem to $Q_\tau(\rho) = q^s(\iota) + \tau r^s(\rho)$, $0 \le \tau \le 1$, and if we decompose $r^s = u^s + v^s$, $v^s = \mathcal{O}(\rho^\infty)$, $u^s = \rho^3\mathcal{O}(s^\infty)$, we are able to solve $H_{Q_\tau} f_\tau = v^s$, for $f_\tau \in I^\infty$. Then the vector field $X_\tau = H_{f_\tau}$ generates a 1-parameter family of canonical transformations κ_τ as in (1.23), and for $\tau = 1$ we get

$$p^s \circ \kappa^s \circ \kappa_1(\rho) = q^s(\iota) + \rho^3\mathcal{O}(s^\infty)$$

which is the normal form for p^s.

3.2 The Lewis-Sternberg Normal Form

As in [IaSj] we extend Theorem 0.2 to the parameter dependent case. For simplicity we just vary one parameter $s \in \operatorname{neigh}(0, \mathbf{R})$. Let $\Phi^s\colon \operatorname{neigh}(0, \mathbf{R}^{2n}) \to \operatorname{neigh}(0, \mathbf{R}^{2n})$, $s \in \operatorname{neigh}(0, \mathbf{R})$, be a smooth family of smooth canonical transformations, leaving fixed $\rho_0 = 0$, and $A^s = d\Phi^s(\rho_0)$. We assume that $\Phi = \Phi^0$ fulfills the assumptions of Proposition 2.4. For small s, A^s is still hyperbolic, but (2.1) need not be verified. We want to investigate to what extent the conclusion of Proposition 2.4 holds for Φ^s, $s \neq 0$, so we look for a smooth, real valued family $p^s(\rho) = \mathcal{O}(\rho^2)$, such that

$$\Phi^s(\rho) = \exp H_{p^s}(\rho) + \rho^2\mathcal{O}(s^\infty). \tag{3.2}$$

By Proposition 2.4, this holds for $s = 0$, with $p^s = p$. Assume for a moment we have found p^s, which fulfills formally (3.2), and consider the family $\Phi_t^s(\rho) = \exp(tH_{p^s})(\rho)$. Since p_s vanishes to second order at ρ_0, the germ of Φ_t^s at ρ_0 is well-defined for all real t. We compute the "logarithmic derivative"

$$(\Phi_t^s)^*\partial_s\Phi_t^s = H_{q_t^s},$$

where

$$q_t^s = \int_0^t \partial_s p^s \circ \Phi_{\tilde t}^s \, d\tilde t.$$

In this formula, we take $t = 1$ (deleting the corresponding subscript) and try to solve

$$q^s(\rho) = \int_0^t \partial_s p^s \circ \Phi_{\tilde t}^s \, d\tilde t \mod \rho^2\mathcal{O}(s^\infty) \tag{3.3}$$

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where $\partial_s p^s$ will be the unknown. We try to achieve this condition at any order in s. At zeroth order, *i.e.,* for $s = 0$, one should have $q^0(\rho) = \int_0^t \big(\partial_s p^s\big)\,|_{s=0} \circ \Phi^0_{\tilde t}\, d\tilde t$, and this can be solved as in (2.11), since condition (2.1) holds for $\Phi^0(\rho)$. We find $\partial_s p^s|_{s=0} = \mathcal{O}(\rho^2)$. If we differentiate (3.3) k times and evaluate at $s = 0$ we get

$$\int_0^1 (\partial_s^{k+1} p^s) \circ \exp tH_{p^s}(\rho)\, dt = \partial_s^k q^s(\rho) + F_k(p^s, \dots, \partial_s^k p^s, \rho), \quad s = 0.$$

If $p^0, \dots, \partial_s^k p^s|_{s=0} = \mathcal{O}(\rho^2)$ have been determined, we get $\partial_s^{k+1} p^s|_{s=0} = \mathcal{O}(\rho^2)$ from this equation. It is then clear that (3.3) has a solution which is unique modulo $\rho^2 \mathcal{O}((\rho, s)^\infty)$. That is the inductive part of the argument. Conversely, define $\widetilde{\Phi}^s = \exp H_{p^s}$. Then

$$(\Phi^s)^* \partial_s \Phi^s = (\widetilde{\Phi}^s)^* \partial_s \widetilde{\Phi}^s + \rho^2 \mathcal{O}(s^\infty), \quad \widetilde{\Phi}^0 = \Phi^0.$$

From identity $\partial_s(\widetilde{\Phi}^s)^{-1}(\rho) = -H_{p^s}\big((\widetilde{\Phi}^s)^{-1}(\rho)\big)$, estimate

$$\partial_s(\Phi^s)^{-1}(\rho) = -H_{p^s}\big((\Phi^s)^{-1}(\rho)\big) + \rho^2 \mathcal{O}(s^\infty),$$

which follows from (3.3), and initial condition $\Phi^0 = \widetilde{\Phi}^0$ we conclude easily that (3.2) holds.

Assume further that for $s = 0$ the μ_j's are rationally independent. Using the parameter dependent Birkhoff transformations as in Proposition A.1, we see that for $s \in \operatorname{neigh}(0, \mathbf{R})$ small enough, there is a smooth family of Hamiltonians $\widetilde{q}^s$, and canonical transformations κ^s, $\kappa^s(\rho_0) = \rho_0$, such that $p \circ \kappa^s = \widetilde{q}^s + \mathcal{O}(\rho^\infty) + \rho^3 \mathcal{O}(s^\infty)$ and $\widetilde{q}^s = \widetilde{q}^s(\iota)$ depend on the action variable only. So we have

$$(\kappa^s)^{-1} \circ \exp H_{p^s} \circ \kappa^s = \exp H_{\widetilde{q}^s} + \mathcal{O}(\rho^\infty) + \rho^2 \mathcal{O}(s^\infty) \tag{3.4}$$

and by (3.2),

$$(\kappa^s)^{-1} \circ \Phi^s \circ \kappa^s = \exp H_{\widetilde{q}^s} + \rho^2 \mathcal{O}(s^\infty). \tag{3.5}$$

It suffices then to apply a parameter dependent version of Theorem 2.3 as in Section 3.1, to get rid of the $\mathcal{O}(\rho^\infty)$ term, and we see that (3.4) and (3.5) imply the following

Proposition 3.1 *Let Φ^s, $s \in \operatorname{neigh}(0, \mathbf{R})$, be a smooth family of smooth canonical transformations, $\Phi^s\colon \operatorname{neigh}(0, \mathbf{R}^{2n}) \to \operatorname{neigh}(0, \mathbf{R}^{2n})$, $\Phi^s(0) = 0$, such that for $s = 0$, $A^0 = d\Phi^0(0)$ is non degenerate, its eigenvalues λ_j, $j = 1 \dots, n$, are non negative and lie outside the unit circle, and $\mu_j = \log \lambda_j$ verify* (2.1)*. Assume further that the μ_j's are rationally independent (*i.e., *the $\lambda_j s$ are non resonant in the strong sense.) Then there are a smooth family of smooth canonical maps κ^s, $s \in \operatorname{neigh}(0, \mathbf{R})$, $\kappa^s(0) = (0)$, $d\kappa^s(0) = \mathrm{Id}$, and a smooth family of smooth functions $q^s(\iota)$ depending on the action variables ι alone, such that*

$$\Phi^s = \exp H_{q^s} + \rho^2 \mathcal{O}(s^\infty).$$

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4 The Complex Case

We present here a rather rough discussion in the almost holomorphic case, *i.e.,* for Hamiltonians whose $\overline{\partial}$ vanishes of infinite order at ρ_0, somewhat in the spirit of [Sj2] and [MeSj]. First we recall some properties concerning symplectic structures in $T\mathbf{C}^n$; then we state the center stable/unstable manifold theorem for almost holomorphic Hamiltonians; at last we prove Theorem 0.4, and conclude with some elementary properties on monodromy.

4.1 Complex Symplectic Geometries

The variables in the complex phase-space $T^*\mathbf{C}^n$ will still be denoted by (x, ξ). As in the real case, we start with some geometric preparations.

First we recall some elementary facts about complex vector fields. If

$$\begin{aligned} v(\rho) &= \sum_{j=1}^{2n} v_j(\rho)\partial_{\rho_j} + v'_j(\rho)\overline{\partial}_{\rho_j} \\ &= \sum_{j=1}^{2n} \big(a_j(\rho)\partial_{x_j} + b_j(\rho)\partial_{\xi_j} + a'_j(\rho)\overline{\partial}_{x_j} + b'_j(\rho)\overline{\partial}_{\xi_j}\big) \in T(T^*\mathbf{C}^n) \end{aligned}$$

is a vector field on $T^*\mathbf{C}^n$, we set $\widehat{v} = 2\operatorname{Re} v = v + \overline{v}$, or

$$\widehat{v}(\rho) = \sum_{j=1}^{2n} \big(v_j(\rho) + \overline{v'_j(\rho)}\big)\,\partial_{\rho_j} + \big(\overline{v_j(\rho)} + v'_j(\rho)\big)\,\overline{\partial}_{\rho_j}.$$

Identifying $\mathbf{C}^n \times \mathbf{C}^n$ with $\mathbf{R}^{2n} \times \mathbf{R}^{2n}$, $\widehat{v}$ is simply the vector

$$\begin{aligned} (v_1 + \overline{v'_1}, \dots, v_{2n} + \overline{v'_{2n}}) &= (a_1 + \overline{a'_1}, \dots, a_n + \overline{a'_n}, b_1 + \overline{b'_1}, \dots, b_n + \overline{b'_n}) \\ &= \big(\operatorname{Re}(a_1 + a'_1), \operatorname{Im}(a_1 - a'_1), \dots, \operatorname{Re}(b_n + b'_n), \operatorname{Im}(b_n - b'_n)\big) \end{aligned}$$

expressed in the basis $B = \big(\partial_{\operatorname{Re} x_1}, \partial_{\operatorname{Im} x_1}, \dots, \partial_{\operatorname{Re}\xi_n}, \partial_{\operatorname{Im}\xi_n}\big)$. In general the identification between $\mathbf{C}^n$ (or $\mathbf{C}^{2n}$) and the underlying real vector space will be expressed as $\Theta(a_1, \dots, a_n) = (\operatorname{Re} a_1, \operatorname{Im} a_1, \dots, \operatorname{Re} a_n, \operatorname{Im} a_n)$.

Let us denote by I the ideal of C^∞ functions in $\mathbf{C}^n$ (or $T^*\mathbf{C}^n$ as will be clear from the context,) that vanish at ρ_0. We assume throughout that $v'_j \in I$, or even $v'_j \in I^\infty$. In that case, we write $v \in T^{(1,0)}(T^*\mathbf{C}^n) \oplus T^{(0,1)}_\infty(T^*\mathbf{C}^n)$. Then $\widehat{v}$ is the (unique) real vector field which gives the same result as v, at the point ρ_0, when applied to a differentiable function u, provided $\overline{\partial}u \in I$. For real t, the flow of $\widehat{v}$ will be denoted by

$$\widehat{\Phi}_t(\rho) = \big(\widehat{x}_t(\rho), \widehat{\xi}_t(\rho)\big) = \exp(t\widehat{v})(\rho).$$

In the case where $v'_j = 0$ (*i.e.,* $v \in T^{(1,0)}(T^*\mathbf{C}^n)$), this is the solution of the system of ODE's

$$\frac{d}{dt}\widehat{(x_j)}_t(\rho) = a_j\big(\widehat{\Phi}_t(\rho)\big), \quad \frac{d}{dt}\widehat{(\xi_j)}_t(\rho) = b_j\big(\widehat{\Phi}_t(\rho)\big), \quad \widehat{\Phi}_0(\rho) = \rho.$$

So it has the property, that if $\nu \in T^{(1,0)}(T^*\mathbf{C}^n)$ has holomorphic coefficients, then $\widehat{\Phi}_t(\rho)$ is the restriction to the real t-axis of the holomorphic flow

$$\Phi_t(\rho) = \big(x_t(\rho), \xi_t(\rho)\big) = \exp(t\nu)(\rho).$$

We recall also that $\mathbf{C}^{2n}$ is endowed with the complex canonical 2-form $\sigma_{\mathbf{C}}$, which makes it a symplectic space, and two real symplectic 2-forms $\operatorname{Re}\sigma_{\mathbf{C}}$ and $\operatorname{Im}\sigma_{\mathbf{C}}$. For convenience, we remove subscript $\mathbf{C}$ from the notations. If p is a smooth complex function on $\mathbf{C}^{2n}$, the Hamiltonian vector field of p is defined as

$$H_p = \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial p}{\partial \overline{\xi}}\frac{\partial}{\partial \overline{x}} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} - \frac{\partial p}{\partial \overline{x}}\frac{\partial}{\partial \overline{\xi}}$$

(note that we have used a different convention from [MeSj, Sj1], where H_p does not contain the antiholomorphic derivatives). If we define the real Hamiltonian vector field $H^{\operatorname{Re}\sigma}$ by $(\operatorname{Re}\sigma)(H_f^{\operatorname{Re}\sigma}, t) = \langle df, t\rangle$, then we have $H_{\operatorname{Re} p}^{\operatorname{Re}\sigma} = \widehat{H}_p$. More precisely, in the basis B,

$$\widehat{H}_p = \Big(\operatorname{Re}\frac{\partial p}{\partial \operatorname{Re}\xi}, -\operatorname{Re}\frac{\partial p}{\partial \operatorname{Im}\xi}, -\operatorname{Re}\frac{\partial p}{\partial \operatorname{Re} x}, \operatorname{Re}\frac{\partial p}{\partial \operatorname{Im} x}\Big).$$

We denote by $\frac{\partial \widehat{H}_p}{\partial \rho}$ the Jacobian (in the real sense) expressed in this basis.

The Poisson bracket associated with $\operatorname{Re}\sigma_{\mathbf{C}}$ is denoted by $\{\cdot,\cdot\}_R$ and coincides with $\{\operatorname{Re}\cdot, \operatorname{Re}\cdot\}$ for the real symplectic structure on $\mathbf{C}^{2n}$ read through Θ.

Let p be a smooth function such that $\overline{\partial} p \in I^\infty$. For real t, the Hamiltonian flow of $\widehat{H}_p$ will be denoted as above by

$$\widehat{\Phi}_t(\rho) = \big(\Phi_{t,x}(\rho), \Phi_{t,\xi}(\rho)\big) = \exp(t\widehat{H}_p)(\rho) \tag{4.1}$$

Let $\widetilde{X}_\rho = \Theta\big(\overline{\partial}_x\Phi_{t,x}, \overline{\partial}_x\Phi_{t,\xi}\big)$, and $\widetilde{Y}_\rho = \Theta\big(\overline{\partial}_\xi\Phi_{t,x}, \overline{\partial}_\xi\Phi_{t,\xi}\big)$ considered as vector fields on $T^*(\mathbf{C}^n)$. In the same way, we write $X_\rho = \Theta\big(\partial_x\Phi_{t,x}, \partial_x\Phi_{t,\xi}\big)$, and $Y_\rho = \Theta\big(\partial_\xi\Phi_{t,x}, \partial_\xi\Phi_{t,\xi}\big)$, where ∂ denotes the holomorphic derivative. We first state a technical Lemma, which follows from a straightforward computation and the fact that p verifies approximately the Cauchy-Riemann equations:

Lemma 4.1 *With p as above, we have:*

$$\partial_t\widetilde{X}_\rho = \frac{\partial \widehat{H}_p}{\partial \rho}(\widehat{\Phi_t})\widetilde{X}_\rho + \mathcal{O}\big(\widehat{\Phi}_t(\rho)^\infty\big)\big(\widetilde{X}_\rho, X_\rho\big) \tag{4.2}$$

$$\partial_t\widetilde{Y}_\rho = \frac{\partial \widehat{H}_p}{\partial \rho}(\widehat{\Phi}_t)\widetilde{Y}_\rho + \mathcal{O}\big(\widehat{\Phi}_t(\rho)^\infty\big)\big(\widetilde{Y}_\rho, Y_\rho\big). \tag{4.3}$$

4.2 The Stable-Unstable-Center Manifold Theorem in the Complex Domain

Our first step is to extend the stable/unstable manifold theorem in the case of almost holomorphic Hamiltonians. To be complete we will actually prove a little bit more than required. We will follow closely the nice geometric argument of [Sj2] in the analytic category, implemented for higher derivatives by an idea we borrowed also from [HeSj1].

So let p such that $\overline{\partial} p \in I^\infty$, have a non degenerate critical point at ρ_0, $p(\rho_0) = 0$. Let $F_{\rho_0}(p)$ as in (1.1) denote the fundamental matrix (in the holomorphic sense), and assume as before that $F_{\rho_0}(p)$ has $2n$ distinct eigenvalues, none purely imaginary. Again let $\Lambda_\pm \subset T_{\rho_0}\mathbf{C}^{2n}$ be the sum of all eigenspaces corresponding to eigenvalues with positive (resp. negative) real parts. We have:

Theorem 4.2 *With the notations above, in a neighborhood of ρ_0, there are $\widehat{H}_p$-invariant, R-Lagrangian manifolds $\mathcal{J}_\pm$ (*i.e.*, Lagrangian for* $\operatorname{Re}\sigma_{\mathbf{C}}$*), passing through ρ_0, such that $T_{\rho_0}(\mathcal{J}_\pm) = \Lambda_\pm$. Within $\mathcal{J}_+$ (resp. $\mathcal{J}_-$), ρ_0 is repulsive (resp. attractive) for $\widehat{H}_p$, and* $\operatorname{Re} p|_{\mathcal{J}_\pm} = 0$*. We can also find* $\operatorname{Re}\sigma_{\mathbf{C}}$*-symplectic coordinates, denoted again by $(x,\xi) = \kappa(y,\eta)$, $\overline{\partial}\kappa \in I^\infty$, such that their differential at ρ_0 verifies $d\kappa(\rho_0) = \mathrm{Id}$, $\kappa^*(\sigma_{\mathbf{C}}) = \sigma_{\mathbf{C}}$ mod I^∞ and $\mathcal{J}_+ = \{\xi = 0\}$, $\mathcal{J}_- = \{x = 0\}$. In these coordinates*

$$\operatorname{Re} p(x,\xi) = \operatorname{Re}\langle A(x,\xi)x, \xi\rangle \tag{4.4}$$

where $A(x,\xi)$ is smooth, has constant term equal to A_0, and $\overline{\partial}A(x,\xi) \in I^\infty$. Moreover,

$$p(x,\xi) = \langle A(x,\xi)x, \xi\rangle \bmod I^\infty. \tag{4.5}$$

(For simplicity, we have written $\langle A(x,\xi)x, \xi\rangle$ instead of

$$\langle A'(x,\xi)(\operatorname{Re} x, \operatorname{Im} x), (\operatorname{Re}\xi, \operatorname{Im}\xi)\rangle$$

where $A'(x,\xi)$ is a $2n \times 2n$ matrix; actually the notation $p(x,\xi) = \langle A(x,\xi)x,\xi\rangle$ makes sense at the level of formal Taylor expansion at ρ_0.)

Outline of Proof We proceed in several steps. In the topological step we start to define, as in Section 1.1, the outgoing/incoming regions relative to $\widehat{H}_p$, and study the flow of Lagrangian manifolds, as $t \to \pm\infty$. This yields, via a compactness argument, to C^0 coordinates where the outgoing (resp. incoming) submanifold $\mathcal{J}_+$ (resp. $\mathcal{J}_-$) is given by $\xi = 0$ (resp. $x = 0$). Then we turn to differentiability and prove the $\mathcal{J}_\pm$ are C^1. Finally we turn to higher derivatives and properties of almost analyticity.

We first choose coordinates where F_{ρ_0} has block-diagonal form. Taking complex linear coordinates as in (1.3), we can make it diagonal. Then the Hamiltonian vector field takes the form

$$H_p = A_0 x \cdot \frac{\partial}{\partial x} - A_0\xi \cdot \frac{\partial}{\partial \xi} + \mathcal{O}\big(\|x,\xi\|^2\big)\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial \xi}\Big) \bmod T_\infty^{(0,1)}(T^*\mathbf{C}^n) \tag{4.6}$$

where we recall $A_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. For real t, let $\widehat{\Phi}_t(\rho)$ be the Hamiltonian flow of $\widehat{H}_p$ as in (4.1). As in Section 1.1 we can construct an hermitian norm $\|\cdot\|_0$ such that identity (1.7) holds if $\|\cdot\|$ and $\|\cdot\|_0$ stand now for the hermitian norms. For $\delta > 0$, we define the outgoing region as

$$\Omega_\delta^{\text{out}} = \{(x,\xi) : \|\xi\|_0 < 2\|x\|_0,\ \|x\|_0^2 + \|\xi\|_0^2 < \delta^2\}$$

and let $\partial\Omega_\delta^{\text{out}}$ denote its boundary. Estimates (4.6) again show that there exists $C > 0$ such that

$$\|x\|_0/(2C) \le \widehat{H}_p\|x\|_0 \le \|x\|_0/C, \quad \rho \in \Omega_\delta^{\text{out}}, \tag{4.7}$$

while

$$-\widehat{H}_p\|\xi\|_0 \ge \|\xi\|_0/C \ \text{ on } \ \partial\Omega_\delta^{\text{out}} \cap \{(x,\xi) : \|\xi\|_0 = 2\|x\|_0\}. \tag{4.8}$$

Let $t \mapsto \widehat{\Phi}_t\big(x(0),\xi(0)\big)$ be an integral curve of $\widehat{H}_p$ with $\rho = \big(x(0),\xi(0)\big) \in \Omega_\delta^{\text{out}}$. Along $\widehat{\Phi}_t$ we have $\partial_t = \widehat{H}_p$, so using (4.7), Gronwall's Lemma, after suitably truncating p outside a neighborhood of ρ_0, shows that

$$e^{t/(2C)}\|x(0)\|_0 \le \|x(t)\|_0 \le e^{t/C}\|x(0)\|_0,\ \rho \in \Omega_\delta^{\text{out}}, t \ge 0, \tag{4.9}$$

which allows us to define the hitting times $T_\pm^{\text{out}}$ as in (1.10) and (1.11) (although we have not yet found the outgoing manifold).

It follows from (4.9) and (4.8) that $\Omega_\delta^{\text{out}}$ is stable under $\widehat{\Phi}_t$, *i.e.,* if $\rho \in \Omega_\delta^{\text{out}}$, then $\exp(t\widehat{H}_p)(\rho) \in \Omega_\delta^{\text{out}}$, for $0 \le t < T_\pm^{\text{out}}(\rho)$, while it never gets back in afterwards.

Similarly, we define the incoming region as in (1.12), and the corresponding hitting times as in (1.13), (1.14).

Now we try to find the outgoing/incoming manifolds for $\widehat{H}_p$, and study the evolution of the complex manifold $\Lambda_t = \{\exp(t\widehat{H}_p)(\rho) : \rho \in \Omega_\delta^{\text{out}}\}$, as $t \to +\infty$. It is convenient to introduce

$$\Lambda^{\text{out}} = \Big\{ \big(t,\tau;\exp(t\widehat{H}_p)(\rho)\big) : \rho \in \Omega_\delta^{\text{out}}, 0 \le t < T_+^{\text{out}}(\rho), \tau = \operatorname{Re} p \circ \exp(t\widehat{H}_p)(\rho)\Big\}$$

By what we have just said, Λ^{out} is a connected submanifold of codimension 1 in the symplectic space $T^*\mathbf{R}^{2n+1}$ endowed with the 2-form $d\tau \wedge dt + \operatorname{Re}\sigma_{\mathbf{C}}$. The vector field $\partial_t + \widehat{H}_p$ is tangent to Λ^{out}, and τ is independent of t. The evolution of a tangent vector $\widehat{X}_\rho(t) = \big(\widehat{X}_x(t), \widehat{X}_\xi(t)\big) \in T(T^*\mathbf{R}^{2n})$ (the ρ-projection of the tangent space to Λ^{out},) is given by the $4n \times 4n$ system:

$$\partial_t \widehat{X}_\rho(t) = \frac{\partial \widehat{H}_p}{\partial \rho}\big(\widehat{\Phi}_t(\rho)\big)\widehat{X}_\rho(t) \tag{4.10}$$

where ∂_ρ denotes the gradient in the real sense.

It is easy to see that the leading term in the $4n \times 4n$ matrix $\partial\widehat{H}_p/\partial\rho$ in the basis B has a hyperbolic structure, each eigenvalue λ_j occurring twice, as well as $-\lambda_j$, $\pm\overline{\lambda}_j$, so that the linear flow is expansive in the $(\operatorname{Re} x, \operatorname{Im} x)$- directions, and contractive in the $(\operatorname{Re}\xi, \operatorname{Im}\xi)$- directions.

So (4.10) shows that if $\varepsilon_0 > 0$ and $\delta > 0$ are sufficiently small, then the outgoing region

$$\|\widehat{X}_\xi(t)\| \le \varepsilon_0 \, \|\widehat{X}_x(t)\| \tag{4.11}$$

is stable along $\widehat{\Phi}_t(\rho)$, $\rho \in \Omega_\delta^{\mathrm{out}}$, as t increases, $t < T_+^{\mathrm{out}}(\rho)$.

Now let $\widetilde{\mathcal{J}}_+ = \{\rho \in \Omega_\delta^{\mathrm{out}}; \xi = 0\}$, $\mathcal{J}_+(t) = \widehat{\Phi}_t(\widetilde{\mathcal{J}}_+) \cap \Omega_\delta^{\mathrm{out}}$, and Λ_+ be its lift in Λ^{out}. This is a submanifold of $T^*\mathbf{R}^{2n+1}$, Lagrangian for $d\tau \wedge dt + \operatorname{Re}\sigma_{\mathbf{C}}$, and its tangent space contains $\partial_t + \widehat{H}_p$. Applying the theorem of constant rank to the projection $\pi\colon \Lambda_+ \to \mathbf{C}_x^n$, (4.11) shows that Λ_+ (or $\mathcal{J}_+(t)$, forgetting about τ which is independent of t, and that we may take equal to 0, since $p(\rho_0) = 0$), is of the form $\xi = g_+(t, x)$ where $g_+ \in C^\infty$ (see for instance [M] for a simple proof). Moreover, $g_+(0, x) = 0$. Since $\widehat{\Phi}_t(\rho) \in \Omega_\delta^{\mathrm{out}}$, we have $\|g_+(t, x)\|_0 \le 2\|x\|_0$ for all $t \ge 0$. By compactness, there is a sequence $t_j \to +\infty$, such that $g_+(t_j, \cdot\,) \to G_+$ in $C^0\big(\{\|x\| < \mathrm{const}\cdot\delta\}\big)$. We put

$$\mathcal{J}_+ = \{(x, G_+(x)) : x \in \operatorname{neigh}(0)\}$$

(the outgoing tail, or outgoing manifold) and proceed to show that $G_+ \in C^1$.

Consider the evolution of a normal vector

$$\widehat{Z}_\rho(t) = \big(\widehat{Z}_x(t), \widehat{Z}_\xi(t)\big) \in N(\mathcal{J}_+(t)) = \big(T(T^*\mathbf{R}^{2n})|_{\mathcal{J}_+(t)}\big)/T(\mathcal{J}_+(t))$$

(the ρ-projection of the normal space to Λ_+). It is given by $\partial_t\widehat{Z}_\rho(t) = M(\widehat{\Phi}_t(\rho))\widehat{Z}_\rho(t)$ where the leading part of $M(x, \xi)$ is obtained from that of $\partial\widehat{H}_p/\partial\rho$ by permuting the eigenvalues with positive and negative real parts. So in Λ_+, the region given by

$$\|\widehat{Z}_\xi(t)\| \ge \|\widehat{Z}_x(t)\|/\,\varepsilon_0 \tag{4.12}$$

is stable under $\widehat{\Phi}_t$.

Now let ρ_t be another integral curve of $\widehat{H}_p$, starting at $\rho \in \Omega_\delta^{\mathrm{out}}$, and not in $\mathcal{J}_+(t)$ (ρ_t lies in Λ^{out}, but we choose the initial condition away from $\widetilde{\mathcal{J}}_+$). Let Γ_t be the orthogonal projection of ρ_t on $\mathcal{J}_+(t)$, $\dot{\Gamma}_t \in N(\mathcal{J}_+(t))$ the normal vector. By (4.12), we see that if γ_t denotes the length of the segment $[\rho_t, \Gamma_t]$, then $\frac{d}{dt}\gamma_t \le -C\gamma_t$, $C > 0$; so the integral curves of $\widehat{H}_p$ approach $\mathcal{J}_+$ exponentially fast as t increases, and the estimate $\|g_+(t, x) - g_+(s, x)\| = \mathcal{O}(e^{-s/C})$, all $t \ge s \ge 0$, shows that $g_+(t, x)$ is Cauchy, and $T(\mathcal{J}_+(t))$ has a limit as $t \to +\infty$ (not only for a sequence t_j). This limit is the tangent space to $\mathcal{J}_+ = \{\xi = G_+(x) : x \in \operatorname{neigh}(0)\}$, and it follows that $\mathcal{J}_+(t)$ tends exponentially fast to $\mathcal{J}_+$ in the C^1 topology. It is easy to see that $\mathcal{J}_+$ is invariant under $\widehat{\Phi}_t$, all t, and characterized as the set of $\rho \in \Omega_\delta^{\mathrm{out}}$ such that $\widehat{\Phi}_t(\rho) \in \Omega_\delta^{\mathrm{out}}$, all $t \le 0$. We have $\widehat{\Phi}_t(\rho) \to \rho_0 = 0$ as $t \to -\infty$, $\rho \in \mathcal{J}_+$. Moreover, $\operatorname{Re} p = \tau = 0$ on $\mathcal{J}_+$.

We are left to show that $\mathcal{J}_+$ is a Lagrangian submanifold for $(T^*\mathbf{C}^n, \operatorname{Re}\sigma_{\mathbf{C}})$. If u_1, u_2 are complex C^1 functions vanishing on $\mathcal{J}_+$, and $\rho \in \mathcal{J}_+$, then $\{u_1, u_2\}_R(\rho) = \{u_1 \circ \Phi_t, u_2 \circ \Phi_t\}_R(\Phi_{-t}(\rho))$. Since integral curves of $\exp t\widehat{H}_p$ approach $\mathcal{J}_+$ exponentially fast, we see that $du_j \circ \Phi_t\big(\Phi_{-t}(\rho)\big)$ tends to 0 as $t \to +\infty$, hence

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$\{u_1, u_2\}_R = 0$, and we have proved that $\mathcal{J}_+$ is involutive. Because $T_{\rho_t}\mathcal{J}_+(t)$ is transversal to $\widetilde{\mathcal{J}}_- = \{\rho \in \Omega_\delta^{\text{out}}, x = 0\}$ (and their intersection is 0) we have also proved, letting $t \to +\infty$, that $\mathcal{J}_+$ is Lagrangian for $\operatorname{Re}\sigma_{\mathbf{C}}$. Furthermore, $T_{\rho_0}(\mathcal{J}_+) = \Lambda_+$. Similarly, we introduce

$$\Lambda^{\text{in}} = \left\{ \left(t, \tau; \exp(-t\widehat{H}_p)(\rho)\right) : \rho \in \Omega_\delta^{\text{in}}, 0 \le t < T_-^{\text{in}}(\rho), \tau = \operatorname{Re} p \circ \exp(-t\widehat{H}_p)(\rho) \right\}$$

Taking the flow of $\widetilde{\mathcal{J}}_-$ through $\widehat{\Phi}(t)$ for negative t, we set $\mathcal{J}_-(t) = \widehat{\Phi}_t(\widetilde{\mathcal{J}}_-) \cap \Omega_\delta^{\text{in}}$, and look for the evolution of a tangent vector to $\mathcal{J}_-(t)$ along an integral curve ρ_t of $\widehat{H}_p$, starting at $\rho \in \Omega_\delta^{\text{in}}$, and not in $\mathcal{J}_-(t)$. Letting $t \to -\infty$, we can see that $\mathcal{J}_-(t)$ tends exponentially fast to $\mathcal{J}_- = \{(G_-(\xi), \xi) : \xi \in \operatorname{neigh}(0)\}$, for some C^1 function $G_-(\xi)$. Then $\mathcal{J}_-$ is again Lagrangian with respect to $\operatorname{Re}\sigma_{\mathbf{C}}$, and we call it the incoming tail, or incoming manifold. Again we have $\operatorname{Re} p = \tau = 0$ on $\mathcal{J}_-$.

It is clear that the invariant manifolds $\mathcal{J}_\pm$ are characterized as the set of $\rho \in \Omega_\delta$ such that $\widehat{\Phi}_{\mp t}(\rho) \in \Omega_\delta$, for all $\pm t \ge 0$.

The higher derivatives cannot apparently be handled with the same method, but by the uniqueness property of the outgoing/incoming manifolds, we can conclude as in [AbRo, App. C] with a fixed point argument, the limits being necessarily $\mathcal{J}_\pm$. An alternative way is to mimick the proof of [HeSj1, Prop. 2.3]. Namely, it follows easily from the previous arguments that $\mathcal{J}_+(t)$ (say) can be parametrized by a phase function $\varphi_t(x, \eta)$, such that the graph of $\exp(t\widehat{H}_p)$, $t \ge 0$, is given by

$$C_t = \left\{ \left(\partial_\eta \varphi_t, \eta, x, \partial_x \varphi_t\right) : (x, \eta) \in \Omega_\delta^{\text{out}} \right\}.$$

Furthermore, φ_t verifies the eikonal equation

$$\frac{\partial \varphi_t}{\partial t} + \operatorname{Re} p\left(x, \frac{\partial \varphi_t}{\partial x}\right) = 0, \ \varphi|_{t=0} = \langle x, \eta \rangle$$

By the previous estimates, we know then that φ_t tends exponentially fast as $t \to +\infty$, to some $\varphi_+(x, \eta)$ in $C^2(\Omega_\delta^{\text{out}})$. Then $\varphi_+(x, \eta)$ verifies again the corresponding stationary eikonal equation, and parametrizes $\mathcal{J}_+$. Using the transport equations verified by $\frac{\partial \varphi_t}{\partial x}$, we can show as in [HeSj1] that this convergence holds actually in $C^\infty(\Omega_\delta^{\text{out}})$. We proceed similarly in $\Omega_\delta^{\text{in}}$.

Once we have found the smooth, involutive invariant manifolds $\mathcal{J}_\pm$, we choose adapted coordinates of the form $(x', \xi') = \left(x - G_-(\xi), \xi - G_+(x)\right)$. By construction, these are smooth symplectic coordinates for $\operatorname{Re}\sigma_{\mathbf{C}}$, where the outgoing (resp. incoming) manifold takes the form $\xi' = 0$ (resp. $x' = 0$.) From now on, we work in these coordinates, which we denote again by (x, ξ), deleting the prime. The same argument as in Section 1 then shows that (1.8) and (1.9) hold for $\rho \in \Omega_\delta$, $t \in \mathbf{R}$, where $(x(t), \xi(t))$ stands for $\widehat{\Phi}_t(\rho)$, and $\|\cdot\|_0$ for the hermitian norm.

We pass now to the almost analyticity property. Using coordinates adapted to $\mathcal{J}_\pm$, this can be done again by combining Lemma 4.1 with the method above, showing that the generating functions verify $\overline{\partial}\varphi_\pm \in I^\infty$. (Alternatively, this can be done by the fixed point argument of [AbRo, App. C].) The Theorem easily follows, since also (4.5) can be recovered from (4.4), using that p verifies the Cauchy-Riemann equations modulo I^∞. ■

4.3 Proof of Theorem 0.4

We proceed exactly as in the real case. Again let $\chi^{\text{out}}+\chi^{\text{in}} = 1$ be a smooth partition of unity in $T^*\mathbf{R}^{2n} \setminus \rho_0$ with supp $\chi^{\text{out}} \subset \{\|\xi\|_0 < 2\|x\|_0\}$, supp $\chi^{\text{in}} \subset \{\|x\|_0 < 2\|\xi\|_0\}$. We start with

Proposition 4.3 *Let p be as above, and $g \in I^\infty$. Let*

$$f^{\text{out}}(\rho) = \int_{-\infty}^{0} \big(\chi^{\text{out}} g\big) \circ \exp(t\widehat{H}_p)(\rho)\, dt, \;\; f^{\text{in}}(\rho) = -\int_{0}^{\infty} \big(\chi^{\text{in}} g\big) \circ \exp(t\widehat{H}_p)(\rho)\, dt$$

Then $f = f^{\text{out}} + f^{\text{in}} \in I^\infty$ solves $\widehat{H}_p f = g$.

We use throughout the C^∞ coordinates as in Theorem 4.2 where $\mathcal{J}_\pm$ are given by $\xi = 0$ and $x = 0$, as we did in Proposition 1.2.

Using again Birkhoff series (in $\mathbf{C}^{2n}$), we know that there is a smooth canonical transform for the complex symplectic structure $(T^*\mathbf{C}^n, \sigma_{\mathbf{C}})$, $\kappa(\rho_0) = \rho_0$, such that

$$p \circ \kappa(x, \xi) = q_0(\iota) + r(x, \xi) \tag{4.13}$$

where $\iota = (\iota_1, \dots, \iota_n)$ are the action variables as in (0.3), and $r \in I^\infty$. The Hamiltonian $q_0(\iota)$ satisfies the same hypotheses as p, and is constructed from the formal Taylor series by a Borel sum of the type $q_0(\iota) = \sum_{k=1}^{\infty} \widetilde{q}_k(\iota)\chi\big(\frac{\iota}{\varepsilon_k}\big)$, $\chi \in C_0^\infty(\mathbf{C}^n)$ equal to 1 near 0, of the form $\chi(z_1, \dots, z_n) = \chi_0(z_1) \otimes \cdots \otimes \chi_0(z_n)$, χ_0 rotation invariant. Of course, $\overline{\partial}_\iota q_0(\iota) = \mathcal{O}(\iota^\infty)$. Using again Borel sums, the canonical transformation is of the form $\kappa = \exp H_{\widetilde{f}}$ for some smooth $\widetilde{f}$, $\overline{\partial}_\rho \kappa = \mathcal{O}(\rho^\infty)$. Now we take real part of (4.13):

$$\operatorname{Re} p \circ \kappa(x, \xi) = q_0'(\iota') + r'(x, \xi),$$

where ι' stands for the real and imaginary part of ι (it is easy to see that these $2n$ new action variables Poisson commute for $\{\cdot, \cdot\}_R$). Following the proof of Theorem 0.1, we consider the family $q_s' = q_0' + sr'$, $0 \le s \le 1$.

As above we look for a family of smooth κ_s preserving $\operatorname{Re}\sigma_{\mathbf{C}}$, satisfying the identity $q_s' \circ \kappa_s = q_0'$ and

$$\partial_s \kappa_s = X_s \circ \kappa_s. \tag{4.14}$$

We look for X_s of the form $X_s = \widehat{H}_{f_s}$, for some family of real valued functions $f_s \in I^\infty$. Since q_s' is real, we get

$$\langle \widehat{H}_{f_s}, q_s' \rangle = -\langle \widehat{H}_{q_s'}, f_s \rangle = -r',$$

and again we are led to solve the homological equation $\langle \widehat{H}_{q_s'}, f_s \rangle = r'$, for which Proposition 4.3 gives $f_s \in I^\infty$. Then (4.14) has a solution of the form $\kappa_s = \operatorname{Id} + \kappa_s'$, $\kappa_s' \in I^\infty$, uniformly for s on compact sets. Furthermore, by construction, κ_s preserves $\operatorname{Re}\sigma_{\mathbf{C}}$, and $(\kappa_s)^*\sigma_{\mathbf{C}} = \sigma_{\mathbf{C}} \bmod I^\infty$. Theorem 0.4 easily follows.

4.4 Remark: Monodromy Along IR-Manifolds

Let p be analytic and have a non degenerate critical point at $\rho = 0$, such that F_{ρ_0} has rationally independent eigenvalues, none purely imaginary, as above. Assume p is real on the real domain. We can apply Theorem 0.1 to $T^*\mathbf{R}^n$ so p is integrable in the C^∞ sense on the real domain, for some real canonical transform $\kappa = \kappa_0$ that takes p into its Birkhoff normal form. We set $\Lambda_0 = T^*\mathbf{R}^n$ and try to move Λ_0 around ρ_0 in the complex domain, so we consider the family of IR-manifolds $\Lambda_s = \exp(isH_p)(\Lambda_0)$, $s \in \mathbf{R}$, which is defined for all real t. (Recall that a submanifold of $T^*\mathbf{C}^n$ is called IR if it is Lagrangian for $\operatorname{Im}\sigma_{\mathbf{C}}$ and symplectic for $\operatorname{Re}\sigma_{\mathbf{C}}$). Then again p is clearly integrable on Λ_s, in the C^∞ sense, *i.e.,* for real times, and one can address the problem of monodromy. The 1–dimensional case has been settled in [HeSj2, App. B], where the authors recover the well-known fact that p is integrable in the holomorphic sense; here κ is univalued, so making a reflection on ρ_0 gives $\Lambda_\pi = \Lambda_{2\pi} = \Lambda_0$. This is actually the way that the "exact Birkhoff normal form" was obtained. In several variables we cannot expect integrability, nor even recovering $\Lambda_s = \Lambda_0$ for some s, since the orbits may never close (see [Ro2] for a more complete study of monodromy).

A Appendix

A.1 The Birkhoff Transformations

We recall here from [KaRo] some formal constructions, using Lie brackets, borrowed essentially from [AbMa, p. 500]. There are of course many alternative proofs, the idea here is just to write formal power series in the most convenient way. Since the procedure is mere algebra, it works equally in the holomorphic, real analytic or C^∞ category, and eigenvalues also can be real or complex. When eigenvalues are complex, and the Hamiltonian real and C^∞, we can recover real asymptotics just by using an appropriate linear symplectic transformation of coordinates. As in [IaSj], we discuss the parameter dependent case. In what follows, $s \in \operatorname{neigh}(0, \mathbf{R}^k)$.

Let $p = p(s)$ depend smoothly on s, and have a non degenerate critical point of hyperbolic type at ρ_s. If $p(s)$ is complex valued, we assume also that $\overline{\partial}_{(z,\zeta)}p$ vanishes of infinite order at ρ_s, so that $p(s)$ has formal Taylor series in (z,ζ) at ρ_s. After a linear symplectic change of coordinates, depending smoothly on s, we may assume that $\rho_s = \rho_0 = 0$, and $p(s)$ has quadratic part $p_2(z,\zeta,s) = \sum_{j=1}^n \lambda_j(s)z_j(s)\zeta_j(s)$. We assume also that $p(0)$ has rationally independent (or non resonant) frequencies $(\lambda_1(0),\dots,\lambda_n(0)) = (\lambda_1,\dots,\lambda_n)$. Using the fact that the symplectic group is connected, we may further perform a symplectic, linear change of coordinates, C^∞ in s, such that $z_j(s), \zeta_j(s)$ become independent of s, and $p_2(z,\zeta,s) = \sum_{j=1}^n \lambda_j(s)z_j\zeta_j$. Of course, the $\lambda_j(s)$'s do not in general verify the non-resonance condition for $s \neq 0$, but we shall investigate up to which accuracy Birkhoff series hold in that case. After reduction of the quadratic part as above, $p(s)$ now takes the form

$$p(z,\zeta,s) = p_2(z,\zeta,s) + \mathcal{O}(|z,\zeta|^3).$$

We want to construct a map $f = f(s)$ between neighborhoods $\mathcal{V}(0)$ of $\rho_0 = 0 \in T^*\mathbf{R}^n$, such that $(\exp H_{f(s)})^* H_{p(s)}$ is resonant, modulo $\rho^3\mathcal{O}(s^\infty)$. Indeed we have:

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Proposition A.1 *Let $p(s) = p(z, \zeta, s)$ as above, and $\rho = (z, \zeta)$. Then there exists a smooth canonical transforms $\kappa(s)\colon \mathcal{V}(0) \to \mathcal{V}(0)$ in $T^*\mathbf{R}^n$, and a smooth function $q(s) = q(\iota, s)$, ι as in (0.3), such that $\kappa(\rho_0, s) = \rho_0 = 0$, $d\kappa(\rho_0, s) = \mathrm{Id}$ and*

$$p(s) \circ \kappa(s) = q(s) + \rho^3 \mathcal{O}\big((\rho, s)^\infty\big) \tag{A.1}$$

Proof For simplicity, we assume $k = 1$, but the general case is similar. We introduce a small ordering parameter ε and rescale coordinates (y, η), as $(\varepsilon y, \varepsilon \eta) = (z, \zeta)$ so that $p(z, \zeta, s) = \varepsilon^2 p_2(y, \eta, s) + \varepsilon^3 p_3(y, \eta, s) + \cdots$ where p_j is homogeneous of degree j. Working first at the level of formal Taylor series, we want to solve (formally), denoting $p = p(s)$, $f = f(s)$:

$$(\exp tH_f)^* H_p = \sum_{j \geq 0} \frac{t^j}{j!} [H_f, [H_f, \ldots, [H_f, H_p] \ldots]] = H_r, \tag{A.2}$$

where $r = r(s)$ is resonant, modulo $\rho^3 \mathcal{O}(s^\infty)$, and $t = \varepsilon^2$. We look also for $f(y, \eta, s) = \varepsilon f_1(y, \eta, s) + \varepsilon^2 f_2(y, \eta, s) + \cdots$ with f_j homogeneous of degree $j + 2$. We proceed by induction. Collecting the ε^3-terms in (A.2), we want to find f_1 such that $H_{p_3} - H_{\{p_2, f_1\}}$ is resonant modulo $\rho^2 \mathcal{O}(s^\infty)$, *i.e.*, $p_3 - \{p_2, f_1\}$ is resonant modulo $\rho^3 \mathcal{O}(s^\infty)$. Writing $p_3(y, \eta, s) = \sum_{|\alpha+\beta|=3} p_{\alpha\beta}(s) y^\alpha \eta^\beta$, $f_1(y, \eta, s) = \sum_{|\alpha+\beta|=3} a_{\alpha\beta}(s) y^\alpha \eta^\beta$ we try to achieve this condition at any order in s. At zeroth order, *i.e.*, for $s = 0$, we take $a_{\alpha\beta}(0) = -p_{\alpha\beta}(0) / \langle \lambda, \alpha - \beta \rangle$ for $\alpha \neq \beta$ and $a_{\alpha\beta}(0) = 0$ otherwise. At first order in s, the condition that $\partial_s \big(p_3 - \{p_2, f_1\} \big) \big|_{s=0}$ is resonant modulo $\rho^3 \mathcal{O}(s^\infty)$ gives

$$\partial_s a_{\alpha\beta}(0) = \frac{\partial_s p_{\alpha\beta}(0) - \langle \partial_s \lambda(0), \alpha - \beta \rangle a_{\alpha\beta}(0)}{\langle \lambda, \alpha - \beta \rangle}$$

when $\alpha \neq \beta$ and say, $\partial_s a_{\alpha\beta}(0) = 0$ otherwise. This process extends by induction to any order in s (note that when s is vector valued, we need to check symmetry for higher derivatives.)

So far we have constructed the formal Taylor series for $a_{\alpha\beta}(s)$ at $s = 0$, and found $f_1(s)$ with an uncertainty $\rho^3 \mathcal{O}(s^\infty)$ (in the original variables). Next we collect the ε^4-terms, which gives:

$$p_4 - H_{p_2} f_2 - H_{p_3} f_1 + \frac{1}{2} \{f_1, \{f_1, p_2\}\} =_{\mathrm{def}} -H_{p_2} f_2 + q_4.$$

We want to find $f_2 = f_2(s)$ such that $-H_{p_2} f_2 + q_4$ is resonant modulo $\rho^2 \mathcal{O}(s^\infty)$. Writing $q_4(y, \eta, s) = \sum_{|\alpha+\beta|=4} q_{\alpha\beta}(s) y^\alpha \eta^\beta$, $f_2(y, \eta, s) = \sum_{|\alpha+\beta|=4} a_{\alpha\beta}(s) y^\alpha \eta^\beta$, we look again for the Taylor series $a_{\alpha\beta}(s) = a_{\alpha\beta}(0) + \partial_s a_{\alpha\beta}(0) s + \frac{1}{2} \partial_s^2 a_{\alpha\beta}(0) s^2 + \cdots$. At zeroth order we may take $a_{\alpha\beta}(0) = 0$ for $\alpha = \beta$, and $a_{\alpha\beta}(0) = q_{\alpha\beta}(0) / \langle \lambda, \alpha - \beta \rangle$ otherwise, then carry on the procedure as above at any order in s. This gives $H_{p_2} f_2 = r_4$, where $r_4(s) = \sum_{|\alpha|=2} q_{\alpha\alpha}(s) y^\alpha \eta^\alpha$ is the resonant part of q_4, modulo $\rho^3 \mathcal{O}(s^\infty)$.

Assume by induction that we have already constructed $f_1, \ldots, f_{N-1}$ homogeneous of degree $3, \ldots, N+1$, so that $f^{(N-1)} = \sum_{j=1}^{N-1} \varepsilon^j f_j$ verifies (A.2) at order ε^{N+1} in ρ, and infinite order in s. Then we try $f^{(N)} = f^{(N-1)} + \varepsilon^N f_N$ to fulfill (A.2) up to order ε^{N+2}, *i.e.,* find $f_N = f_N(s)$ such that

$$(A.3)\quad H_p + [H_f, H_p] + \cdots + \frac{t^{N-1}}{(N-1)!}[H_f,[H_f,\ldots,[H_f,H_p]\cdots]] + \frac{t^N}{N!}[H_f,[H_f,\ldots,[H_f,H_p]\cdots]]$$

is resonant modulo $\rho^2\mathcal{O}(s^\infty)$. Each of the terms of that sum are expanded to order ε^{N+2}. The last one is an N-fold bracket and contains only $[H_{f_1},[H_{f_1},\ldots,[H_{f_1},H_{p_2}]\cdots]]$ to this order; other terms are j-fold brackets containing $f_1,\ldots,f_{N-1}$, and f_N occurs only in $[H_{f_N},H_{p_2}]$. Writing $f_N(y,\eta,s) = \sum_{|\alpha+\beta|=N+2} a_{\alpha\beta}(s)y^\alpha\eta^\beta$, we can find $a_{\alpha\beta}(s)$ as before so that $-H_{p_2}f_N + q_{N+2} = r_{N+2}$ where $q_{N+2} = q_{N+2}(s)$ and $r_{N+2} = r_{N+2}(s)$ are of degree $N+2$ (or $r_{N+2}(s) = 0$, according to the parity of N), and $r_{N+2}(s)$ is resonant modulo $\rho^3\mathcal{O}(s^\infty)$.

Summing up, we have found $f_j,\ r_j$, $\deg(f_j) = j+2$, $\deg(r_j) = j$ such that

$$(\exp tH_{(\varepsilon f_1+\varepsilon^2 f_2+\cdots)})^* H_{(\varepsilon^2 p_2+\varepsilon^3 p_3+\varepsilon^4 p_4+\cdots)} = H_{\varepsilon^4 r_4+\cdots}$$

so (A.2) is verified at the level of formal power series. In the original variables $(z,\zeta) = \varepsilon(y,\eta)$, so by homogeneity: $(\exp H_{f(s)})^* H_{p(s)} = H_{r(s)}$.

All this computation can be implemented at the level of C^∞ germs of functions at $\rho = \rho_0(0,0), s = 0$ if we apply Borel's theorem to the $f_j(s)$ and $r_j(s)$. Hence the relation $(\exp H_{f(s)})^* H_{p(s)} = H_{r(s)}$ holds at the level of C^∞ germs, with $r(s)$ resonant modulo $\rho^3\mathcal{O}(s^\infty)$, *i.e.,* asymptotic to a C^∞ function of $(z_1\zeta_1,\ldots,z_n\zeta_n)$. Since $(\exp H_f)^* H_p = H_{p\circ\exp H_f}$ [AbMa, p. 194], we get $H_{p\circ\exp H_f} = H_r$, and so $p\circ\exp H_f = r$ is resonant modulo $\rho^3\mathcal{O}(s^\infty)$. So we have proved the Proposition with $\kappa(s) = \exp H_{\widetilde{f}(s)}$, where $\widetilde{f}(s)$ is a Borel sum for $f(z,\zeta,s)$. ∎

A.2 Families of Fourier Integral Operators

We review the most fundamental properties of FIOs needed in the main text, following essentially the book by V. Ivrii [Iv, Section 1].

First item composing our toolbox is the class $S^m(T^*\mathbf{R}^n)$ of smooth symbols in h of order $m \in \mathbf{Z}$ on $T^*\mathbf{R}^n$, *i.e.,* $h^{-m}a(\rho,h) = a_0(\rho) + ha_1(\rho) + \cdots$ (in the sense of asymptotic series in h), where a_j are C^∞ functions defined in a (fixed) neighborhood of ρ_0. Of course, a may depends on other parameters, and this dependence will also be smooth. We shall always work microlocally near ρ_0, which roughly means that symbols are compactly supported near $\rho_0 = (x_0,\xi_0)$, and only defined modulo $\mathcal{O}(h^\infty)$. We call also "amplitude" an asymptotic sum $a(x,y,\theta,h)$ depending on the position variable $x, y \in \mathbf{R}^n$, defined near $x = x_0$, $y = y_0$, and possibly on other phase variables $\theta \in \mathbf{R}^N$. Their class is again denoted by $S^m = S^m(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^N)$, etc.

Next item is the class of smooth (real) phase functions $\phi(x,y,\theta)$, $(x,y,\theta) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^N$, nondegenerate in the sense that $d\phi'_{\theta_1},\ldots,d\phi'_{\theta_N}$ are linearly independent on the critical set $C_\phi = \{(x,y,\theta) : \phi'_\theta = 0\}$. If ϕ is nondegenerate, the map $\iota\colon (x,y,\theta) \in$

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$C_\phi \mapsto (x, \phi'_x; y, -\phi'_y)$ is a (local) diffeomorphism onto its range Λ_ϕ. Then $\Lambda = \Lambda_\phi$ is a Lagrangian submanifold of $T^*\mathbf{R}^{2n}$ for the 2-form $d\xi \wedge dx + d\eta \wedge dy$, and the graph of a canonical transform κ. Conversely, if Λ is Lagrangian and $\pi\colon \Lambda \to T^*\mathbf{R}^n$, $(x, \xi; y, \eta) \mapsto (x, \eta)$ is non-degenerate, then Λ is the graph of a canonical map, and if we consider a generating function φ, then $\Lambda = \Lambda_\phi$, with the standard phase $\phi(x, y, \theta) = \varphi(x, \eta) - y\eta$, $\theta = \eta$. We say usually that ϕ quantizes, or parametrizes κ. These objects may not defined everywhere, but we shall always assume that Λ_ϕ contains a neighborhood of $(\kappa(\rho_0), \rho_0,)$. On C_ϕ there is a natural half-density $\delta_C^{1/2}$, and the inertial index sgn Φ, where Φ is the Hessian of ϕ, with respect to all variables, is a well-defined integer.

Given an amplitude $a \in S^0$ and a non-degenerate phase function ϕ as above, a FIO is a linear operator A on $C_0^\infty(\mathbf{R}^n)$ with Schwartz kernel of the form

$$K_A(x, y) = I(a, \phi)(x, y) = (2\pi h)^{-(n+N)/2} \int e^{i\phi(x,y,\theta)/h} a(x, y, \theta, h)\, d\theta.$$

Again we say that A quantizes κ, thinking of the case where A is (formally) unitary. The principal symbol of A is the function on Λ_ϕ defined by

$$a^0(\kappa(\rho), \rho) = e^{i\frac{\pi}{4}\operatorname{sgn}\Phi} a_0 \delta_C^{1/2} \circ \iota^{-1}(\kappa(\rho), \rho).$$

Again, such an FIO is only defined "microlocally near $(\kappa(\rho_0), \rho_0)$"; in the present case where $\kappa(\rho_0) = \rho_0$ we simply say that A is defined microlocally near ρ_0. The relevant setup is the notion of "frequency set", and we refer to [Iv, Section 1] for details.

Objects such as the canonical transform κ or the principal symbol a^0 are intrinsically attached to A, but not the phase or amplitude, which gives some degrees of freedom for writing an FIO. Namely, if A is defined through $I(a, \phi)$, and $\widetilde{\phi}(x, y, \theta)$ is another phase function parametrizing κ (the number of phase variables θ need not be the same as for ϕ), then there exists another amplitude $\widetilde{a}(x, y, \theta)$ such that $I(a, \phi) = I(\widetilde{a}, \widetilde{\phi})$, microlocally near ρ_0.

In particular, if A_s, $0 \le s \le 1$, is a smooth family of FIOs associated with the same canonical transformation κ, $\kappa(\rho_0) = \rho_0$, with $K_{A_0} = I(a_0, \phi_0)$ then there is a smooth family of amplitudes $a_s(x, y, \theta, h)$ such that $K_{A_s} = I(a_s, \phi_0)$, microlocally near ρ_0.

There exists a nice calculus of FIOs. They compose according to their canonical relation, in particular if the principal symbol of A is non vanishing, then A is invertible (microlocally near ρ_0), and we can choose for A^{-1} the phase $-\phi(y, x, \theta)$, which parametrizes κ^{-1}. Let A, B quantize κ and κ^{-1} respectively, and P be a h-PDO (a h-PDO is a particular FIO with $\kappa = \mathrm{Id}$; we shall always use Weyl quantization of symbols) then $Q = BPA$ is again a h-PDO. Denoting by $P(\rho, h) = p_0(\rho) + hp_1(\rho) + \cdots$ and $Q(\rho, h) = q_0(\rho) + hq_1(\rho) + \cdots$ their Weyl symbol, the following relation holds:

$$q_k = \sum_{j=0}^{k} \ell_{k-j}(x, \xi, \partial_x, \partial_\xi) p_j \circ \kappa \tag{A.4}$$

where ℓ_j are linear differential operators of degree $2j$. In particular, we have Egorov's Theorem:

$$q_0(\rho) = b^0\big(\rho, \kappa(\rho)\big)\,(p_0 \circ \kappa)(\rho) a^0\big(\kappa(\rho), \rho\big)\,. \tag{A.5}$$

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References

[AbMar] R. Abraham, J. Marsden, *The foundations of mechanics*. Benjamin, N.Y. Revised edition, 1978.

[AbRob] R. Abraham, J. Robbin (with an Appendix by A. Kelley), *Transversal mappings and flows*. Benjamin, New York, 1967.

[Ar] V. Arnold, *Les méthodes mathématiques de la mécanique classique*. Éditions Mir, Moscow, 1976.

[ArNo] V. Arnold, and S. Novikov,eds., *Dynamical systems III-IV*. Encyclopaedia of Mathematics. Springer-Verlag Berlin, 1988–1990.

[ArVaGo] V. Arnold, A. Varchenko, S. Goussein-Zadé, *Singularités des applications différentiables I*. Éditions Mir, Moscow, 1986.

[Au] M. Audin, *Les systèmes Hamiltoniens et leur intégrabilité*. Soc. Math. France **8**(2001).

[BamGraPa] D. Bambusi, S. Graffi, Th. Paul, *Normal forms and quantization formulae*. Comm. Math. Phys. **207**(1999) 173–195.

[BaLlWa] A. Banyaga, R. de La Llave, C. Wayne, *Cohomology equations near hyperbolic points and geometric versions of Sternberg linearization theorem*. J. Geom. Anal. **690**(1996), 613–649, .

[BeKo1] G. Belitskii, A. Kopanskii, *Sternberg theorem for equivariant Hamiltonian vector fields.* Nonlinear Anal. **47**(2001), 4491–4499, .

[BeKo2] ———, *Sternberg-Chen theorem for equivariant Hamiltonian vector fields.* In: Symmetry and perturbation theory III – SPT2001, D. Bambusi, M. Cadoni and G. Gaeta eds., World Scientific, River Edge, NJ, 2001.

[Bi] G. D. Birkhoff, *Dynamical systems*. Amer. Math. Soc. Colloquium Publ. 1927, revised ed. 1966.

[BrKo] I. Bronstein and A. Kopanskii, *Normal forms of vector fields satisfying certain geometric conditions*. In: Nonlinear dynamical systems and chaos. Birkhäuser, Basel, 1996, pp. 79–101.

[Bru] F. Bruhat, *Travaux de Sternberg*. Séminaire Bourbaki **6**,(1995), 179–196.

[Ch1] K.-T. Chen, *Collected papers of K.-T. Chen*, Birkhäuser Boston, Boston, MA, 2001.

[Ch2] ———, *Equivalence and decomposition of vector fields about an elementary critical point.* Amer. J. Math. **85**(1963), 693–722 (reprinted in [Ch1]).

[CuB] R. Cushman, L. Bates, *Global aspects of classical integrable systems.* Birkhäuser-Verlag, Basel, 1997.

[Ec] J. Ecalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Hermann, Paris, 1992.

[El] L. H. Eliasson, *Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case*. Comment. Math. Helv. **65**(1990), 4–35, .

[Fr] J. P. Françoise, *Propriétés de généricité des transformations canoniques*. In: Geometric dynamics, J. Palis, (ed.), Springer-Verlag, 1983, pp. 216–260.

[Gal] G. Gallavotti, *The Elements of mechanics*. Springer-Verlag, New York, 1983.

[GeSj] C. Gérard and J. Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type*. Comm. Math. Phys. **108**(1987), 391–421, .

[GiDeFoGaSim] A. Giorgilli, A. Delsham, E. Fontich, L. Galgani and C. Simò, *Effective stability for a Hamiltonian system near an equilibrium point with an application to the restricted three-body problem*. J. Differential Equations **77**(1989), 167–198, .

[Gr] S. Graff, *On the conservation of hyperbolic tori for Hamiltonian systems.* J. Differential Equations **15**(1974), 1–69,

[GuSc] V. Guillemin and D. Schaeffer, *On a certain class of fuchsian partial differential equations*. Duke Math. J. **44**(1977), 157–199, .

[Ha] P. Hartman, *Ordinary differential equations.* Wiley, New York, 1964.

[HeSj1] B. Helffer and J. Sjöstrand, *Multiple wells in the semi-classical limit III. Interaction through non-resonant wells.* Math. Nachr. **124**(1985), 263–313.

[HeSj2] ———, *Semi-classical analysis for Harper's equation III.* Soc. Math. France, Mém. (N.S.) **39**(1989).

[HiPuSh] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds.* Lecture Notes in Mathematics, 583, Springer-Verlag, Berlin, 1977.

[IaSj] A. Iantchenko and J. Sjöstrand, *Birkhoff normal forms for Fourier integral operators II.* Amer. J. Math. **124**)2002), 817–850.

[It] H. Ito, *Integrable symplectic maps and their Birkhoff normal form*. Tohoku Math. J. **49**(1997), 73–114.

[Iv] V. Ivrii, *Microlocal analysis and precise spectral asymptotics.* Springer-Verlag, Berlin, 1998.

[KaRo] N. Kaidi and M. Rouleux, *Quasi-invariant tori and semi-excited states for Schrödinger operators I. Asymptotics.* Comm. Partial Differential Equations **27**(2002), 1695–1750.

[M] P. Malliavin, *Géométrie différentielle intrinsèque.* Hermann. Paris, 1972.

[MaSo] A. Martinez and V. Sordoni, *Microlocal WKB expansions.* J. Funct. Anal. **168**(1999), 380–402.

[MeSj] A. Melin and J. Sjöstrand, *Determinants of pseudo-differential operators and complex deformations of phase space*. Methods Appl. Anal. **9**(2002), 177–237.

[Mo] J. Moser, *On the generalization of a theorem of A. Lyapunoff.* Comm. Pure Appl. Math. **11**(1958), 257–271, .

[Ne] E. Nelson, *Topics in dynamics I: Flows*. Princeton University Press, Princeton, NJ, 1969.

[Ro1] M. Rouleux, *Quasi-invariant tori and semi-excited states for Schrödinger operators II. Tunneling.* In preparation.

[Ro2] ———, *Integrability of an holomorphic Hamiltonian near a hyperbolic fixed point.* In preparation.

[Si1] C. L. Siegel, *Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung*. Nachr. Akad. Wiss. Göttingen (1952), 21–30.

[Si2] ———, *Über die Existenz einer Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung* Math. Ann. **128**(1954), 144-170.

[SiMo] C. L. Siegel and J. Moser, *Lectures on celestial mechanics*, Springer-Verlag, Berlin, 1971.

[Sie] S. Siegmund, *Normal forms for nonautonomous differential equations.* J. Differential Equations **178**(2001), 541–573.

[Sj1] J. Sjöstrand, *Singularités analytiques microlocales.* Astérisque **95**(1982).

[Sj2] ———, *Analytic wavefront sets and operators with multiple characteristics*. Hokkaido Math. J. **12**(1983), 392–433.

[Sj3] ———, *Semi-excited states in nondegenerate potential wells.* Asymptotic Anal. **6**(1992), 29–43, .

[SjZw] J. Sjöstrand and M. Zworski, *Quantum monodromy and semiclassical trace formulae.* J. Math. Pures Appl. **81**(2002), 1–33.

[St] S. Sternberg, *The structure of local diffeomorphisms III*. Amer. J. Math. **81**(1959), 578–604.

[Vi] M. Vittot, *Birkhoff expansions in Hamiltonian mechanics: a simplification of the combinatorics*. In: Non-linear dynamics, G. Turchetti, (ed.) World Scientific, Teaneck, NJ, 1989, pp. 276–286.

[Vu1] S. Vu Ngoc, *Sur le spectre des systèmes complètement intégrables semi-classiques avec singularités*. Ph.D. Thesis, Université de Grenoble, 1998.

[Vu2] ———, *On semi-global invariants for focus-focus singularities*. Topology **42**(2003), 365–380.

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