EFFECT OF COSSERATS' COUPLE-STRESSES ON THE STRESS DISTRIBUTION IN A SEMI-INFINITE MEDIUM WITH VARYING MODULUS OF ELASTICITY

GUNADHAR PARIA

(Received 24 June 1965)

Summary

The theory of Cosserats' couple-stresses is briefly described in a cartesian system of coordinates, and is applied to the problem of stress distribution in a semi-infinite medium which possesses a non-homogeneous elastic property of an exponential type. Effects of couple-stresses on the stress concentration factors are determined both in homogeneous and non-homogeneous materials.

1. Introduction

The classical theory of deformation of an elastic solid body has been developed on the understanding that the stress tensor is symmetric. This is because of the fact that only in special types of media, for example, granular soils [1], the stress tensor is found to be non-symmetric. But, under certain circumstances, the common materials in which stresses are believed to be symmetric may behave as those having non-symmetric properties for the stresses. These are the cases where moments, for example magnetic moments, are acting at each point of the medium. A theory which takes into account these moments, called the couple-stresses, and which introduces as a consequence the non-symmetric properties of the stress tensor, was developed by Cosserats [2] long ago. It is only in recent years [3—14] that the theory is being revived. The development and applications of this theory will no doubt open up a new branch in the theory of deformations of solid bodies.

The present paper is concerned with the application of the Cosserats' theory of couple-stresses to the problem of stress distribution in a semi-infinite solid which possesses the non-homogeneous elastic properties [15] of a specified nature, and is stressed by surface tractions. A method of successive approximations is adopted for the solution of the problem. The effects of couple stresses on the anti-symmetric part of the stress components are shown graphically in both the cases when the material is homogeneous and also when it is non-homogeneous to an approximation of the first order.

2. Cosserats' equations of equilibrium

We write the equations of equilibrium in cartesian coordinates that hold in a stressed body when the couple stresses are taken into account, besides the usual normal and shearing stresses [8]. In two dimensional problems, let $(\sigma_{xx}, \sigma_{xy})$ denote as usual the normal and shearing stresses on a plane perpendicular to the x-axis, while μ_x denotes the couple-stress per unit area on this plane. Similarly, we denote by $(\sigma_{yy}, \sigma_{yx})$ the normal and shearing stresses on a plane perpendicular to the y-axis, and the couple-stress on this plane by μ_y. The consideration of the equilibrium of forces, as done by Mindlin [8], parallel to the x and y axes gives the equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \tag{2.1}$$

if the body forces are neglected. The consideration of the equilibrium of moments implies

$$\frac{\partial \mu_x}{\partial x} + \frac{\partial \mu_y}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0, \tag{2.2}$$

if the body-couples are omitted. Equations (2.1) and (2.2) are called Cosserats' equations of equilibrium for the stressed body. Equation (2.2) shows that the usual assumptions of the symmetric property $\sigma_{xy} = \sigma_{yx}$ need not be necessarily true if the couple stresses are taken into consideration. The solutions of these equations in terms of stress functions may be obtained as follows.

Equations (2.1) are satisfied if

$$\begin{aligned} \sigma_{xx} &= \frac{\partial \varphi_1}{\partial y}, \qquad & \sigma_{yx} &= -\frac{\partial \varphi_1}{\partial x}, \\ \sigma_{yy} &= \frac{\partial \varphi_2}{\partial x}, \qquad & \sigma_{xy} &= -\frac{\partial \varphi_2}{\partial y}, \end{aligned} \tag{2.3}$$

and then the equation (2.2) is satisfied if

$$\mu_x = \frac{\partial \varphi}{\partial y} - \varphi_1, \qquad \mu_y = -\frac{\partial \varphi}{\partial x} + \varphi_2. \tag{2.4}$$

Hence all the components of stresses and stress-couples may be expressed in terms of three stress functions φ, φ_1 and φ_2.

3. Kinematics of deformations

In a state of plane strain parallel to the xy-plane, the non-vanishing components (u, v) of the displacement vector are functions of x and y. The strain components

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

are associated with the stress components σ_{ij}, while the local rotation component

$$\omega = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

will be associated with the couple-stresses μ_x and μ_y. The couple-stresses μ_x and μ_y will produce a curvature κ_x of the material fibre parallel to the x-axis. Similarly, μ_y will give rise to the curvature κ_y of a fibre parallel to the y-axis. From geometry, it has been shown by Mindlin [8] that

$$\kappa_x = \frac{\partial \omega}{\partial x}, \qquad \kappa_y = \frac{\partial \omega}{\partial y}. \tag{3.1}$$

The compatibility condition between the strains is

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}, \tag{3.2}$$

while that between the curvatures is obtained from (3.1) as

$$\frac{\partial \kappa_x}{\partial y} = \frac{\partial \kappa_y}{\partial x}. \tag{3.3}$$

A compatibility condition between κ_x (or κ_y) and the strain components may be found by eliminating ω, so that

$$\kappa_x = \tfrac{1}{2}\frac{\partial e_{xy}}{\partial x} - \frac{\partial e_{xx}}{\partial y}. \tag{3.4}$$

It may be noted that other compatibility relations may be deduced from the three relations (3.1), (3.2), (3.3), and may be used if required.

4. Constitutive relations

We assume that the curvatures are proportional to the couple-stresses so that [8]

$$\kappa_x = \frac{1}{4B}\mu_x, \qquad \kappa_y = \frac{1}{4B}\mu_y, \tag{4.1}$$

https://doi.org/10.1017/S1446788700004742 Published online by Cambridge University Press

where B is a modulus of curvature characteristic of the material. By putting $B = 0$ we can recover the results for the corresponding cases when couple-stresses are neglected, that is, when the stresses are symmetric.

In the case of plane strain the normal strains (e_{xx}, e_{yy}) are related to the normal stresses $(\sigma_{xx}, \sigma_{yy})$ as

$$
\begin{aligned}
e_{xx} &= \frac{1+\nu}{E}\,[\sigma_{xx}-\nu(\sigma_{xx}+\sigma_{yy})], \\
e_{yy} &= \frac{1+\nu}{E}\,[\sigma_{yy}-\nu(\sigma_{xx}+\sigma_{yy})],
\end{aligned} \tag{4.2}
$$

where E is Young's modulus and ν is the Poisson ratio. The symmetric part

$$\sigma_s = \tfrac{1}{2}(\sigma_{xy}+\sigma_{yx})$$

of the shear stresses $(\sigma_{xy}, \sigma_{yx})$ produces the shear strain e_{xy} and hence

$$e_{xy} = \frac{1}{G}\,\sigma_s = \frac{1+\nu}{E}\,(\sigma_{xy}+\sigma_{yx}), \tag{4.3}$$

where $G = \frac{1}{2}E/(1+\nu)$ is the shear modulus. The anti-symmetric part

$$\sigma_A = \tfrac{1}{2}(\sigma_{xy}-\sigma_{yx}) \tag{4.4}$$

of the shear stresses produce the rotation ω, and the relation between σ_A and ω may be obtained with the help of equations (2.2), (4.1), and (3.1).

5. Fundamental equations in non-homogeneous material

The material coefficients E, ν and B are in general functions of coordinates in a non-homogeneous material [16]. In the present discussion we shall consider ν and B as constants while $E(x, y)$ will be taken as a function of position coordinates (x, y) and later of y only. Equations (3.3) and (4.1) then show that we may take

$$\mu_x = \frac{\partial \psi}{\partial x}, \qquad \mu_y = \frac{\partial \psi}{\partial y}, \tag{5.1}$$

so that ψ is the potential function for the couple stresses. The functions φ_1, φ_2 in equations (2.3) can be eliminated with the help of (2.4) and (5.1) as

$$
\begin{aligned}
\sigma_{xx} &= \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x\,\partial y}, & \sigma_{yy} &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x\,\partial y}, \\
\sigma_{xy} &= -\frac{\partial^2 \varphi}{\partial x\,\partial y} - \frac{\partial^2 \psi}{\partial y^2}, & \sigma_{yx} &= -\frac{\partial^2 \varphi}{\partial x\,\partial y} + \frac{\partial^2 \psi}{\partial x^2}.
\end{aligned} \tag{5.2}
$$

Thus the stresses and stress-couples can be expressed in terms of two functions φ and ψ. If relations (4.2), (4.3) and (5.2) are utilised the compatibility equations (3.2) and (3.4) yield respectively

$$
\begin{aligned}
&\frac{1-\nu}{E}\nabla^4\varphi-\nu\nabla^2\varphi\nabla^2\left(\frac{1}{E}\right)+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\right)\\
&\quad+\frac{\partial^2\psi}{\partial x\,\partial y}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{E}\right)+\left\{2\frac{\partial^2\varphi}{\partial x\,\partial y}-\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right\}\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{E}\right)\\
&\quad+\left\{2(1-\nu)\frac{\partial}{\partial x}(\nabla^2\varphi)+\frac{\partial}{\partial y}(\nabla^2\psi)\right\}\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\\
&\quad+\left\{2(1-\nu)\frac{\partial}{\partial y}(\nabla^2\varphi)-\frac{\partial}{\partial x}(\nabla^2\psi)\right\}\frac{\partial}{\partial y}\left(\frac{1}{E}\right)=0,
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&\frac{1}{2B(1+\nu)}\frac{\partial\psi}{\partial x}-\frac{1}{E}\frac{\partial}{\partial x}(\nabla^2\psi)+\frac{2(1-\nu)}{E}\frac{\partial}{\partial y}(\nabla^2\varphi)\\
&\quad+\left(2\frac{\partial^2\varphi}{\partial x\,\partial y}+\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right)\frac{\partial}{\partial x}\left(\frac{1}{E}\right)\\
&\quad+2\left(\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\psi}{\partial x\,\partial y}-\nu\nabla^2\varphi\right)\frac{\partial}{\partial y}\left(\frac{1}{E}\right)=0.
\end{aligned}
\tag{5.4}
$$

Equations (5.3) and (5.4) determine φ and ψ if the boundary conditions are prescribed.

6. Non-homogeneous semi-infinite media with surface loads

In order to illustrate the foregoing theory and the effect of couple-stresses on the non-homogeneity of the material, we consider the problem of a semi-infinite elastic medium with a surface load. Let $y = 0$ be the bounding plane and suppose the medium occupies the space $y \geqq 0$. Let the surface load on $y = 0$ be equal to $(-A_0 \cos mx)$, and the shearing force and the couple-stress on it be zero. Thus the boundary conditions on $y = 0$ are

$$
\begin{aligned}
\sigma_{yy}&=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\psi}{\partial x\,\partial y}=-A_0\cos mx,\\
\sigma_{yx}&=-\frac{\partial^2\varphi}{\partial x\,\partial y}+\frac{\partial^2\psi}{\partial x^2}=0,\\
\mu_y&=\frac{\partial\psi}{\partial y}=0.
\end{aligned}
\tag{6.1}
$$

Moreover, all physical quantities must tend to zero as y tends to infinity.

https://doi.org/10.1017/S1446788700004742 Published online by Cambridge University Press

Let the law of variation for Young's modulus be assumed as

$$\frac{1}{E} = \frac{1}{E_0}\,(1+\alpha e^{-\beta y}) \tag{6.2}$$

where β is positive, but α may be positive or negative, so that E varies from the value $E_0/(1+\alpha)$ at $y = 0$ to E_0 at $y \to \infty$, Such assumptions are very frequent in the literature on non-homogeneous materials [16]. By putting $\alpha = 0$, the results for homogeneous materials may be deduced.

Now if the relation (6.2) is used in equations (5.3) and (5.4) we obtain

$$\begin{aligned}(1-\nu)(1+\alpha e^{-\beta y})\nabla^4\varphi - \alpha\beta e^{-\beta y}\Big\{\nu\beta\nabla^2\varphi - \beta\frac{\partial^2\varphi}{\partial y^2}\\ + \beta\frac{\partial^2\psi}{\partial x\,\partial y} + 2(1-\nu)\frac{\partial}{\partial y}(\nabla^2\varphi) - \frac{\partial}{\partial x}(\nabla^2\psi)\Big\} = 0,\end{aligned} \tag{6.3}$$

$$\begin{aligned}\frac{1}{l^2}\frac{\partial\psi}{\partial x} - 2\alpha\beta e^{-\beta y}\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\,\partial y} - \nu\nabla^2\varphi\right)\\ -(1+\alpha e^{-\beta y})\left\{\frac{\partial}{\partial x}(\nabla^2\psi) - 2(1-\nu)\frac{\partial}{\partial y}(\nabla^2\varphi)\right\} = 0,\end{aligned} \tag{6.4}$$

where

$$l^2 = \frac{2(1+\nu)B}{E_0}, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{6.5}$$

It can be shown that l is of the dimension of length [8]. Let us take

$$\varphi = F(y)\cos mx, \qquad \psi = f(y)\sin mx. \tag{6.6}$$

Then the equations (6.3) and (6.4) yield

$$(D^2-m^2)^2F = -\alpha e^{-\beta y}\left[P_1(D)F + \frac{m\beta}{1-\nu}P_3(D)f\right], \tag{6.7}$$

$$\begin{aligned}(D^2-m_1^2)f - \frac{2(1-\nu)}{m}D(D^2-m^2)F\\ = \frac{1}{m}\alpha e^{-\beta y}[2(1-\nu)P_2(D)F - mP_4(D)f],\end{aligned} \tag{6.8}$$

where

$$m_1^2 = m^2 + \frac{1}{l^2}, \qquad D = \frac{d}{dy},$$

and we have introduced the operater functions

https://doi.org/10.1017/S1446788700004742 Published online by Cambridge University Press

$$
\begin{aligned}
P_1(D) &= (D^2-m^2)^2-2\beta D(D^2-m^2)+\beta^2(D^2-m^2)+\frac{1}{1-\nu}m^2\beta^2, \\
P_2(D) &= D(D^2-m^2)-\beta\left(D+\frac{\nu}{1-\nu}m^2\right), \\
P_3(D) &= D^2-\beta D-m^2, \\
P_4(D) &= D^2-2\beta D-m^2.
\end{aligned} \tag{6.9}
$$

The boundary conditions (6.1) now imply

$$
F(0)=\frac{A_0}{m^2}, \qquad DF(0)-mf(0)=0, \qquad Df(0)=0. \tag{6.10}
$$

It is important to note that $m_1 \to \infty$ as $l \to 0$ and hence the results in the following sections are to be taken when $l \neq 0$, that is, when m_1 is finite.

7. Solution of the problem

Equations (6.7) and (6.8) will be solved by a method of successive approximations. Assuming that the parameter α characterising the non-homogeneity of the material is small, we take

$$
F(y)=\sum_{r=0}^{\infty}\alpha^r F_r(y), \qquad f(y)=\sum_{r=0}^{\infty}\alpha^r f_r(y). \tag{7.1}
$$

Substituting these values in equations (6.7) and (6.8) and equating like powers of α, we obtain the approximations of the zero-th order as

$$
(D^2-m^2)^2F_0=0, \qquad (D^2-m_1^2)f_0-\frac{2(1-\nu)}{m}D(D^2-m^2)F_0=0, \tag{7.2}
$$

while the approximations of the first order are as follows.

$$
\begin{aligned}
(D^2-m^2)^2F_1 &= -e^{-\beta y}\left[P_1(D)F_0+\frac{m\beta}{1-\nu}P_3(D)f_0\right], \\
(D^2-m_1^2)f_1-\frac{2(1-\nu)}{m}D(D^2-m^2)F_1 &= \frac{1}{m}e^{-\beta y}[2(1-\nu)P_2(D)F_0 \\
&\qquad -mP_4(D)f_0].
\end{aligned} \tag{7.3}
$$

Similarly, the rth order approximations are

$$
\begin{aligned}
(D^2-m^2)^2F_r &= -e^{-\beta y}\left[P_1(D)F_{r-1}+\frac{m\beta}{1-\nu}P_3(D)f_{r-1}\right], \\
(D^2-m_1^2)f_r-\frac{2(1-\nu)}{m}D(D^2-m^2)F_r &= \frac{1}{m}e^{-\beta y}[2(1-\nu)P_2(D)F_{r-1} \\
&\qquad -mP_4(D)f_{r-1}].
\end{aligned} \tag{7.4}
$$

The boundary conditions (6.10) now imply

$$F_0(0) = \frac{A_0}{m^2}, \qquad DF_0(0) - mf_0(0) = 0, \qquad Df_0(0) = 0 \tag{7.5}$$

and,

$$F_\kappa(0) = 0, \qquad DF_\kappa(0) - mf_\kappa(0) = 0, \qquad Df_\kappa(0) = 0 \qquad \kappa = 1, 2, \cdots \tag{7.6}$$

Solutions for F_0 and f_0 are obtained from (7.2) as

$$\begin{aligned} F_0 &= (a_0 + b_0 y)e^{-my}, \\ f_0 &= C_0 e^{-m_1 y} - 4(1-\nu)l^2 m b_0 e^{-my}, \end{aligned} \tag{7.7}$$

if the vanishing conditions for $y \to \infty$ are used. The boundary conditions (7.5) yield

$$a_0 = \frac{A_0}{m^2}, \qquad b_0 = \frac{A_0}{m\varDelta}, \qquad c_0 = 4(1-\nu)l^2 \left(\frac{m}{m_1 \varDelta}\right) A_0, \tag{7.8}$$

where

$$\varDelta = 1 + 4(1-\nu)l^2 m^2 \left(1 - \frac{m}{m_1}\right),$$

so that

$$F_0 = \frac{A_0}{m^2}\left(1 + \frac{my}{\varDelta}\right) e^{-my}, \qquad f_0 = 4(1-\nu)l^2(A_0|\varDelta) \left[\frac{m}{m_1} e^{-m_1 y} - e^{-my}\right]. \tag{7.9}$$

Using these values in the first of the equations (7.3) and then integrating we obtain

$$\begin{aligned} F_1 = {} & (a_1 + b_1 y)e^{-my} + \frac{A_0 e^{-(\beta+m)y}}{(\beta+2m)^2 \varDelta} \left\{2\left(1 + 2m^2 l^2 + 2\frac{m}{\beta}\right)\right. \\ & \left. - \frac{\varDelta}{1-\nu} - \frac{4m(m+\beta}{(1-\nu))\beta(\beta+2m)} - \frac{m}{1-\nu} y\right\} \\ & - \frac{4A_0 m^2 l^2 (1+\beta m_1 l^2)}{\beta m_1 (1 + 2\beta m_1 l^2 + \beta^2 l^2)\varDelta} e^{-(\beta+m_1)y}. \end{aligned} \tag{7.10}$$

Now, the second of equations (7.3) gives

$$\begin{aligned} f_1 = {} & Ge^{-m_1 y} + \frac{2(1-\nu)D(D^2-m^2)}{m(D^2-m_1^2)} F_1 \\ & + \frac{1}{m(D^2-m_1^2)} [e^{-\beta y}\{2(1-\nu)P_2(D)F_0 - mP_4(D)f_0\}]. \end{aligned} \tag{7.11}$$

https://doi.org/10.1017/S1446788700004742 Published online by Cambridge University Press

To evaluate the second term in the right hand side of this equation, we use the expression for F_1 as given by (7.10). It is seen after a little lengthy calculation that

$$\frac{D(D^2-m^2)}{D^2-m_1^2}F_1 = -2m^2l^2b_1e^{-my}+\frac{A_0l^2e^{-(m+\beta)y}}{(\beta+2m)^2(1-2m\beta l^2-\beta^2l^2)\Delta}$$
$$\times\left\{\beta(\beta+2m)(m+\beta)\left(k_1-\frac{m}{1-\nu}y\right)+\frac{m}{1-\nu}(2m^2+6m\beta+3\beta^2)\right\}$$
$$+\frac{4A_0m^2(1+\beta m_1l^2)(m_1+\beta)}{\beta^2m_1(\beta+2m_1)\Delta}e^{-(m_1+\beta)y},$$

where

$$k_1 = 2\left(1+2m^2l^2+2\frac{m}{\beta}\right)-\frac{\Delta}{1-\nu}-\frac{2m(m+\beta)}{1-\nu}\left\{\frac{2}{\beta(\beta+2m)}-\frac{l^2}{1-2m\beta l^2-\beta^2l^2}\right\}.$$

Also, we have the identity

$$\frac{1}{m(D^2-m_1^2)}[e^{-\beta y}\{2(1-\nu)P_2(D)F_0-mP_4(D)f_0\}]$$
$$=\frac{2(1-\nu)A_0l^2e^{-(m+\beta)y}}{m(1-2m\beta l^2-\beta^2l^2)\Delta}\left\{\frac{\beta\Delta}{1-\nu}-2(m+\beta)+4\beta m^2l^2\right.$$
$$\left.+\frac{\beta m}{1-\nu}\left(y-\frac{2(m+\beta)l^2}{1-2m\beta l^2-\beta^2l^2}\right)\right\}+4(1-\nu)A_0\frac{m}{m_1\Delta}\cdot\frac{1+2\beta m_1 l}{\beta(\beta+2m_1)}e^{-(m_1+\beta)y}.$$

Substituting these values in (7.11) we obtain

$$f_1 = c_1e^{-m_1y}-4(1-\nu)b_1ml^2e^{-my}+\frac{2A_0l^2e^{-(m+\beta)y}}{m(\beta+2m)(1-2m\beta l^2-\beta^2l^2)\Delta}$$
$$\times\left[\beta m^2y-\frac{2m^3}{\beta+2m}+4(1-\nu)\beta m^2l^2\left\{m\left(4-\frac{m}{m_1}\right)+2\beta\right\}\right.$$
$$\left.-\frac{2\beta(m+\beta)m^2l^2}{1-2m\beta l^2-\beta^2l^2}\right]+\frac{8(1-\nu)A_0m(1+\beta m_1l^2)(m_1+\beta)}{\beta^2m_1(\beta+2m_1)\Delta}\cdot e^{-(m_1+\beta)y}$$
$$+\frac{4(1-\nu)A_0m(1+2\beta m_1l^2)}{\beta m_1(\beta+2m_1)\Delta}\cdot e^{-(m_1+\beta)y}. \tag{7.12}$$

To evaluate the constants of integration a_1, b_1, c_1 involved in (7.10) and (7.12) we use the conditions (7.6) for the case $k = 1$. These yield the following equations.

$$a_1+\frac{A_0}{(\beta+2m)^2\varDelta}\left\{2\left(1+2m^2l^2+2\,\frac{m}{\beta}\right)-\frac{\varDelta}{1-\nu}-\frac{4m(m+\beta)}{(1-\nu)\beta(\beta+2m)}\right\}+\frac{4A_0m^2l^2(1+\beta m_1l^2)}{\beta m_1(1+2\beta m_1l^2+\beta^2l^2)\varDelta}=0, \tag{7.13}$$

$$ma_1-\{1+4(1-\nu)m^2l^2\}b_1+mc_1+a_2=0, \tag{7.14}$$

$$4(1-\nu)m^2l^2b_1-m_1c_1+c_2=0, \tag{7.15}$$

where a_2 and c_2 are defined by

$$\begin{aligned} a_2=&\frac{A_0}{(\beta+2m)^2\varDelta}\left[(\beta+m)\left\{2\left((1+m^2l^2+2\,\frac{m}{\beta}\right)-\frac{\varDelta}{1-\nu}-\frac{4m(m+\beta)}{(1-\nu)\beta(\beta+2m)\varDelta}\right\}\right.\\ &\left.+\frac{m}{1-\nu}\right]-\frac{4A_0m^2l^2(1+\beta m_1l^2)(\beta+m_1)}{\beta m_1(1+2\beta m_1l^2+\beta^2l^2)\varDelta}\\ &+\frac{2A_0l^2}{(\beta+2m)(1-2m\beta l^2-\beta^2l^2)\varDelta}\left[4(1-\nu)\beta m^2l^2\left\{m\left(4-\frac{m}{m_1}\right)+2\beta\right\}\right.\\ &\left.-\frac{2m^3}{\beta+2m}-\frac{2\beta(m+\beta)m^2l^2}{1-2m\beta l^2-\beta^2l^2}\right]+\frac{8(1-\nu)A_0m^2(1+\beta m_1l^2)(m_1+\beta)}{\beta^2m_1(\beta+2m_1)\varDelta}\\ &+\frac{4(1-\nu)A_0m^2(1+2\beta m_1l^2)}{\beta m_1(\beta+2m_1)\varDelta}, \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} c_2=&\frac{2A_0l^2}{m(\beta+2m)(1-2m\beta l^2-\beta^2l^2)\varDelta}\left[(m+\beta)\left\{\frac{2m^3}{\beta+2m}\right.\right.\\ &\left.\left.-4(1-\nu)\beta m^2l^2\left(4m+2\beta-\frac{m^2}{m_1}\right)+\frac{2\beta(m+\beta)m^2l^2}{1-2m\beta l^2-\beta^2l^2}\right\}-\beta m^2\right]\\ &-\frac{8(1-\nu)A_0m(1+\beta m_1l^2)(m_1+\beta)^2}{\beta^2m_1(\beta+2m_1)\varDelta}\\ &-\frac{4(1-\nu)A_0m(1+2\beta m_1l^2)(m_1+\beta)}{\beta m_1(\beta+2m_1)\varDelta}. \end{aligned} \tag{7.17}$$

Equation (7.13) determines a_1, and then equations (7.14) and (7.15) give b_1 and c_1. Thus the problem is formally solved. Detailed discussion of the effects of couple-stresses is given in the subsequent sections.

Since the value of c_1 will be required afterwards, we determine it from (7.14) and (7.15) as

$$c_1=\frac{4(1-\nu)m^2l^2(ma_1+a_2+c_2)+c_2}{m_1+4(1-\nu)m^2l^2(m_1-m)}. \tag{7.18}$$

https://doi.org/10.1017/S1446788700004742 Published online by Cambridge University Press

8. Effect of couple-stresses on the stresses in a homogeneous material

The functions F_0 and f_0 given by (7.9) correspond to the solution of the problem for the homogeneous material. To obtain an estimate of the effect of couple-stresses in the homogeneous material we consider the anti-symmetric part of the stress tensor, which is most important from the standpoint of the present discussion. This is given by the equation (4.4) together with the last two equations in (5.2), that is,

$$\sigma_A = \tfrac{1}{2}(\sigma_{xy} - \sigma_{xy}) = -\tfrac{1}{2}\nabla^2 \psi. \tag{8.1}$$

Recalling that $f = f_0$ in the homogeneous case, we obtain, after substitution from (6.6) and (7.9),

$$\sigma_A = -\frac{2(1-\nu)A_0 m}{m_1 \Delta} e^{-m_1 y} \sin mx. \tag{8.2}$$

Since $\sigma_{yx} = 0$ on $y = 0$, this gives

$$(\sigma_A)_{y=0} = (\sigma_{xy})_{y=0} = -2(1-\nu)A_0 m\gamma \sin mx, \tag{8.3}$$

where

$$\frac{1}{\gamma} = m_1 \Delta = m_1 \left[1 + 4(1-\nu)l^2 m^2 \left(1 - \frac{m}{m_1}\right)\right]. \tag{8.4}$$

Thus γ is the influence factor for couple-stresses. It depends upon the Poisson ratio ν and the load distribution factor m as well as the parameter l. It is easily seen that $\gamma \to 0$ when $l \to 0$, while $\gamma \to 1/m(3-2\nu)$ when $l \to \infty$, so that γ varies from the value zero to the asymptotic value $1/m(3-2\nu)$. In fact, $\gamma \simeq l$ for small values of l, while

$$\gamma \simeq \frac{1-(1/2l^2m^2)}{m(3-2\nu)},$$

for large l. Table I gives the values of γ for different values of l and ν, where we have taken $m = 1$ in the non-dimensional coordinates. Figure 1 gives their graphical representations. It is however to be noted that l is generally small, so that the graph in the neighbourhood of the origin is most representative.

TABLE I

Values of γ for various values of l and ν ($m = 1.0$)

ν \ l	0.0	0.2	0.4	0.6	0.8	1.0	5.0
0.0	0.0	0.173	0.251	0.304	0.319	0.326	0.333
0.3	0.0	0.180	0.294	0.347	0.375	0.389	0.416
0.5	0.0	0.185	0.315	0.383	0.422	0.446	0.498

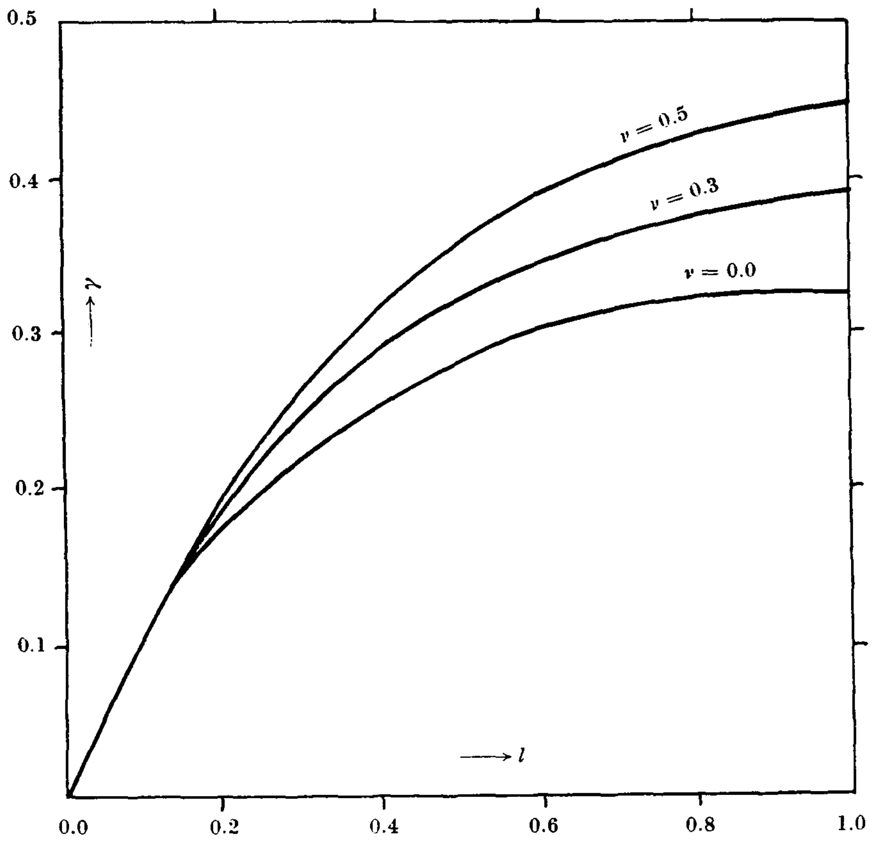

Figure 1

9. Effect of couple-stresses on the stresses in a non-homogeneous material

Equation (7.1) shows that the functions F_1 and f_1 characterize the effect of the couple-stresses on the stress distribution in a non-homogeneous material to a first approximation. As in the previous section, we consider the anti-symmetric part of the stress tensor, that is,

$$\sigma_A = \tfrac{1}{2}(\sigma_{xy} - \sigma_{yx}) = -\tfrac{1}{2}\nabla^2\psi.$$

Since $f = \alpha f_1$ in the present case, the substitution from (6.6) in the above relation gives

$$\sigma_A = -\tfrac{1}{2}\alpha(D^2 f_1 - m^2 f_1) \sin mx. \tag{9.1}$$

Since $\sigma_{yx} = 0$ on $y = 0$, we have

(9.2) $$(\sigma_A)_{y=0} = (\sigma_{xy})_{y=0} = -\tfrac{1}{2}\alpha N \sin mx$$

where

(9.3) $$N = (D^2 f_1 - m^2 f_1)_{y=0}$$

represents the effect of the non-homogeneity to the first approximation as seen from (7.1), and also contain the effect of couple-stresses. Substituting the value of f_1 from (7.12) we obtain

(9.4)
$$\begin{aligned} N = \frac{c_1}{l^2} &+ \frac{2A_0\beta l^2}{m(1-2m\beta l^2-\beta^2 l^2)\Delta}\left[4(1-\nu)\beta m^2 l^2\left(4m+2\beta-\frac{m^2}{m_1}\right)\right. \\ &\left.-\frac{2m^2(2m-\beta)}{\beta+2m}-\frac{2\beta(m+\beta)m^2l^2}{1-2m\beta l^2-\beta^2l^2}\right] \\ &+\frac{8(1-\nu)A_0 m(1+\beta m_1 l^2)(m_1+\beta)}{\beta^2 m_1(\beta+2m_1)\Delta}\cdot\frac{1+2m_1\beta l^2+\beta^2l^2}{l^2} \\ &+\frac{4(1-\nu)A_0 m(1+2\beta m_1 l^2)}{\beta m_1(\beta+2m_1)\Delta}\cdot\frac{1+2m_1\beta^2+\beta^2 l^2}{l^2}, \end{aligned}$$

where c_1 is determined by (7.18).

The influence stress N is thus a function of l and other parameters such as the load factor m and the exponential factor β. In non-dimensional coordinates we can always take $m = 1$. Moreover, there will not be much loss of generality by taking $\beta = 1$ for detailed discussion of the value of N. The parameter l is generally small and hence we may neglect l^3 and higher powers of l in comparison with unity. Then we obtain

$$m_1 = \frac{\sqrt{1+l^2}}{l}, \qquad \Delta = 1+4(1-\nu)l^2, \qquad \frac{1}{\Delta} = 1-4(1-\nu)l^2,$$

and relations (7.13), (7.16) and (7.17) simplify to

(9.5)
$$\begin{aligned} a_1 &= -\tfrac{1}{9}A_0\left\{6-\frac{11}{3(1-\nu)}\right\}\{1-4(1-\nu)l^2\}, \\ a_2 &= \tfrac{1}{9}A_0\left\{12-\frac{19}{3(1-\nu)}\right\}\{1-4(1-\nu)l^2\}-\tfrac{4}{9}A_0 l^2-2(1-\nu)A_0 l^2, \\ c_2 &= -\tfrac{2}{9}A_0 l^2-2(1-\nu)A_0[2+4l+l^2-4(1-\nu)l^2][l+\sqrt{1+l^2}] \end{aligned}$$

Hence from (7.18)

(9.6) $$\frac{c_1}{l^2} = 4(1-\nu)A_0 l^2\left\{\frac{7}{6}-\frac{8}{27(1-\nu)}\right\}+\frac{c_2}{l}.$$

https://doi.org/10.1017/S1446788700004742 Published online by Cambridge University Press

Moreover, (9.4) now simplifies to

$$N = 4(1-\nu)A_0 l\left\{\frac{7}{6}-\frac{8}{27(1-\nu)}\right\}+\frac{c_2}{l}+2(1-\nu)A_0\left[\frac{2}{l}+4+l-4(1-\nu)l -\frac{7}{4}l^2-4(1-\nu)l^2\right][1+2l\sqrt{1+l^2}+l^2]. \tag{9.7}$$

By the statement in Section 6, the value of N given by (9.7) is valid if l is a finite non-zero (small) quantity. As l tends to zero, it may be seen that N tends to a finite value. In fact, $[N/\{4(1-\nu)A_0\}] \to 1$. But putting $l = 0$ in earlier equations (6.4) or (7.12), it is found that the function ψ and hence the functions f_r are all identically zero. That is, the non-homogeneity alone cannot produce non-symmetric stresses, as is also well known. The discontinuity of N at $l = 0$ shows that, if the material is assumed to be even slightly non-homogeneous, the effect of couple-stresses becomes immediately pronounced.

The values of $N/2{\cdot}8A_0$ for different values of l are given in Table II and the graphical representation in Figure 2, where we have taken $\nu = 0{\cdot}3$ for definiteness. As seen from the figure, the non-symmetric surface stress increases with the increase of the parameter of the couple-stresses.

TABLE II

($\nu = 0.3$)

l	0.001	0.01	0.05	0.1	0.2	0.5
$N/2{\cdot}8A_0$	1.000	1.025	1.125	1.336	1.535	1.637

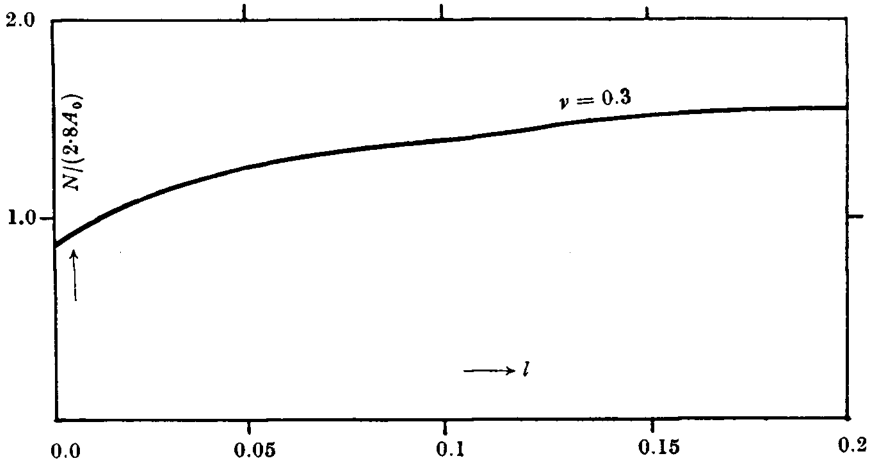

Figure 2

References

[1] G. Paria, "Love waves in granular medium", *Bull. Calcutta Math. Soc.* 52 (1960), 195—203.

[2] E. Cosserat and F. Cosserat, "Théorie des Corps déformables" (A. Herman et Fils, Paris, 1909).

[3] C. Truesdell and R. Toupin, "The classical field theories", in *Handbuch der Physik*, Vol. III/1, edited by S. Flügge (Springer-Verlag, 1960).

[4] R. Toupin, "Elastic materials with couple-stresses", *Arch. Rat. Mech. Anal.* 11 (1962), 385—414.

[5] R. D. Mindlin and H. F. Tiersten, "Effects of couple-stresses in linear elasticity", *Arch. Rat. Mech. Anal.* 11 (1962), 415—448.

[6] S. Kaliski, Z. Plochocki and D. Rogula, "The asymmetry of the stress tensor and the angular momentum conservation for a combined mechanical and electromagnetic field in a continuous medium", *Bull. Acad. Polo. Sci. ser. Sci. Tech.* 10 (1962), 135—141.

[7] W. Gunther, Miscellaneous topics in applied mechanics, editor W. Tollmien (Akad-Verlag, 1962).

[8] R. D. Mindlin, "Influence of couple-stresses on stress concentrations", *Exper. Mech.* 3 (1963), 1—7.

[9] S. Kaliski, "On a model of the continuum with an essentially non-symmetric tensor of mechanical stress", *Arch. Mech. Stos.* 15 (1963), 33—45.

[10] R. V. Kuvshinskii and E. L. Aéro, "Continuum theory of asymmetric elasticity", *Fiz. Tverd. Tela.* 5 (1963), 2591—2598 (in Russian). English translation: Soviet Physics, Solid State, 5 (1964), 1892—1897.

[11] M. Misicu, "Theory of viscoelasticity with couple-stresses and some reductions to two dimensional problems I", *Rev. Mec. Appl.* 8 (1963), 921—952.

[12] R. D. Mindlin, "Micro-structure in linear elasticity", *Arch. Rat. Mech. Anal.* 16 (1964), 51—78.

[13] R. Hayart, "Une application simple des formules de Murnaghan généralisées", *C. R. Acad. Sci. Paris*, 258 (1964), 1712—1715.

[14] W. T. Koiter, "Couple-stresses in the theory of elasticity, I and II", *Nederl. Akad. Wetensch. Proc.* ser. B67 (1964), 17—29 and 30—44.

[15] G. Paria, "Radial vibration of a sphere of varying modulus of elasticity", *J. Sci. Eng. Res.* 7 (1963), 1—4.

[16] W. Olszak (editor), "Non-homogeneity in elasticity and plasticity", *IUTAM Symposium* (Pergamon Press, 1958).

Department of Applied Mathematics
Shri Govindram Seksaria Technological Institute
Indore (M.P.), India