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Positive Solutions of Impulsive Dynamic System on Time Scales

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Abstract. In this paper, some criteria for the existence of positive solutions of a class of systems of impulsive dynamic equations on time scales are obtained by using a fixed point theorem in cones.

1 Introduction

Let $\mathbf{T}$ be a time scale, *i.e.,* $\mathbf{T}$ is a nonempty closed subset of R. Let $T > 0$ be fixed and $0, T$ be points in $\mathbf{T}$. An interval $(0, T)_{\mathbf{T}}$ denote time scales interval, that is, $(0, T)_{\mathbf{T}} := (0, T) \cap \mathbf{T}$. Other types of intervals are defined similarly.

The theory of impulsive differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real world phenomena in physics, biology, engineering, *etc.*, (see [11, 20, 24, 31–33, 37, 38]). At the same time, the boundary value problems for impulsive differential equations and impulsive difference equations have received much attention [2, 18, 19, 25, 26, 29–31, 39]. Recently, the theory of dynamic equations on time scales has become important (see, for example, [1, 7, 8, 17, 21]). There are also some papers ([3–6, 22]) about dynamic equations on time scales that should be cited here. In [3], R. P. Agarwal et al. considered a class of singular second-order dynamic equations with homogeneous Dirichlet boundary conditions that includes those problems related to the negative exponent Emden–Fowler equation. Some sufficient conditions for the existence of multiple positive solutions were obtained by using perturbation and variational techniques. In [22], a monotone sequence of solutions of linear problems converging uniformly and quadratically to a solution of a class of second order, nonlinear, three-point, time scale boundary value problems was obtained by means of the method of upper and lower solutions and the generalized quasilinearization technique. To enlarge the field of applications of the dynamic equations on time scales and to have more theoretical opportunities M. U. Akhmet and M. Turan ([5,6]) proposed to generalize the transition operator and to investigate differential equations on time scales with transition condition and differential equations on variable time scales with transition condition. In [4], Agarwal et al. studied the Wirtinger-type inequalities for the Lebesgue $\triangle$-integral on an arbitrary time scale.

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Naturally, some authors have focused their attention on the BVPs of impulsive dynamic equations on time scales [9, 10, 12, 14–16, 27, 28, 36]. In particular, for the first order impulsive dynamic equations on time scales

$$\text{(1.1)}\quad \begin{cases} y^{\triangle}(t) + p(t)y(\sigma(t)) = f(t, y(t)),\ t \in J := [a, b],\ t \neq t_k, k = 1, 2, \dots, m, \\ y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, 2, \dots, m, \\ y(a) = \eta, \end{cases}$$

where $\mathbf{T}$ is a time scale that has at least finitely-many right-dense points, $[a, b] \subset \mathbf{T}$, p is regressive and right-dense continuous, $f\colon \mathbf{T} \times R \to R$ is a given function, $I_k \in C(R, R)$. The paper [9] obtained the existence of one solution to the problem (1.1) by using the nonlinear alternative of Leray-Schauder type.

In [15], Geng et al. considered the following impulsive periodic boundary value problem on time scales $\mathbf{T}$:

$$\text{(1.2)}\quad \begin{cases} y^{\triangle}(t) = f(t, y(t)), \quad t \in J := [0, T]_{\mathbf{T}}, \quad t \neq t_k, k = 1, 2, \dots, m, \\ \operatorname{Imp}(y)(t_k) := I_k(y(t_k^-)), \quad k = 1, 2, \dots, m, \\ y(0) = y(\sigma(T)), \end{cases}$$

where $f\colon J \times (-\infty, \infty) \to (-\infty, \infty)$ is continuous in the second variable, $I_k\colon (-\infty, \infty) \to (-\infty, \infty)$ is continuous, $t_k \in (0, T)_{\mathbf{T}}$ and $0 < t_1 < \cdots < t_m < T$, $\operatorname{Imp}(y)(t_k) = y(t_k^+) - y(t_k^-)$. The existence of extremal solutions to the problem (1.2) was obtained by virtue of the method of lower and upper solutions coupled with monotone iterative technique.

In [36], the author considered the following first-order impulsive periodic boundary value problem on time scales $\mathbf{T}$:

$$\text{(1.3)}\quad \begin{cases} x^{\triangle}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))),\ t \in J := [0, T]_{\mathbf{T}},\ t \neq t_k,\ k = 1, 2, \dots, m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) = x(\sigma(T)). \end{cases}$$

The existence of positive solutions to the problem (1.3) was obtained by means of the well-known Guo–Krasnoselskii fixed point theorem [13].

However, to the best of our knowledge, there is little work concerning the system of impulsive dynamic equations on time scales.

In this paper, we are concerned with the existence of positive solutions for the following system of impulsive dynamic equations on time scale:

$$\text{(1.4)}\quad \begin{cases} x^{\triangle}(t) + P(t)x(\sigma(t)) = F(t, x(\sigma(t))),\ t \in J := [0, T]_{\mathbf{T}},\ t \neq t_k,\ k = 1, 2, \dots, m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) = x(\sigma(T)), \end{cases}$$

where ($\mathcal{T}$ stands for the transpose)

$$x = (x_1, x_2, \dots, x_n)^{\mathcal{T}}, \quad P(t) = \operatorname{diag}[p_1(t), p_2(t), \dots, p_n(t)],$$
$$F = (f_1, f_2, \dots, f_n)^{\mathcal{T}}, \quad I_k = (I_k^1, I_k^2, \dots, I_k^n)^{\mathcal{T}}.$$

For $i \in \{1, 2, \ldots, n\}$, $p_i\colon [0, T]_{\mathbf{T}} \to (0, \infty)$ is right-dense continuous (that is $p_i \in \mathcal{R}^+$, where $\mathcal{R}^+$ will be defined in section 2), $f_i\colon J \times [0, \infty)^n \to [0, \infty)$ is continuous, $I_k^i\colon [0, \infty)^n \to [0, \infty)$ is continuous; $t_k \in (0, T)_{\mathbf{T}}$, $0 < t_1 < \cdots < t_m < T$, and for each $k = 1, 2, \ldots, m$, $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$. For each $x = (x_1, x_2, \ldots, x_n)^{\mathcal{T}} \in R^n$, the norm of x is defined as $|x| = \sum_{i=1}^n |x_i|$.

The main results in this paper are proved by means of a fixed point theorem [23] that is different from those used in [9, 15, 36]. Note that papers [34, 35] discussed problem (1.4) and obtained some results about the existence of solution or positive solution to problem (1.4) when $n = 1$ and $I_k = 0$, $k = 1, 2, \ldots, m$. Moreover, for the case of $n = 1$, problem (1.4) reduces to the problem (1.3).

In the remainder of this section, we state a fixed point theorem [23].

Theorem 1.1 ([23]) *Let X be a Banach space with a cone K. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $\Phi\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

(i) *$\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$ (or $x \in K \cap \partial\Omega_2$), and*

(ii) *there exists $\psi \in K \setminus \{0\}$ such that $x \neq \Phi x + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ (or $x \in K \cap \partial\Omega_1$) and $\lambda > 0$.*

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Remark 1.2 In Theorem 1.1, the use of (ii) gives better results than using the common assumption $\|\Phi x\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$(or $x \in K \cap \partial\Omega_1$).

2 Some Results on Time Scales

In this section, we state some fundamental definitions and results concerning time scales, so that the paper is self-contained. For more details, one can refer to [1, 7, 8, 17, 21].

Definition 2.1 Assume that $x\colon \mathbf{T} \to R$ and fix $t \in \mathbf{T}$ (if $t = \sup \mathbf{T}$, we assume t is not left-scattered). Then x is called delta differentiable at $t \in \mathbf{T}$ if there exists a $\theta \in R$ such that for any given $\varepsilon > 0$, there is an open neighborhood U of t such that

$$\big|x(\sigma(t)) - x(s) - \theta|\sigma(t) - s|\big| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

In this case, θ is called the delta derivative of x at $t \in \mathbf{T}$ and is denoted by $\theta = x^{\triangle}(t)$.

If $F^{\triangle}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s) \triangle s = F(t) - F(a).$$

Definition 2.2 A function $f\colon \mathbf{T} \to R$ is called rd-continuous provided it is continuous at right-dense points in $\mathbf{T}$, and its left-sided limits exist at left-dense points in $\mathbf{T}$. The set of rd-continuous $f\colon \mathbf{T} \to R$ will be denoted by C_{rd}.

Lemma 2.3 *If $f \in C_{rd}$ and $t \in \mathbf{T}^k$, then $\int_t^{\sigma(t)} f(s)\triangle s = \mu(t)f(t)$, where $\mu(t) = \sigma(t) - t$ is the* graininess *function.*

Lemma 2.4 *If $f^{\triangle} \geq 0$, then f is increasing.*

Lemma 2.5 *Assume that $f, g\colon \mathbf{T} \to R$ are delta differentiable at t, then*

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) = f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).$$

Definition 2.6 A function $p\colon \mathbf{T} \to R$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbf{T}^k.$$

The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$.

Definition 2.7 We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbf{T}\}.$$

Definition 2.8 If $p \in \mathcal{R}$, then the delta exponential function is given by $e_p(t,s) = \exp(\int_s^t g(\tau)\triangle\tau)$, where

$$g(\tau) = \begin{cases} p(\tau), & \text{if } \mu(\tau) = 0, \\ \frac{1}{\mu(\tau)} \operatorname{Log}(1 + p(\tau)\mu(\tau)), & \text{if } \mu(\tau) \neq 0, \end{cases}$$

where Log is the principal logarithm.

Lemma 2.9 *If $p \in \mathcal{R}$, then*

(i) $e_p(t,t) \equiv 1$;
(ii) $e_p(t,s) = \frac{1}{e_p(s,t)}$;
(iii) $e_p(t,u)e_p(u,s) = e_p(t,s)$;
(iv) $e_p^{\triangle}(t,t_0) = p(t)e_p(t,t_0)$, *for $t \in \mathbf{T}^k$ and $t_0 \in \mathbf{T}$.*

Lemma 2.10 *If $p \in \mathcal{R}^+$ and $t_0 \in \mathbf{T}$, then $e_p(t,t_0) > 0$ for all $t \in \mathbf{T}$.*

3 Main Results

Throughout the rest of this paper, we will always assume that the points of impulse t_k are right-dense for each $k = 1, 2, \ldots, m$.

We define

$$PC = \{x \in [0, \sigma(T)]_{\mathbf{T}} \to R^n : x^k \in C(J_k, R^n), \quad k = 1, 2, \ldots, m, \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k^-) = x(t_k),\ k = 1, 2, \ldots, m\},$$

where x^k is the restriction of x to $J_k = (t_k, t_{k+1}]_{\mathbf{T}} \subset (0, \sigma(T)]_{\mathbf{T}}$, $k = 1, 2, \ldots, m$ and $J_0 = [0, t_1]_{\mathbf{T}}$, $J_{m+1} = \sigma(T)$.

Let

$$X = \Big\{ x(t) : x(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big)^{\mathcal{T}} \in PC,\ x(0) = x\big(\sigma(T)\big) \Big\}$$

with the norm $\|x\| = \sum_{i=1}^{n} |x_i|_0$, where $|x_i|_0 = \sup_{t\in[0,\sigma(T)]_{\mathbf{T}}} |x_i(t)|$. Then X is a Banach space.

Definition 3.1 A function $y \in PC\cap C^1(J\backslash\{t_1, t_2, \dots, t_m\}, R)$ is said to be a solution of PBVP (1.4) when $n = 1$ if and only if y satisfies the dynamic equation

$$y^{\triangle}(t) + p(t)y(\sigma(t)) = f(t, y(\sigma(t))) \text{ everywhere on } J\backslash\{t_1, t_2, \dots, t_m\},$$

the impulsive conditions

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, 2, \dots, m,$$

and the periodic boundary condition $y(0) = y(\sigma(T))$.

Lemma 3.2 ([36]) *Suppose $h\colon [0, T]_{\mathbf{T}} \to R$ is rd-continuous, then y is a solution of*

$$y(t) = \int_0^{\sigma(T)} G(t, s)h(s)\triangle s + \sum_{k=1}^{m} G(t, t_k)I_k(y(t_k)), \quad t \in [0, \sigma(T)]_{\mathbf{T}},$$

where

$$G(t, s) = \begin{cases} \dfrac{e_p(s, t)e_p(\sigma(T), 0)}{e_p(\sigma(T), 0) - 1}, & 0 \le s \le t \le \sigma(T), \\ \dfrac{e_p(s, t)}{e_p(\sigma(T), 0) - 1}, & 0 \le t < s \le \sigma(T), \end{cases}$$

if and only if y is a solution of the boundary value problem

$$\begin{cases} y^{\triangle}(t) + p(t)y(\sigma(t)) = h(t),\ t \in J := [0, T]_{\mathbf{T}},\ t \ne t_k,\ k = 1, 2, \dots, m, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\ k = 1, 2, \dots, m, \\ y(0) = y(\sigma(T)). \end{cases}$$

Remark 3.3 When $\mathbf{T} = R$, Lemma 3.2 is reduced to [25, Lemma 2.1].

Lemma 3.4 *For $i \in \{1, 2, \dots, n\}$, let $G_i(t, s)$ be defined by*

$$G_i(t, s) = \begin{cases} \dfrac{e_{p_i}(s, t)e_{p_i}(\sigma(T), 0)}{e_{p_i}(\sigma(T), 0) - 1}, & 0 \le s \le t \le \sigma(T), \\ \dfrac{e_{p_i}(s, t)}{e_{p_i}(\sigma(T), 0) - 1}, & 0 \le t < s \le \sigma(T), \end{cases}$$

then

$$A_i \triangleq \frac{1}{e_{p_i}(\sigma(T), 0) - 1} \le G_i(t, s) \le \frac{e_{p_i}(\sigma(T), 0)}{e_{p_i}(\sigma(T), 0) - 1} \triangleq B_i.$$

Define $A \triangleq \min_{1\le i\le n} A_i$, $\quad B \triangleq \max_{1\le i\le n} A_i$, and let

$$K = \left\{ x(t) : x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathcal{T}} \in X : x_i(t) \ge \delta |x_i|_0,\ i = 1, 2, \dots, n \right\},$$

where $\delta = \frac{A}{B} \in (0,1)$. Obviously, K is a cone in X.

We define an operator $\Phi\colon K \to X$ as follows:

$$(\Phi x)(t) = \big((\Phi_1 x)(t),\ (\Phi_2 x)(t), \dots,\ (\Phi_n x)(t) \big)^{\mathcal{T}},$$

where

$$(\Phi_i x)(t) = \int_0^{\sigma(T)} G_i(t,s) f_i(s, x(\sigma(s))) \triangle s + \sum_{k=1}^{m} G_i(t,t_k) I_k^i(x(t_k)), \quad t \in [0, \sigma(T)]_{\mathbf{T}}.$$

By Lemma 3.2, it is easy to see that fixed points of Φ are the solutions to the system (1.4).

Lemma 3.5 *$\Phi\colon K \to K$ is completely continuous.*

Proof Suppose $x \in K$; it is easy to see that $\Phi x \in X$. Then for all $x \in K$, by Lemma 3.4 we get

$$|\Phi_i x|_0 \le B_i \int_0^{\sigma(T)} f_i\big(s, x(\sigma(s))\big) \triangle s + B_i \sum_{k=1}^{m} I_k^i\big(x(t_k)\big).$$

So,

$$\begin{aligned}
(\Phi_i x)(t) &= \int_0^{\sigma(T)} G_i(t,s) f_i\big(s, x(\sigma(s))\big) \triangle s + \sum_{k=1}^{m} G_i(t,t_k) I_k^i\big(x(t_k)\big) \\
&\ge A_i \int_0^{\sigma(T)} f_i\big(s, x(\sigma(s))\big) \triangle s + A_i \sum_{k=1}^{m} I_k^i(x(t_k)) \\
&= \frac{A_i}{B_i}\Big[B_i \int_0^{\sigma(T)} f_i\big(s, x(\sigma(s))\big) \triangle s + B_i \sum_{k=1}^{m} I_k^i\big(x(t_k)\big) \Big] \ge \delta |\Phi_i x|_0.
\end{aligned}$$

This shows that $\Phi\colon K \to K$. Furthermore, with similar arguments as in [36], we can prove that $\Phi\colon K \to K$ is completely continuous. ∎

Notation Let

$$F^a = \lim_{x\in K,\ \|x\|\to a} \sup \frac{\int_0^{\sigma(T)} F(s,x) \triangle s}{\|x\|}, \quad F_a = \lim_{x\in K,\ \|x\|\to a} \inf \frac{\int_0^{\sigma(T)} F(s,x) \triangle s}{\|x\|}$$

and

$$I^a = \lim_{x\in K,\ \|x\|\to a} \sup \frac{\sum_{j=1}^m I_j(x)}{\|x\|}, \quad I_a = \lim_{x\in K,\ \|x\|\to a} \inf \frac{\sum_{j=1}^m I_j(x)}{\|x\|},$$

where a denotes either 0 or ∞.

Now we state our main results.

Theorem 3.6 *Assume that the following conditions are satisfied:*

(H_1) $0 < F^0, I^0 < \frac{1}{2B}$;

(H_2) $\frac{1}{2A} < F_\infty, I_\infty < \infty$.

Then system (1.4) *has at least one positive solution.*

Proof Since $0 < F^0, I^0 < \frac{1}{2B}$, we may choose $\rho_1 > 0$ such that

(3.1) $$\int_0^{\sigma(T)} F(s,x)\triangle s \le \frac{\rho_1}{2B}, \quad \sum_{j=1}^{m} I_j(x) \le \frac{\rho_1}{2B} \text{ for } 0 \le \|x\| \le \rho_1.$$

Set $\Omega_1 = \{x \in X : \|x\| < \rho_1\}$, then $\Omega_1$is a bounded open subset of X and $0 \in \Omega_1$. Thus, if $x \in K \cap \partial\Omega_1$, then from (3.1), we have

$$\begin{aligned}\|\Phi x\| &= \sum_{i=1}^{n} |\Phi_i x|_0 \le B \sum_{i=1}^{n} \int_0^{\sigma(T)} |f_i(s, x(\sigma(s)))|\triangle s + B \sum_{i=1}^{n}\sum_{k=1}^{m} |I_k^i(x(t_k))| \\ &= B\int_0^{\sigma(T)} F(s, x(\sigma(s)))\triangle s + B\sum_{k=1}^{m} I_k(x(t_k)) \le B\cdot\frac{\rho_1}{2B} + B\cdot\frac{\rho_1}{2B} = \rho_1 = \|x\|.\end{aligned}$$

This implies

(3.2) $$\|\Phi x\| \le \|x\| \text{ for } x \in K \cap \partial\Omega_1.$$

On the other hand, in view of $\frac{1}{2A} < F_\infty, I_\infty < \infty$, there exists $\eta > \rho_1$ such that for $\|x\| \ge \eta$

$$\int_0^{\sigma(T)} F(s,x)\triangle s \ge (F_\infty - \varepsilon)\|x\|, \quad \sum_{j=1}^{m} I_j(x) \ge (I_\infty - \varepsilon)\|x\|,$$

where ε is chosen so that $0 < \varepsilon < \frac{1}{2}\big(F_\infty + I_\infty - \frac{1}{A}\big)$. Let $\rho_2 = \frac{\eta}{\delta}$ and $\Omega_2 = \{x \in X : \|x\| < \rho_2\}$. Obviously, Ω_2 is an open subset of X with $\overline{\Omega}_1 \subset \Omega_2$. Choose $\psi = (1, 1, \dots, 1)^{\mathfrak{T}} \in K\backslash\{0\}$, then we can claim that for any $x \in K \cap \partial\Omega_2$ and $\lambda > 0$,

(3.3) $$x \ne \Phi x + \lambda\psi.$$

In fact, if not, there exist $\overline{x} \in K \cap \partial\Omega_2$ and $\overline{\lambda} > 0$ such that $\overline{x} = \Phi\overline{x} + \overline{\lambda}\psi$. Then $\|\overline{x}\| = \|\Phi\overline{x} + \overline{\lambda}\psi\|$, that is

$$\begin{aligned}\rho_2 = \|\overline{x}\| &= \sum_{i=1}^{n} |\Phi_i\overline{x} + \overline{\lambda}|_0 \\ &\ge A\sum_{i=1}^{n}\int_0^{\sigma(T)} |f_i(s, \overline{x}(\sigma(s)))|\triangle s + A\sum_{i=1}^{n}\sum_{k=1}^{m} |I_k^i(\overline{x}(t_k))| + n\overline{\lambda} \\ &= A\int_0^{\sigma(T)} F(s, \overline{x}(\sigma(s)))\triangle s + A\sum_{k=1}^{m} I_k(\overline{x}(t_k)) + n\overline{\lambda} \\ &\ge A\rho_2(F_\infty + I_\infty - 2\varepsilon) + n\overline{\lambda} > \rho_2 + n\overline{\lambda}.\end{aligned}$$

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This is a contradiction.

Therefore, by Theorem 1.1, it follows from Lemma 3.5, (3.2), and (3.3) that Φ has a fixed point $x^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $\rho_1 \leq \|x^*\| \leq \rho_2$ which is a positive solution of system (1.4). ■

Remark 3.7 Using the following (h_1^*) instead of (H_1) and (H_2), the conclusion of Theorem 3.6 is true.

(h_1^*) $F^0 + I^0 < \frac{1}{B}$ and $F_\infty + I_\infty < \frac{1}{A}$.

Theorem 3.8 *Assume that the following conditions are satisfied:*

(H_3) $F^0 = I^0 = 0$;

(H_4) $F_\infty = I_\infty = \infty$.

Then system (1.4) *has at least one positive solution.*

Since the proof similar to that of Theorem 3.6, we omit it here.

Theorem 3.9 *Assume that the following conditions are satisfied:*

(H_5) $0 < F^\infty, I^\infty < \frac{1}{2B}$;

(H_6) $\frac{1}{2A} < F_0, I_0 < \infty$.

Then system (1.4) *has at least one positive solution.*

Proof Since $0 < F^\infty, I^\infty < \frac{1}{2B}$, we may choose $\rho_3 > 0$ such that

$$\int_0^{\sigma(T)} F(s,x)\triangle s \leq \frac{\|x\|}{2B}, \quad \sum_{j=1}^{m} I_j(x) \leq \frac{\|x\|}{2B} \text{ for } \|x\| \geq \rho_3. \tag{3.4}$$

Set $\Omega_3 = \{x \in X : \|x\| < \rho_3\}$, then Ω_3 is a bounded open subset of X and $0 \in \Omega_3$.Thus, if $x \in K \cap \partial\Omega_3$, then from (3.4), we have

$$\begin{aligned}\|\Phi x\| &= \sum_{i=1}^{n} |\Phi_i x|_0 \leq B \sum_{i=1}^{n} \int_0^{\sigma(T)} |f_i(s, x(\sigma(s)))|\triangle s + B \sum_{i=1}^{n}\sum_{k=1}^{m} |I_k^i(x(t_k))| \\ &= B\int_0^{\sigma(T)} F(s, x(\sigma(s)))\triangle s + B\sum_{k=1}^{m} I_k(x(t_k)) \leq B\cdot\frac{\|x\|}{2B} + B\cdot\frac{\|x\|}{2B} = \|x\|.\end{aligned}$$

This implies

$$\|\Phi x\| \leq \|x\| \text{ for } x \in K \cap \partial\Omega_3. \tag{3.5}$$

On the other hand, in view of $\frac{1}{2A} < F_0, I_0 < \infty$, there exists $0 < \rho_4 < \rho_3$ such that for $0 \leq \|x\| \leq \rho_4$,

$$\int_0^{\sigma(T)} F(s,x)\triangle s \geq (F_0 - \varepsilon_0)\|x\|, \quad \sum_{j=1}^{m} I_j(x) \geq (I_0 - \varepsilon_0)\|x\|,$$

where ε_0 is chosen so that $0 < \varepsilon_0 < \frac{1}{2}(F_0 + I_0 - \frac{1}{A})$.

Let $\Omega_4 = \{x \in X : \|x\| < \rho_4\}$. Obviously, Ω_4 is an open subset of X with $\overline{\Omega}_4 \subset \Omega_3$. Choose $\psi = (1, 1, \dots, 1)^{\mathcal{T}} \in K\backslash\{0\}$, then we can claim that for any $x \in K \cap \partial\Omega_4$ and $\mu > 0$

$$x \neq \Phi x + \mu\psi. \tag{3.6}$$

In fact, if not, there exist $\overline{x} \in K \cap \partial\Omega_4$ and $\overline{\mu} > 0$ such that $\overline{x} = \Phi\overline{x} + \overline{\mu}\psi$. Then $\|\overline{x}\| = \|\Phi\overline{x} + \overline{\mu}\psi\|$, that is,

$$\begin{aligned}
\|\overline{x}\| &= \sum_{i=1}^{n} |\Phi_i\overline{x} + \overline{\mu}|_0 \\
&\geq A\sum_{i=1}^{n}\int_0^{\sigma(T)} |f_i(s, \overline{x}(\sigma(s)))|\triangle s + A\sum_{i=1}^{n}\sum_{k=1}^{m}|I_k^i(\overline{x}(t_k))| + n\overline{\mu} \\
&= A\int_0^{\sigma(T)} F(s, \overline{x}(\sigma(s)))\triangle s + A\sum_{k=1}^{m} I_k(\overline{x}(t_k)) + n\overline{\mu} \\
&\geq A(F_0 - \varepsilon_0)\|\overline{x}\| + A(I_0 - \varepsilon_0)\|\overline{x}\| + n\overline{\mu} > \|\overline{x}\| + n\overline{\mu}.
\end{aligned}$$

This leads to a contradiction.

Therefore, by Theorem 1.1 it follows from Lemma 3.5, (3.5), and (3.6) that Φ has a fixed point $x^* \in K \cap (\overline{\Omega}_3 \setminus \Omega_4)$ with $\rho_4 \leq \|x^*\| \leq \rho_3$ which is a positive solution of system (1.4). ∎

Remark 3.10 Using the following (h_2^*) instead of (H_5) and (H_6), the conclusion of Theorem 3.9 is true.

(h_2^*) $F^\infty + I^\infty < \frac{1}{B}$ and $F_0 + I_0 < \frac{1}{A}$.

Theorem 3.11 *Assume that the following conditions are satisfied:*

(H_7) $F^\infty = I^\infty = 0$;

(H_8) $F_0 = I_0 = \infty$.

Then system (1.4) *has at least one positive solution.*

Since the proof similar to that of Theorem 3.9, we omit it here.

4 Example

Example 4.1 Let $\mathbf{T} = [0, 1] \cup [2, 3]$. We consider the following PBVP on $\mathbf{T}$

$$\begin{cases} x^{\triangle}(t) + x\big(\sigma(t)\big) = F\big(t, x(\sigma(t))\big), \ t \in [0, 3]_{\mathbf{T}}, \ t \neq \frac{1}{2}, \\ x\big(\frac{1}{2}^+\big) - x\big(\frac{1}{2}^-\big) = I\big(x\big(\frac{1}{2}\big)\big), \\ x(0) = x(3), \end{cases} \tag{4.1}$$

where $n = 2$, $P(t) = \operatorname{diag}[p_1(t), p_2(t)] \equiv \operatorname{diag}[1, 1]$, $T = 3$, and

$$f_1(t, x) = (1 + t)x_1^2, \quad f_2(t, x) = (1 + t)x_2^3, \quad I^1(x) = x_1^2, \quad I^2(x) = x_2^3.$$

Then it is easy to see that $F^0 = I^0 = 0$, $F_\infty = I_\infty = \infty$. Therefore, together with Theorem 3.8, it follows that system (4.1) has at least one positive solution.

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