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ON DISCONTINUITY OF L^2-ANGLE

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Abstract

In this note the L^2-angle between two concentric rings and between the ring and the exterior of the disc in the complex plane are calculated. In the second part we prove that the L^2-angles between domains A and B and between $A \times C$ and $B \times C$ are equal. We give also some examples of discontinuity of the L^2-angle between domains.

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Introduction

The alternating projections and L^2-angle in the theory of the Bergman function were introduced by M. Skwarczyński [4, 5]. The application of this procedure leads in some cases to the explicit-analytic calculations of L^2-angle between domains in $\mathbb{C}^N$ (see [2, 5]).

Let A and B be two domains in $\mathbb{C}^N$, and put $D = A \cup B$. Set $F = L^2H(D) := \{f \in L^2(D) \colon f \text{ is holomorphic in } D\}$. Denote by F_i, $i = 1, 2$, the subspaces of $L^2(D)$ consisting of functions holomorphic in A and B respectively. Assume that $m(A \backslash B) > 0$ and $m(B \backslash A) > 0$ (here, as well as in the rest of the paper, m denotes the Lebesgue measure in $\mathbb{C}^N$). The L^2-angle $\gamma(A, B) \in [0, \frac{\pi}{2}]$ between A and B is given by (see [5, Section 1 (1)])

$$(1) \qquad \cos\gamma(A, B) = \sup\left\{\frac{|\langle f_1, f_2\rangle|}{\|f_1\|\,\|f_2\|} \colon f_i \in F_i \backslash \{0\},\ f_i \perp F,\ i = 1, 2\right\}.$$

Under the additional assumption that $L^2H(A) \neq 0$ or $L^2H(B) \neq 0$, one can prove (see [5]) that

$$(2)\quad \cos^2\gamma(A,B) = \sup\left\{ \frac{\|f\|^2_{A\setminus B} + \|\hat{f}\|^2_B}{\|f\|^2_A + \|f\|^2_{B\setminus A}} : f \in F_1\setminus\{0\},\ f \perp F, \text{ and } f \text{ holomorphic in } \operatorname{Int}(B\setminus A)\right\}.$$

(Here $\hat{f}$ is the Bergman projection of $f|_B$ in B.)

In this note we calculate the L^2-angle between two concentric rings on the complex plane, and between the ring and the exterior of the disc. We prove also a result on the L^2-angle between cartesian products of domains. Moreover, we give examples of discontinuity of the L^2-angle.

1. The case of rings

Let $A = \{z \in \mathbb{C}: 0 < r_1 < |z| < r_2\}$, $B = \{z \in \mathbb{C}: R_1 < |z| < R_2\}$, where $r_1 < R_1 < r_2 < R_2$. Let $F_1 = \{f \in L^2(D): f \in \operatorname{Hol}(A)\}$, $F_2 = \{f \in L^2(D): f \in \operatorname{Hol}(B)\}$.

THEOREM 1.
$$\cos^2\gamma(A,B) = \frac{\ln(R_1/r_1)\ln(R_2/r_2)}{\ln(R_2/R_1)\ln(r_2/r_1)}$$

PROOF. Consider a function f such that $f \in F_1\setminus\{0\}$,

$$(3)\qquad \langle f, g\rangle = 0 \quad \text{for every } g \in L^2H(D),$$

and f is holomorphic in $\operatorname{Int}(B\setminus A)$. This function on A and $B\setminus A$ has power series expansions

$$f|_A(z) = \sum_{n\in Z} a_n z^n, \qquad f|_{B\setminus A}(z) = \sum_{n\in Z} b_n z^n.$$

Denote $\|z^n\|^2_U = U(n)$. We calculate the expression in (2)

$$\|f\|^2_{A\setminus B} = \sum_{n\in Z} |a_n|^2 (A\setminus B)(n),$$
$$\|f\|^2_A = \sum_{n\in Z} |a_n|^2 A(n),$$
$$\|f\|^2_{B\setminus A} = \sum_{n\in Z} |b_n|^2 (B\setminus A)(n).$$

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Denote by P the Bergman projection in D. Since f is orthogonal to $L^2H(D)$, we have $Pf = 0$. Therefore, for every $n \in Z$, $0 = \langle Pf, z^n\rangle = \langle f, z^n\rangle = a_n A(n) + b_n (B\backslash A)(n)$, and so

$$b_n = -a_n \frac{A(n)}{(B\backslash A)(n)}. \tag{4}$$

Hence

$$\|f\|^2_{B\backslash A} = \sum_{n\in Z} |a_n|^2 \frac{A^2(n)}{(B\backslash A)(n)}.$$

Similarly, if $\hat{f} = \sum_{n\in Z} c_n z^n$ in B, then for every $n \in Z$, $c_n B(n) = \langle \hat{f}, z^n\rangle_B = \langle f|_B, z^n\rangle_B = a_n (A\cap B)(n) + b_n (B\backslash A)(n)$, and so

$$\begin{aligned}\|\hat{f}\|^2_B &= \sum_{n\in Z} |c_n|^2 B(n) \\ &= \sum_{n\in Z} \frac{|a_n(A\cap B)(n) + b_n(B\backslash A)(n)|^2}{B^2(n)} B(n) \\ &= \sum_{n\in Z} |a_n|^2 \left|\frac{(A\cap B)(n) - A(n)}{B(n)}\right|^2 B(n)\end{aligned}$$

by (4). Therefore the numerator and denominator in (2) are respectively

$$\|f\|^2_{A\backslash B} + \|\hat{f}\|^2_B = \sum_{n\in Z} |a_n|^2 \left\{\left|\frac{(A\cap B)(n) - A(n)}{B(n)}\right|^2 B(n) + (A\backslash B)(n)\right\}$$

and

$$\|f\|^2_A + \|f\|^2_{B\backslash A} = \sum_{n\in Z} |a_n|^2 \left\{A(n) + \left|\frac{A(n)}{(B\backslash A)(n)}\right|^2 (B\backslash A)(n)\right\}.$$

Note that $((A\cap B)(n) - A(n))^2 = (A(n) - (A\cap B)(n))^2 = ((A\backslash B)(n))^2$. Now we have

$$\begin{aligned}\frac{\|f\|^2_{A\backslash B} + \|\hat{f}\|^2_B}{\|f\|^2_A + \|f\|^2_{B\backslash A}} &\le \sup_{n\in Z} \frac{\frac{(A\backslash B)^2(n)}{B(n)} + (A\backslash B)(n)}{A(n) + \frac{A^2(n)}{(B\backslash A)(n)}} \\ &= \sup_{n\in Z} \frac{(A\backslash B)(n)}{A(n)} \left[\frac{1 + \frac{(A\backslash B)(n)}{B(n)}}{1 + \frac{A(n)}{(B\backslash A)(n)}}\right] = \sup_{n\in Z} \frac{(A\backslash B)(n)}{A(n)} \frac{(B\backslash A)(n)}{B(n)}.\end{aligned}$$

For $n \neq -1$,

$$\begin{aligned}(A\backslash B)(n) = \|z^n\|^2_{A\backslash B} &= \int_0^2 \int_{r_1}^{R_1} s s^{2n}\, ds\, d\varphi \\ &= \frac{\pi}{n+1}((R_1^2)^{n+1} - (r_1^2)^{n+1}).\end{aligned}$$

Analogously,

$$A(n) = \frac{\pi}{n+1}((r_2^2)^{n+1} - (r_1^2)^{n+1}),$$
$$(B\backslash A)(n) = \frac{\pi}{n+1}((R_2^2)^{n+1} - (r_2^2)^{n+1}),$$
$$B(n) = \frac{\pi}{n+1}((R_2^2)^{n+1} - (R_1^2)^{n+1}).$$

For $n = -1$,

$$(A\backslash B)(-1) = \|z^{-1}\|_{A\backslash B} = 2\pi \int_{r_1}^{R_1} \frac{1}{s}\,ds = 2\pi \ln \frac{R_1}{r_1},$$
$$A(-1) = 2\pi \ln \frac{r_2}{r_1}, \quad (B\backslash A)(-1) = 2\pi \ln \frac{R_2}{r_2}, \quad B(-1) = 2\pi \ln \frac{R_2}{R_1}.$$

Setting $a = r_1^2$, $b = R_1^2$, $c = r_2^2$, $d = R_2^2$, we obtain

$$\text{(5)} \quad \cos^2 \gamma(A, B) \le \sup \left(\frac{\ln(b/a)\ln(d/c)}{\ln(d/b)\ln(c/a)}, \frac{b^m - a^m}{d^m - b^m} \cdot \frac{d^m - c^m}{c^m - a^m} \right), \qquad m = n + 1 \neq 0.$$

On the other hand, if we define the functions f_n by setting f_n equal to z^n in A and to $-A(n)/(B\backslash A)(n)z^n$ in $B\backslash A$, then f_n satisfy the conditions from (2), and it is easy to check that $\sup_{n\in Z}\{(\|f\|^2_{A\backslash B} + \|\hat{f}\|^2_B)/(\|f_n\|^2_A + \|f_n\|^2_{B\backslash A})\}$ is equal to the right-hand side of (5). Therefore we have the equality in (5). It is easy to see that

$$\frac{b-a}{d-b}\frac{d-c}{c-a} \ge \frac{b^m - a^m}{d^m - b^m}\frac{d^m - c^m}{c^m - a^m}, \qquad m = 1, 2, \dots .$$

We want to show that

$$\frac{\ln(b/a)\ln(d/c)}{\ln(d/b)\ln(c/a)} > \frac{b-a}{d-b}\frac{d-c}{c-a} \quad \text{or} \quad \frac{\frac{\ln(b/a)}{b-a}}{\frac{\ln(d/b)}{d-b}} \frac{\frac{\ln(d/c)}{d-c}}{\frac{\ln(c/a)}{c-a}} > 1, \qquad a < b < c < d.$$

For $b = c$, the left-hand side is 1, so it suffices to show that the first factor on the left is increasing with respect to b on the interval (a, c). Without loss of generality we may assume that $a = 1$, and consider the expression

$$\frac{(\ln b)/(b-1)}{(\ln(d/b))/(d-b)}, \qquad 1 < b < d.$$

We shall show more: this expression is decreasing for $b \in (1, d)$. Substituting $b = d^s$, $s \in (0, 1)$, it suffices to show that

$$\frac{(s \ln d)(d^s - 1)}{((1-s)\ln d)/(d - d^s)}$$

is decreasing for $s \in (0, 1)$, or equivalently that

$$\frac{s}{1-s}\frac{d - d^s}{d^s - 1}$$

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is decreasing for $s \in (0,1)$. Assuming $0 < s < t < 1$, we want to show that

$$\frac{s}{1-s}\frac{d-d^s}{d^s-1} > \frac{t}{1-t}\frac{d-d^t}{d^t-1}$$

of (after dividing both sides by s) that

$$\frac{s}{1-s}\frac{1-d^{s-1}}{d^s-1} > \frac{t}{1-t}\frac{1-d^{t-1}}{d^t-1}.$$

Without loss of generality we may assume that $s = k/n$, $t = m/n$ are rational, with n equal to some power of 2, and k, m even. Then $1 - s = (n/k)/n$, $1 - t = (n-m)/n$. Let $p = d^{1/n} > 1$. We shall prove that for $k < m$,

$$\frac{k/n}{(n-k)/n}\frac{1-p^{k-n}}{p^k-1} > \frac{m/n}{(n-m)/n}\frac{1-p^{m-n}}{p^m-1},$$

or equivalently that

$$\frac{k}{n-k}\frac{p^{n-k}-1}{p^k-1} > \frac{m}{n-m}p^{m-k}\frac{p^{n-m}-1}{p^{n-k}-1}.$$

Note that by dividing by $p^{(m-k)/2}$ we obtain the inequality

$$\frac{k}{n-k}\frac{p^m-1}{p^k-1}\frac{1}{p^{(m-k)/2}} > \frac{m}{n-m}p^{(m-k)/2}\frac{p^{n-m}-1}{p^{n-k}-1}. \tag{6}$$

It is known that

$$\frac{p^m-1}{p^k-1}\frac{1}{p^{(m-k)/2}} = \frac{p^{m-1}+p^{m-2}+\cdots+1}{p^{(m+k)/2-1}+\cdots+p^{(m-k)/2}}$$

is increasing for $p \in (1,\infty)$ and that

$$p^{(m-k)/2}\frac{p^{n-m}-1}{p^{n-k}-1} = \frac{p^{n-\frac{m}{2}-\frac{k}{2}-1}+\cdots+p^{\frac{m}{2}-\frac{k}{2}}}{p^{n-k-1}+\cdots+1}$$

is decreasing for $p \in (1,\infty)$ [5]. Hence it suffices to verify (6) for $p = 1$, in which case it reduces to

$$\frac{k}{n-k}\frac{m}{k} = \frac{m}{n-m}\frac{n-m}{n-k},$$

which is obvious.

2. The case of ring and exterior of a disc

Let

$$A_r = \{z \in \mathbb{C} : r < |z| < 1\}, \qquad B_R = \{z \in \mathbb{C} : R < |z|\}, \qquad 0 < r < R < 1.$$

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THEOREM 2.

$$\cos^2 \gamma(A_r, B_R) = \frac{\ln(R/r)}{\ln(1/r)}.$$

PROOF. Now the considered function f (see equation (2)) has power series expansions

$$f|_{A_r}(z) = \sum_{n \in Z} c_n z^n, \qquad f|_{B_R \setminus A_r}(z) = \sum_{n \leq -1} d_n z^n.$$

Similar calculations to those in part 1 (separately for $n < -1$, $n = -1$ and $n > -1$) lead to the following expression:

$$\cos^2 \gamma(A_r, B_R) = \sup\left(\frac{\ln(R/r)}{\ln(1/r)}, \frac{(R^2)^m - (r^2)^m}{1 - (r^2)^m}\right), \qquad m = 1, 2, \dots.$$

Put $\alpha = (R/r)^2$, $\beta = (1/r)^2$. Then $1 < \alpha < \beta$ and

$$\cos^2 \gamma(A_r, B_R) = \max\left(\frac{\ln \alpha}{\ln \beta}, \frac{\alpha^m - 1}{\beta^m - 1}\right).$$

Since $(\alpha - 1)/(\beta - 1) \geq (\alpha^m - 1)/(\beta^m - 1)$, $m = 1, 2, \dots$, we have

$$\cos^2 \gamma(A_r, B_R) = \max\left(\frac{\ln \alpha}{\ln \beta}, \frac{\alpha - 1}{\beta - 1}\right).$$

It is easy to see that the first number on the right-hand side realizes the maximum, which completes the proof.

Denote by C_r the disc with radius r and by D_R the ring with radii 1 and R, $1 < r < R$. Since the mapping $z \to 1/z$ transforms $C_r \setminus \{0\}$ and D_R biholomorphically onto $B_{1/r}$ and $A_{1/R}$ (where $B_{1/r}$ and $A_{1/R}$ have the same meaning as in Theorem 2), and since $L^2H(C_r \setminus \{0\}) = L^2H(C_r)$ (more general results of this type were obtained by J. Siciak in [3]), we have by Theorem 2 and by [2, Theorem 3], that

$$\cos^2 \gamma(C_r, D_R) = \cos^2 \gamma(A_{1/R}, B_{1/r}) = \frac{\ln(R/r)}{\ln R}.$$

If $R \to \infty$, this expression tends to 1. On the other hand, if $D = \lim_{R\to\infty} D_R = \bigcup_{R>1} D_R = \{z \in \mathbb{C} \colon |z| > 1\}$, then by [2, page 658] and by [5, Theorem 3], $\cos \gamma(C_r, D) = 1/r$. Therefore, the following holds:

THEOREM 3 (Discontinuity of the L^2-angle).

$$\lim_{R\to\infty} \cos \gamma(C_r, D_R) \neq \cos \gamma\left(C_r, \lim_{R\to\infty} D_R\right).$$

Note that in contrast to the above result, if we consider the domains $A_r = \{z \in \mathbb{C} \colon r < |z| < 1\}$ and $B_R = \{Z \in \mathbb{C} \colon R < |z| < \rho\}$, $0 < r < R < 1 < \rho$,

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and let ρ tend to infinity, then by Theorem 1,

$$\cos^2\gamma(A_r, B_{R\rho}) = \frac{\ln(R/r)\ln\rho}{\ln(\rho/R)\ln(1/r)} \to \frac{\ln(R/r)}{\ln(1/r)}$$

which is equal, by Theorem 2, to $\cos^2\gamma(A_r, B_R)$. Hence in this case the "continuity" of the L^2-angle holds.

3. Higher dimension case

In this section we prove a result on the L^2-angle between cartesian products of domains. We give then some examples of "discontinuity" in a higher dimension case. In order to make the presentation more concise, we use the following notations: given domains $A, B, C, \ldots$ in $\mathbb{C}^n$ and G in $\mathbb{C}^m$, we will write $A_G, B_G, C_G, \ldots$ instead of $A \times G, B \times G, C \times G, \ldots$.

THEOREM 4. *Let A and B be two domains in $\mathbb{C}^n$ such that $m(A\backslash B) > 0$ and $m(B\backslash A) > 0$ and*

$$m((B\backslash A)\backslash \mathrm{Int}(B\backslash A)) = 0, \tag{7}$$

(*where m denotes the Lebesgue measure in $\mathbb{C}^n$*), *and let G be a domain in $\mathbb{C}^m$ such that $L^2H(G) \neq 0$. Then the L^2-angle between A_G and B_G is defined and satisfies the equality*

$$\cos\gamma(A_G, B_G) = \cos\gamma(A, B).$$

We need the following lemma:

LEMMA 5. *Suppose that domains $A, B \subset \mathbb{C}^n$ are such that*

$$m((B\backslash A)\backslash \mathrm{Int}(B\backslash A)) = 0,$$

and let G be a domain in $\mathbb{C}^m$. Set $D = A \cup B$. Let $h \in L^2(D_G)$ be such that h is holomorphic in A_G and in $\mathrm{Int}(B_G\backslash A_G)$, *and h is orthogonal to $L^2H(D_G)$. Then for every $w \in G$, the function $h(\cdot, w)$ belongs to $L^2(D)$, is holomorphic in A and in* $\mathrm{Int}(B\backslash A)$, *and is orthogonal to $L^2H(D)$.*

PROOF. Since h is square-integrable and holomorphic in A_G and $\mathrm{Int}(B_G\backslash A_G)$, it is well known that for each $w \in G$, the function $h(\cdot, w)$ is holomorphic and square-integrable in A and $\mathrm{Int}(B\backslash A)$. Since

$$m((B\backslash A)\backslash \mathrm{Int}(B\backslash A)) = 0$$

by assumption, we have also $h(\cdot, w) \in L^2(D)$. In order to prove that $h(\cdot, w)$ is orthogonal to $L^2H(D)$ take arbitrary functions f and g from $L^2H(D)$ and

$L^2H(G)$ respectively. Define the function h_f on G by $h_f(w) = \langle h(\cdot, w), f\rangle_D$, $w \in G$. (This definition makes sense, because $h(\cdot, w) \in L^2(D)$). Since, by the Schwarz inequality,

$$\int_G |h_f(w)|^2\,dm(w) \le \int_G \left(\int_D |h(z,w)|\left|\overline{f(z)}\right|\,dm(z)\right)^2 dm(w)$$
$$\le \int_G \left(\int_D |h(z,w)|^2\,dm(z) \int_D |f(z)|^2\,dm(z)\right) dm(w) = \|f\|_D^2\|h\|_{D_G}^2$$

the function h_f is in $L^2(G)$. Let $\{D_n\}_{n=1}^{\infty}$ be an increasing sequence of compact subsets of D such that $\bigcup_{n=1}^{\infty} D_n = D$. It is then well known that the funtions $(h_f)_n(w) = \langle h(\cdot,w), f\rangle_{D_n} = \int_{D_n} h(z,w)\overline{f(z)}\,dm(z)$ are holomorphic in G, and by an estimate similar to that in (8) we see that they are also in $L^2(G)$. Moreover, by the same manner as in (8), we obtain the inequality $\|h_f - (h_f)_n\|^2 \le \|f\|_{D_n}^2\|h\|_{(D\setminus D_n)_G}^2$. Since the last expression tends to zero as $n \to \infty$, and since $(h_f)_n$ are in $L^2H(G)$, which is a closed subspace of $L^2(G)$, h_f belongs to $L^2H(G)$. The function $f(z)g(w)$ is in $L^2H(D_G)$. Therefore, by the assumption on h,

$$0 = \langle h, f(z)g(w)\rangle = \int_G \left(\int_D h(z,w)\overline{f(z)}\,dm(z)\right)\overline{g(w)}\,dm(w) = \langle h_f, g\rangle_G.$$

This means that h_f is orthogonal to $L^2H(G)$, and thus since h_f is itself in $L^2H(G)$, $h_f = 0$. Hence, for every $w \in G$, $h_f(w) = \langle h(\cdot,w), f\rangle_D = 0$. Since $f \in L^2H(D)$ was taken arbitrary, we conclude that $h(\cdot, w)$ is orthogonal to $L^2H(D)$.

PROOF OF THEOREM 4. We show first that $\cos\gamma(A,B) \le \cos\gamma(A_G,B_G)$. As in the introduction, let F_1 (respectively F_2) denote the subspace of $L^2(D)$, consisting of functions which are holomorphic in A (respectively in B). Similarly, let G_1 and G_2 be the subspaces of those functions from $L^2(D_G)$, which are holomorphic respectively in A_G and in B_G. Take any $f_i \in F_i\setminus\{0\}$ with $f_i \perp L^2H(D)$, $i = 1,2$. Let g be an arbitrary function from $L^2H(G)\setminus\{0\}$. Then the functions $f_1(z)g(w)$ and $f_2(z)g(w)$ are in $G_1\setminus\{0\}$ and $G_2\setminus\{0\}$ respectively. Moreover, since every function from $L^2H(D_G)$ can be approximated in the L^2-norm by functions of the form $h_1(z)g_1(w)+\cdots+h_n(z)g_n(w)$ with $h_i \in L^2H(D)$ and $g_1 \in L^2H(G)$, we conclude from the orthogonality conditions on f_i that $f_i(z)g(w) \perp L^2H(D_G)$, $i = 1,2$. Hence,

$$\text{(9)} \qquad \cos\gamma(A_G,B_G) = \frac{|\langle f_1(z)g(w), f_2(z)g(w)\rangle_{D_G}|}{\|f_1(z)g(w)\|_{D_G}\|f_2(z)g(w)\|_{D_G}}$$
$$= \frac{|\langle f_1, f_2\rangle_D|\|g\|_G^2}{\|f_1\|_D\|f_2\|_D\|g\|_G^2} = \frac{|\langle f_1, f_2\rangle_D|}{\|f_1\|_D\|f_2\|_D}.$$

Since $\cos\gamma(A,B)$ is the supremum over all expressions occuring in the right-hand side of (9) with f_1 and f_2 as described above, we are done.

In order to prove the opposite inequality, suppose first that $L^2H(A)$ is nontrivial. (The case when $L^2H(B) \neq 0$ is treated analogously.) Then (see [5, Theorem 2]) the formula

$$(10)\quad \cos^2\gamma(A_G,B_G) = \sup\left\{ \frac{\|h\|^2_{(A\setminus B)_G} + \|P_{B_G}h\|^2_{B_G}}{\|h\|^2_{D_G}} : h \in G_1\setminus\{0\}, h \perp L^2H(D_G), h \text{ is holomorphic in } \operatorname{Int}(B_G\setminus A_G) \right\}$$

holds (here P_{B_G} denotes the Bergman projection in B_G). By virtue of Lemma 5, for every $w \in G$, the function $f(\cdot,w)$ belongs to F_1, is holomorphic in $\operatorname{Int}(B\setminus A)$, and is orthogonal to $L^2H(D)$. Therefore, again by [5, Theorem 2], for every $w \in G$ we have the inequality

$$(11)\qquad \frac{\|h(\cdot,w)\|^2_{A\setminus B} + \|P_Bh(\cdot,w)\|^2_B}{\|h(\cdot,w)\|^2_D} \le \cos^2\gamma(A,B),$$

(with P_B the Bergman projection in B). Let K_B, K_G and K_{B_G} denote the Bergman functions for domains B, G and B_G respectively, and set $S = (B\setminus A)\setminus\operatorname{Int}(B\setminus A)$. By Bremermann's theorem,

$$K_{B_G}(s,t;z,w) = K_B(s,z)K_G(t,w).$$

Moreover $h(s,\cdot) \in L^2H(G)$ for every $s \in B\setminus S$, and $m(S) = 0$. Therefore, we have

$$\begin{aligned} P_{B_G}h(z,w)7 &= \int_{B_G} \overline{K_{B_G}(s,t;z,w)}h(s,t)\,dm(s)\,dm(t) \\ &= \int_{B\setminus S} \overline{K_B(s,z)} \left(\int_G \overline{K_G(t,w)}h(s,t)\,dm(t)\right) dm(s) \\ &= \int_{B\setminus S} \overline{K_B(s,z)}h(s,w)\,dm(s) = (P_Bh(\cdot,w))(z), \end{aligned}$$

$z \in B$, $w \in G$. Thus

$$\frac{\|h\|^2_{(A\setminus B)_G} + \|P_{B_G}h\|^2_{B_G}}{\|h\|^2_{D_G}} = \frac{\int_G(\|h(\cdot,w)\|^2_{A\setminus B} + \|P_Bh(\cdot,w)\|^2_B)\,dm(w)}{\int_G \|h(\cdot,w)\|^2_D\,dm(w)}.$$

Because of (11), this last expression does not exceed $\cos^2\gamma(A,B)$. Taking the supremum of those expressions over all h as above, we obtain by (10), that $\cos^2\gamma(A_G,B_G) \le \cos^2\gamma(A,B)$. At least, if $L^2H(A) = L^2H(B) = \{0\}$, then also $L^2H(A_G) = L^2H(B_G) = \{0\}$, and thus, by [5, Theorem 1], $\cos\gamma(A,B) = \cos\gamma(A_G,B_G) = 0$. This completes the proof.

We give now some further examples of "discontinuity" of the L^2-angle. Let C_r and D_R have the same meaning as in Theorem 3. Let G be any domain of holomorphy in some $\mathbb{C}^m$ with $L^2H(G) \neq \{0\}$. Then, by (the proof of) Theorem 3, and by Theorem 4, we have $\lim_{R\to\infty} \cos\gamma((C_r)_G, (D_R)_G) = 1$, and similarly, if $D = \lim_{R\to\infty} D_R = \{z \in \mathbb{C}: |z| > 1\}$, then $\cos\gamma((C_r)_G, D_G) = 1/r$. Thus the above example exhibits the discontinuity of the L^2-angle, as in Theorem 3. Note that $(C_r)_G$, $(D_R)_G$ and D_G are domains of holomorphy.

Another example of discontinuity of L^2-angle is the following. Let $\Pi_+ = \{(z, w) \in \mathbb{C}^2: \operatorname{Re} z > 1\}$, $\Pi_- = \{(z, w) \in \mathbb{C}^2: \operatorname{Re} z < -1\}$. Set also

$$A = \Pi_- \cup \{\operatorname{Re} z < 1, (\operatorname{Im} z)^2 + |w|^2 < 1\},$$
$$A_0 = \{-2 < \operatorname{Re} z < 1, (\operatorname{Im} z)^2 + |w|^2 < 1\},$$
$$B = \Pi_+ \cup \{-1 < \operatorname{Re} z, (\operatorname{Im}^2 + |w|^2 < 1\}.$$

Since $L^2H(C) = \{0\}$, then also $L^2H(\Pi_+) = L^2H(\Pi_-) = \{0\}$, and so $L^2H(A) = L^2H(B) = \{0\}$. Also $L^2H(A \cup B) = \{0\}$, F_1 is orthogonal to F_2, and so $\cos\gamma(A, B) = 0$ by (1). On the other hand, consider the sequence $\{A_n\}_{n=0}^{\infty}$ of bounded domain in $\mathbb{C}^2$, such that $A_0 \subset A_1 \subset \dots$, $\bigcup_{n=0}^{\infty} A_n = A$, and $A_n \cap B = A_0 \cap B$ for every $n = 0, 1, 2, \dots$. Set $F^{(i)} = L^2H(A_i \cup B) = \{0\}$, $F_1^{(i)} = \{f \in L^2(A_i \cup B): f \text{ is holomorphic in } A_i\}$, and $F_2^{(i)} = \{f \in L^2(A_i \cup B): f$ is holomorphic in $B\}$. Let f_i be a function which is equal to one in A_i and to zero in Π_+ and let g_i be equal to one in $A_i \backslash B$ and to zero in B. Since the domains A_i are bounded, the supports of the functions f_i and g_i have finite measure, and so we have $f_i \in F_1^{(i)} \backslash \{0\}$ and $g_i \in F_2^{(i)} \backslash \{0\}$. Then

$$\cos\gamma(A_i, B) \geq \frac{|\langle f_i, g_i \rangle|}{\|f_i\| \, \|g_i\|} = \frac{m(A_i \backslash B)}{\sqrt{m(A_i) m(A_i \backslash B)}},$$

where m denotes the Lebesgue measure in $\mathbb{C}^2$. This last expression tends to one as i tends to infinity.

Note that in contrast to the previous example, the cosine of the L^2-angle between limit domains in the present situation is equal to zero. In the last example the considered domains are not domains of holomorphy.

Remark. Note that in all the aforementioned examples of the discontinuity of the L^2-angle, the space $L^2H(A \cup B)$ is trivial. It would be interesting to find some sufficient conditions, under which the continuity of the L^2-angle holds; for example it is not known to us whether the condition $L^2H(A \cup B) \neq \{0\}$ would be sufficient.

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