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A GAMMA ACTIVITY TIME PROCESS WITH NONINTEGER PARAMETER AND SELF-SIMILAR LIMIT

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Abstract

We construct a process with gamma increments, which has a given convex autocorrelation function and asymptotically a self-similar limit. This construction validates the use of long-range dependent t and variance-gamma subordinator models for actual financial data as advocated in Heyde and Leonenko (2005) and Finlay and Seneta (2006), in that it allows for noninteger-valued model parameters to occur as found empirically by data fitting.

Keywords: Gamma process; variance-gamma distribution; t distribution; subordinator model; construction; long-range dependence; self-similarity

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1. Introduction

Heyde and Leonenko (2005) and Finlay and Seneta (2006) respectively constructed discrete time t and variance-gamma (VG) distributed subordinator models which exhibit long-range dependence (LRD) of squared returns, a desirable property for asset price models. This LRD comes from asymptotically self-similar reciprocal gamma (RΓ) and gamma (Γ) based 'activity time' $\{T_t\}$ processes, respectively, and in particular is driven by the LRD of the increment processes, denoted by $\tau(t) = T_t - T_{t-1},\ t = 1, 2, \dots$. (A continuous-time process $\{Y_t\}$ is said to be self-similar with parameter H if $Y_{ct} \stackrel{\mathrm{D}}{=} c^H Y_t$, where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution; LRD of a discrete-time stationary process with ultimately nonnegative autocorrelations $\{\gamma_k\}$ is said to hold if $\sum_{k=1}^{\infty} \gamma_k = \infty$; and activity time, as opposed to standard clock time, is the increasing stochastic process over which security prices are taken to evolve.) The $\{T_t\}$ processes are scaled such that their increments over unit time have unit expectation, with the $\tau(t)$ taken to be R$\Gamma(\nu/2, \nu/2 - 1),\ \nu > 4$, and $\Gamma(\nu/2, \nu/2),\ \nu \geq 1$, distributed, respectively, having the following probability density functions for $x > 0$:

$$f_{\mathrm{R}\Gamma}(x) = \frac{(\nu/2 - 1)^{\nu/2}}{\Gamma(\nu/2)} x^{-\nu/2-1} \mathrm{e}^{(1-\nu/2)/x} \quad \text{and} \quad f_{\Gamma}(x) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} \mathrm{e}^{-(\nu/2)x}.$$

The processes constructed in Heyde and Leonenko (2005) and Finlay and Seneta (2006) are restricted to integer values of ν, however, a condition not consistent with estimation with actual data. The purpose of this note is therefore to extend the constructions to allow for noninteger ν.

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This extension validates the use of the models for real data. Section 2 details this for the Γ-based process, while Section 4 details this for the RΓ-based process. We also include a brief description of two other possible activity time constructions which result in LRD, given in Section 5. In the interest of brevity we exclude any nonessential details or references; more background information can be found in the two papers mentioned.

2. Noninteger ν: the Γ case

For $\mathbb{N} = \{1, 2, 3, \dots\}$, let $\{\eta_i(t),\ t \in \mathbb{N}\}$, $i = 1, \dots, \lfloor \nu \rfloor$, $\nu \geq 1$ (where $\lfloor \cdot \rfloor$ denotes the integer-part function), be independent and identically distributed (i.i.d.) stationary Gaussian processes with zero mean, unit variance, and autocorrelation function (ACF) $\rho(s)$, $s \in \mathbb{N}$. Define the stationary process $\{\tau_{\lfloor \nu \rfloor}(t),\ t \in \mathbb{N}\}$ by

$$\tau_{\lfloor \nu \rfloor}(t) = \frac{\eta_1^2(t) + \cdots + \eta_{\lfloor \nu \rfloor}^2(t)}{\lfloor \nu \rfloor}.$$

Then we can set

$$T_t = \sum_{i=1}^{t} \tau_{\lfloor \nu \rfloor}(i),$$

so that, for each integer $t \geq 1$,

$$T_t - T_{t-1} = \tau_{\lfloor \nu \rfloor}(t) \stackrel{\mathrm{D}}{=} \Gamma\bigg(\frac{\lfloor \nu \rfloor}{2}, \frac{\lfloor \nu \rfloor}{2}\bigg)$$

with

$$\operatorname{cov}(\tau_{\lfloor \nu \rfloor}(t), \tau_{\lfloor \nu \rfloor}(t+s)) = \frac{2}{\lfloor \nu \rfloor}\rho^2(s). \tag{1}$$

Here we have set $T_0 = \tau_{\lfloor \nu \rfloor}(0) = \eta_1(0) = \cdots = \eta_{\lfloor \nu \rfloor}(0) = 0$. This is the discrete $\{T_t\}$ process that Heyde and Leonenko (2005) and Finlay and Seneta (2006) worked with and showed, after appropriate norming, to converge weakly to a continuous-time, self-similar 'Rosenblatt' process.

Assumption 1. *Set* $Z(s) = \rho^2(s) - \rho^2(s+1)$*, and assume that* $Z(s-1) - Z(s) \geq 0$ *for* $s \in \mathbb{N}$*, which is equivalent to* $\rho^2(s)$ *being convex on the integers. We also require that* $Z(s) \geq 0$.

Theorem 1. *Under Assumption 1 there exists a process* $\tau_\nu(t)$, $t \in \mathbb{N}$, *with noninteger* $\nu > 0$ *and marginal* $\Gamma(\nu/2, \nu/2)$ *distribution such that* $\operatorname{cov}(\tau_\nu(t), \tau_\nu(t+s)) = (2/\nu)\rho^2(s)$ *for* $s \in \mathbb{N}$*, parallel to (1).*

Setting $T_t = \sum_{i=1}^{t} \tau_\nu(i)$ and choosing $\rho(s)$ such that $\tau_\nu(t)$ is LRD results in a discrete LRD VG process with noninteger ν parameter. In this section we aim to prove Theorem 1, with the main steps set out in Lemmas 1, 2 and 3.

First we construct two τ_νs such that they have covariance of the form given by (1). Fix $n \in \mathbb{N}$ and set $\iota = (\nu - \lfloor \nu \rfloor)/2$ and $Y_{i,*}^n \stackrel{\mathcal{D}}{=} Y_{i,\circ}^n \stackrel{\mathcal{D}}{=} \Gamma(\iota/n, 1/2)$, $i = 1, \dots, n$, all independent and independent of the ηs. Then set

$$X_*^n := \sum_{i=1}^{n} Y_{i,*}^n \stackrel{\mathrm{D}}{=} \Gamma\bigg(\iota, \frac{1}{2}\bigg) \quad \text{and} \quad X_\circ^n := \sum_{i=1}^{k} Y_{i,*}^n + \sum_{i=1}^{n-k} Y_{i,\circ}^n \stackrel{\mathrm{D}}{=} \Gamma\bigg(\iota, \frac{1}{2}\bigg), \tag{2}$$

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so there is an overlap of k of the $Y^n_{i,*}$s between X^n_* and $X^n_\circ$. Now set

$$\tau_\nu(t) = \frac{\eta_1^2(t) + \cdots + \eta_{\lfloor\nu\rfloor}^2(t) + X^n_*}{\nu}$$

and

$$\tau_\nu(t+s) = \frac{\eta_1^2(t+s) + \cdots + \eta_{\lfloor\nu\rfloor}^2(t+s) + X^n_\circ}{\nu},$$

both $\Gamma(\nu/2, \nu/2)$ random variables.

Lemma 1. *For any fixed $t \in \mathbb{N}$ and fixed single temporal lag $s \in \mathbb{N}$, $\tau_\nu(t)$ and $\tau_\nu(t+s)$ as defined above with $k = \lfloor n\rho^2(s)\rfloor$ result in $\operatorname{cov}(\tau_\nu(t), \tau_\nu(t+s)) \to (2/\nu)\rho^2(s)$ as $n \to \infty$, with the error bounded by $4\iota/n\nu^2$ independent of t and s. (In this case we do not need Assumption 1.)*

Proof. From (2),

$$\operatorname{cov}(X^n_*, X^n_\circ) = \operatorname{var}(Y^n_{1,*} + \cdots + Y^n_{k,*}) = \frac{4\iota k}{n},$$

so that

$$\begin{aligned}\operatorname{cov}(\tau_\nu(t), \tau_\nu(t+s)) &= \frac{1}{\nu^2}\operatorname{cov}\left(\sum_{i=1}^{\lfloor\nu\rfloor}\eta_i^2(t), \sum_{i=1}^{\lfloor\nu\rfloor}\eta_i^2(t+s)\right) + \frac{1}{\nu^2}\operatorname{cov}(X^n_*, X^n_\circ)\\ &= \frac{2}{\nu}\rho^2(s) + \frac{4\iota}{\nu^2}\frac{\lfloor n\rho^2(s)\rfloor - n\rho^2(s)}{n}.\end{aligned}$$

The above shows how we construct a process τ_ν with the desired correlation structure at lag s. Constructing a stationary process τ_ν that has the correct correlation at all lags is more involved. We now give a procedure to this end.

Again fix $n \in \mathbb{N}$, set

$$\iota = \frac{\nu - \lfloor\nu\rfloor}{2} \quad \text{and} \quad Y^n_{i,j} \stackrel{\mathcal{D}}{=} \Gamma\left(\frac{\iota}{n}, \frac{1}{2}\right),$$

$i = 1, \ldots, \lfloor n\rho^2(1)\rfloor$ for $j = 0$, and $i = 1, \ldots, n - \lfloor n\rho^2(1)\rfloor$ for $j = 1, 2, \ldots$, with all the $Y^n_{i,j}$s mutually independent. Then set

$$X^n_t = \sum_{i=1}^{\lfloor n\rho^2(t)\rfloor} Y^n_{i,0} + \sum_{j=1}^{t}\left(\sum_{i=1}^{\lfloor n\rho^2(t-j)\rfloor - \lfloor n\rho^2(t-j+1)\rfloor} Y^n_{i,j}\right) \stackrel{\mathrm{D}}{=} \Gamma\left(\iota, \frac{1}{2}\right) \quad \text{for } t = 1, 2, \ldots \tag{3}$$

(assuming $Z(s) \geq 0$, setting $X^n_0 = 0$, and noting that $\rho^2(0) = 1$), and

$$\begin{aligned}\tau_\nu(t) &= \frac{\eta_1^2(t) + \cdots + \eta_{\lfloor\nu\rfloor}^2(t) + X^n_t}{\nu}\\ &\stackrel{\mathrm{D}}{=} \Gamma\left(\frac{\nu}{2}, \frac{\nu}{2}\right),\\ T_t &= \sum_{i=1}^{t}\tau_\nu(i). \end{aligned}\tag{4}$$

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Lemma 2. *Under Assumption 1, for any time $t \in \mathbb{N}$ and temporal lag $s \in \mathbb{N}$, with $\tau_\nu(t)$ and $\tau_\nu(t+s)$ as defined above,* $\operatorname{cov}(\tau_\nu(t), \tau_\nu(t+s)) \to (2/\nu)\rho^2(s)$ *as $n \to \infty$.*

Proof. Consider any X_t^n and X_{t+s}^n for $t, s \in \mathbb{N}$. Then, for any j such that $1 \le j \le t$, X_t^n contains the first $\lfloor n\rho^2(t-j)\rfloor - \lfloor n\rho^2(t-j+1)\rfloor$ of the $Y_{i,j}^n$s, while X_{t+s}^n contains the first $\lfloor n\rho^2(t+s-j)\rfloor - \lfloor n\rho^2(t+s-j+1)\rfloor$ of the same $Y_{i,j}^n$s. But $s > 0$, so by Assumption 1 $\lfloor n\rho^2(t+s-j)\rfloor - \lfloor n\rho^2(t+s-j+1)\rfloor \le \lfloor n\rho^2(t-j)\rfloor - \lfloor n\rho^2(t-j+1)\rfloor$ for large n, so the overlap of $Y_{i,j}^n$s between X_t^n and X_{t+s}^n is simply $\lfloor n\rho^2(t+s-j)\rfloor - \lfloor n\rho^2(t+s-j+1)\rfloor$. For $j > t$, X_t^n contains none of the $Y_{i,j}^n$s while, for $j = 0$, X_t^n contains the first $\lfloor n\rho^2(t)\rfloor$ of the $Y_{i,0}^n$s, while X_{t+s}^n contains the first $\lfloor n\rho^2(t+s)\rfloor$ of the $Y_{i,0}^n$s. Hence, the total number of overlapping $Y_{i,j}^n$s between X_t^n and X_{t+s}^n is

$$\sum_{j=1}^{t}(\lfloor n\rho^2(t+s-j)\rfloor - \lfloor n\rho^2(t+s-j+1)\rfloor) + \lfloor n\rho^2(t+s)\rfloor = \lfloor n\rho^2(s)\rfloor.$$

But from Lemma 1 this delivers the correct correlation.

Lemma 3. *Under Assumption 1, $\{X_t^n\}$ for $t \in \mathbb{N}$, as defined by (3), converges weakly to a well-defined stochastic process $\{X_t\}$ as $n \to \infty$.*

Proof. Fix $p \in \mathbb{N}$, and let $a_1, \dots, a_p \in \mathbb{R}$. To ease notation set

$$f(t) = \lfloor n\rho^2(t)\rfloor \quad \text{and} \quad g(t) = \lfloor n\rho^2(t-1)\rfloor - \lfloor n\rho^2(t)\rfloor.$$

Then starting from (3), we can show that $\sum_{t=1}^{p} a_t X_t^n$ is given by

$$\begin{aligned}
&\sum_{i=1}^{f(p)}\biggl(\biggl(\sum_{t=1}^{p} a_t\biggr)Y_{i,0}^n\biggr) + \sum_{j=1}^{p-1}\biggl(\sum_{i=f(j+1)+1}^{f(j)}\biggl(\biggl(\sum_{t=1}^{j} a_t\biggr)Y_{i,0}^n\biggr)\biggr)\\
&\quad+\sum_{k=1}^{p}\biggl(\sum_{i=1}^{g(k)}\biggl(\biggl(\sum_{t=p-k+1}^{p} a_t\biggr)Y_{i,p-k+1}^n\biggr)\biggr)\\
&\quad+\sum_{k=1}^{p-1}\biggl(\sum_{j=1}^{p-k}\biggl(\sum_{i=g(j+1)+1}^{g(j)}\biggl(\biggl(\sum_{t=k}^{k+j-1} a_t\biggr)Y_{i,k}^n\biggr)\biggr)\biggr).
\end{aligned}\tag{5}$$

Now each $Y_{i,j}^n$ is independent and $\Gamma(\iota/n, 1/2)$ distributed, so the characteristic function (CF) of $(X_1^n, \dots, X_p^n)$ is given by

$$\begin{aligned}
\phi_p^n(a_1,\dots,a_p) &= \biggl(\biggl(1 - 2\mathrm{i}\biggl(\sum_{t=1}^{p} a_t\biggr)\biggr)^{-\iota f(p)/n}\biggr)\biggl(\prod_{j=1}^{p-1}\biggl(1 - 2\mathrm{i}\biggl(\sum_{t=1}^{j} a_t\biggr)\biggr)^{-\iota g(j+1)/n}\biggr)\\
&\quad\times\biggl(\prod_{k=1}^{p}\biggl(1 - 2\mathrm{i}\biggl(\sum_{t=p-k+1}^{p} a_t\biggr)\biggr)^{-\iota g(k)/n}\biggr)\\
&\quad\times\biggl(\prod_{k=1}^{p-1}\biggl(\prod_{j=1}^{p-k}\biggl(1 - 2\mathrm{i}\biggl(\sum_{t=k}^{k+j-1} a_t\biggr)\biggr)^{-\iota(g(j)-g(j+1))/n}\biggr)\biggr).
\end{aligned}$$

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As $n \to \infty$, we have

$$\frac{f(t)}{n} \to \rho^2(t) \quad \text{and} \quad \frac{g(t)}{n} \to \rho^2(t-1) - \rho^2(t) = Z(t-1),$$

so that $\phi_p^n(a_1, \dots, a_p)$ converges to a function $\phi_p(a_1, \dots, a_p)$ given by

$$\begin{aligned}\phi_p(a_1, \dots, a_p) = {} & \bigg(\bigg(1 - 2\mathrm{i}\bigg(\sum_{t=1}^{p} a_t\bigg)\bigg)^{-\iota\rho^2(p)}\bigg)\bigg(\prod_{j=1}^{p-1}\bigg(1 - 2\mathrm{i}\bigg(\sum_{t=1}^{j} a_t\bigg)\bigg)^{-\iota Z(j)}\bigg) \\ & \times \bigg(\prod_{k=1}^{p}\bigg(1 - 2\mathrm{i}\bigg(\sum_{t=p-k+1}^{p} a_t\bigg)\bigg)^{-\iota Z(k-1)}\bigg) \\ & \times \bigg(\prod_{k=1}^{p-1}\bigg(\prod_{j=1}^{p-k}\bigg(1 - 2\mathrm{i}\bigg(\sum_{t=k}^{k+j-1} a_t\bigg)\bigg)^{-\iota(Z(j-1)-Z(j))}\bigg)\bigg), \end{aligned} \tag{6}$$

which is clearly continuous about the origin (we can also verify that $X_t \overset{\mathcal{D}}{=} \Gamma(\iota, \frac{1}{2})$ for $t = 1, 2, \dots$ by considering $\phi_t(a_1, \dots, a_t)$ and choosing $a_1 = \dots = a_{t-1} = 0$). Weak convergence follows from Theorem 7.6 of Billingsley (1968) (see also the second paragraph on p. 30 of Billingsley (1968)).

If we choose $\rho(s) = (1 + \omega|s|^\alpha)^{(H-1)/\alpha}$ for $\omega > 0$, $0 < \alpha \le 2$, and $\frac{1}{2} < H < 1$ (i.e. an autocorrelation function from the so-called Cauchy family detailed in Gneiting (2000)), our construction will lead to an LRD VG model. Actual data estimation results in noninteger-valued ν estimates, so it is important to show that such LRD VG processes do actually exist.

Now, from (4) we can take our activity time process T_t as the sum of two independent parts:

$$T_t = \frac{1}{\nu}\sum_{j=1}^{\lfloor \nu \rfloor}\sum_{i=1}^{t} \eta_j^2(i) + \frac{1}{\nu}\sum_{i=1}^{t} X_i = A_t + B_t, \tag{7}$$

say. Using Taqqu (1975), we can show that $\operatorname{var}(A_k)$ and $\operatorname{var}(B_k)$ are both $\mathcal{O}(k^{2H})$, and that $(A_{\lfloor kt \rfloor} - \mathrm{E}\, A_{\lfloor kt \rfloor})/k^H$ converges weakly as k tends to ∞ to a self-similar process with parameter H (Heyde and Leonenko (2005) and Finlay and Seneta (2006) showed this for $\omega = 1$ and $\alpha = 2$, but the proof can be extended to cover the more general case where $\omega > 0$ and $0 < \alpha \le 2$).

We give a proof in Theorem 2, below, that $(1/k^H)(B_k - \mathrm{E}\, B_k)$ converges in probability to 0, which is enough to demonstrate that our new discrete-time $\{T_t\}$ process (7) has asymptotically a self-similar limit structurally coincident with that of the original $\{T_t\}$ process used in Finlay and Seneta (2006).

3. Convergence of the add-on term

Theorem 2. *When $\rho(s)$ is given by a member of the Cauchy family, the sequence*

$$\zeta_k = \bigg(\frac{1}{k^H}\bigg)\sum_{i=1}^{k}(X_i - \mathrm{E}\, X_i) \quad \textit{for } k = 1, 2, \dots,$$

converges in distribution, and therefore probability, to 0 *as* k *tends to* ∞.

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Proof. We give the proof taking $\rho(s)$ as any member of the Cauchy family which satisfies Assumption 1, in order to use (6) from Lemma 2.

First consider the CF $\phi_p^*(a_1, \dots, a_p)$ of $(X_1 - \mathrm{E}\, X_1, \dots, X_p - \mathrm{E}\, X_p)$. By replacing each $Y_{i,j}^n$ in (5) with $Y_{i,j}^n - \mathrm{E}\, Y_{i,j}^n = Y_{i,j}^n - 2\iota/n$, we can show that $\phi_p^*(a_1, \dots, a_p)$ is given by (6), but with each expression of the form $(1 - 2\mathrm{i}x)^{-\iota y}$ replaced by $(1 - 2\mathrm{i}x)^{-\iota y}\mathrm{e}^{-2\mathrm{i}x\iota y}$. Now, for $a \in \mathbb{R}$, the CF of ζ_k is given by

$$\varphi_k(a) = \mathrm{E}\left\{\exp\left(\frac{\mathrm{i}a}{k^H}\sum_{j=1}^{k}(X_j - 2\iota)\right)\right\} = \phi_k^*\left(\frac{a}{k^H}, \dots, \frac{a}{k^H}\right).$$

From (6), $\varphi_k(a)$ is a product comprising the following four factors:

$$(1 - 2\mathrm{i}ak^{1-H})^{-\iota\rho^2(k)}\exp(-2\mathrm{i}ak^{1-H}\iota\rho^2(k)), \tag{8}$$

$$\prod_{j=1}^{k-1}((1 - 2\mathrm{i}jak^{-H})^{-\iota Z(j)}\exp(-2\mathrm{i}jak^{-H}\iota Z(j))), \tag{9}$$

$$\prod_{j=1}^{k}((1 - 2\mathrm{i}jak^{-H})^{-\iota Z(j-1)}\exp(-2\mathrm{i}jak^{-H}\iota Z(j-1))), \tag{10}$$

$$\prod_{m=1}^{k-1}\prod_{j=1}^{k-m}((1 - 2\mathrm{i}jak^{-H})^{-\iota(Z(j-1)-Z(j))}\exp(-2\mathrm{i}jak^{-H}\iota(Z(j-1) - Z(j)))). \tag{11}$$

We shall use Markov's inequality to show that the random variables whose CFs are given by (8), (9), and (10) converge in probability to 0 as k tends to ∞, and show directly that the moment generating function (MGF) of the random variable with CF given by (11) converges to 1 as k tends to ∞, thus establishing the result.

First note that, for a nonnegative random variable Y_k, say, Markov's inequality states that, for any fixed $\varepsilon > 0$,

$$\mathrm{P}(Y_k > \varepsilon) \le \frac{\mathrm{E}\, Y_k}{\varepsilon},$$

so that if $\mathrm{E}\, Y_k \to 0$ then $Y_k \xrightarrow{\mathrm{P}} 0$ and, therefore, $(Y_k - \mathrm{E}\, Y_k) \xrightarrow{\mathrm{P}} 0$, where '$\xrightarrow{\mathrm{P}}$' denotes convergence in probability. Note also that each of (8), (9), and (10) represent the CF of a sum (mean-corrected) of independent and nonnegative Γ random variables, so that if we show that the mean of each such sum (before mean correction) converges to 0, we have completed the proof.

Now (8) before mean-correction is the CF of a $\Gamma(\iota\rho^2(k), 1/(2k^{1-H}))$ random variable with mean of $2k^{1-H}\iota\rho^2(k)$. For $\rho(s) = (1 + \omega|s|^\alpha)^{(H-1)/\alpha}$, $\rho^2(k) = \mathcal{O}(k^{2H-2})$, so that

$$2k^{1-H}\iota\rho^2(k) = \mathcal{O}(k^{H-1}) \to 0$$

as $k \to \infty$. Similarly the mean of the sum of Γ random variables with CF (9) is given by $2k^{-H}\iota\sum_{j=1}^{k-1} jZ(j)$. Here we have

$$0 \le k^{-H}\sum_{j=1}^{k-1} jZ(j) = k^{-H}\left(\left(\sum_{j=1}^{k-1}\rho^2(j)\right) - (k-1)\rho^2(k)\right),$$

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but both $k^{-H}(k-1)\rho^2(k)$ and $k^{-H}\sum_{j=1}^{k-1}\rho^2(j)$ are $\mathcal{O}(k^{H-1}) \to 0$ as $k \to \infty$, so that $2k^{-H}\iota\sum_{j=1}^{k-1} jZ(j) \to 0$. A similar result holds for (10).

Finally consider (11). In this case the mean is $\mathcal{O}(k^{1-H}) \to \infty$ and so we cannot use Markov's inequality. Instead change the order of multiplication to write the MGF of the *negative* of the random variable with CF (11) as

$$M_k(a) = \exp\biggl(\iota\sum_{j=1}^{k-1}(Z(j-1)-Z(j))(k-j)(2jak^{-H}-\log(1+2jak^{-H}))\biggr). \tag{12}$$

Working with the MGF instead of the CF simplifies matters, since $M_k(a)$ is well defined for all $a \ge 0$ and, from Theorem 2 of Mukherjea *et al.* (2006), pointwise convergence of $M_k(a)$ in some fixed interval (b, d), $0 < b < d < \infty$, as k tends to ∞, to the MGF $M(a)$ of some random variable implies weak convergence to the associated limit distribution. Thus, if $M_k(a)$ converges to 1, the MGF of 0, we have completed the proof. Now,

$$x \ge x - \log(1+x) \ge 0 \quad\text{and}\quad x^2 \ge x-\log(1+x) \ge 0 \qquad \text{for } x \ge 0,$$

so that

$$\begin{aligned} 0 &\le \sum_{j=1}^{k-1}(Z(j-1)-Z(j))(k-j)(2jak^{-H}-\log(1+2jak^{-H})) \\ &\le \sum_{j=1}^{\lfloor k^H\rfloor-1}(Z(j-1)-Z(j))(k-j)(2jak^{-H})^2 \\ &\quad + \sum_{j=\lfloor k^H\rfloor}^{k-1}(Z(j-1)-Z(j))(k-j)2jak^{-H} \\ &\le c_1k^{-2H}\sum_{j=1}^{\lfloor k^H\rfloor-1} j^{2H-2}(k-j) + c_2k^{-H}\sum_{j=\lfloor k^H\rfloor}^{k-1} j^{2H-3}(k-j) \end{aligned} \tag{13}$$

for constants c_1 and c_2, since $Z(j-1)-Z(j) = \mathcal{O}(j^{2H-4})$ by repeated application of the mean value theorem, using Assumption 1 and the explicit form of $\rho^2(s)$. But

$$k^{-2H}\int_1^{k^H} j^{2H-2}(k-j)\,\mathrm{d}j = \frac{k^{1-3H+2H^2}-k^{1-2H}}{2H-1} - \frac{k^{2H(H-1)}-k^{-2H}}{2H}$$

converges to 0 as k tends to ∞, since each exponent of k is negative for $\frac{1}{2} < H < 1$, and

$$k^{-H}\int_{k^H}^{k} j^{2H-3}(k-j)\,\mathrm{d}j = \frac{k^{H-1}-k^{1-3H+2H^2}}{2H-2} - \frac{k^{H-1}-k^{2H(H-1)}}{2H-1}$$

converges to 0 as k tends to ∞, so that (13) converges to 0 and (12) converges to 1.

Finally, recall that, for $Z(s) = \rho^2(s)-\rho^2(s+1)$, we require $Z(s) \ge 0$ and $Z(s)$ decreasing with s. It is clear that all members of the Cauchy family satisfy the first property, but the same is not true of the second. For example, $\alpha = \omega = 2$ satisfies the second property for any $\frac{1}{2} < H < 1$, whereas $\alpha = 2$ and $\omega = 1$ for $0.648 < H < 1$ does not. In the latter case the ACF value for $\{X_t\}$ at lag 1 will be $1 - \rho^2(1) + \rho^2(2)$ instead of the larger $\rho^2(1)$, but ACF values at larger lags will be unaffected (at lags greater than 1, the requirement on $Z(s)$ is satisfied if

$$1 - \alpha + \omega |s|^{\alpha}(3 - 2H) \geq 0,$$

which, for any given $\omega > 0$, $0 < \alpha \leq 2$, and $\frac{1}{2} < H < 1$, will be the case for sufficiently large values of s). Before leaving this section, we briefly discuss how our results are affected when the second property fails to hold for the first few lags s.

As touched on above, if $Z(s)$ does not decrease with s for the first m lags, say, then Lemma 2 will fail and the first m ACF values will be lower than those given by $\rho^2(\cdot)$ (ACF values at lags greater than m will be unaffected).

Considering Lemma 3, we undertook to partition $\sum_{t=1}^{p} a_t X_t^n$ into groups of i.i.d. $Y_{i,j}^n$s with the same coefficients (some sum of a_ts). Now, for each t, and ignoring rounding issues associated with taking the integer part, from (3) the first $nZ(t-j)$ of the $Y_{i,j}^n$s for $j = 1, \dots, t$ are included in X_t. Hence, the relative sizes of $Z(0), Z(1), \dots, Z(p-1)$ determine which of the $Y_{i,j}^n$s are included in which X_t, and so determine the groupings of $Y_{i,j}^n$s. For example, if we set

$$Z_0 = \max(Z(0), Z(1), \dots, Z(p-1)),$$

with Z_1 the next biggest through to

$$Z_{p-1} = \min(Z(0), Z(1), \dots, Z(p-1)),$$

then there would be Z_{p-1} of the $Y_{i,j}^n$s with coefficient $\sum_{t=1}^{p} a_t$, $Z_{p-2} - Z_{p-1}$ of the $Y_{i,j}^n$s with a coefficient of all bar one of the a_ts, and $Z_0 - Z_1$ of the $Y_{i,j}^n$s with a coefficient of only 1 of the a_ts. If Assumption 1 holds then $Z_j = Z(j)$, but if it does not hold then the order of the largest few Z_js may change. With this in mind, we can modify (5) by replacing each $g(j)$ by g_j defined analogously to the Z_js (note that the composition of the sum of a_ts associated with each $Y_{i,j}^n$ will change too, but the total number of a_ts summed will not change). From (5) we can carry through the changes to arrive at a new version of (6), with the Z_js in place of the $Z(j)$s and the composition of the sums of the a_ts changed for the last two factors, but Lemma 3 otherwise unaffected.

Next consider Theorem 2 in light of our new CF (6). Both (10) and (11) will change, but the mean of the random variable with CF (10) will still be $\mathcal{O}(k^{H-1}) \to 0$ as $k \to \infty$, while the first few $Z_{j-1} - Z_j$ values will still be bounded by $c^* j^{2H-4}$ for some constant c^*, and so (13) will be unaffected and (12) will still converge to 1. Hence, the activity time process generated from a member of the Cauchy family that does not satisfy Assumption 1 will still be LRD and have a self-similar limit, but the first few ACF values will be lower than in the integer ν construction.

4. Noninteger ν: the RΓ case

To construct their LRD t process, Heyde and Leonenko (2005) started with Γ distributed increments with integer ν. Their construction from then on does not require integer ν however; see Heyde and Leonenko (2005, Sections 3.3 and 5.1). As such, our Γ distributed increments from Section 2 can be 'plugged in' to their construction to show that LRD asymptotically self-similar t processes with noninteger ν values also exist. (Note that Heyde and Leonenko also required that $\nu > 4$ to ensure that $\mathrm{var}(\tau(t)) < \infty$, although Sly (2006) has since developed an approach which allows for $2 < \nu \leq 4$.)

5. A brief outline of other possible activity time processes

We briefly describe two other activity time constructions which lead to LRD subordinator models with a self-similar limit. The aim is only to introduce other possible approaches, with proofs and greater detail available in the papers mentioned.

From Sly (2006) (see also Taqqu (1979)), for $\eta_1(t)$ as in Section 2, set $\Phi(\cdot)$ as the distribution function of a standard normal, and set F as the distribution function of a Γ or RΓ random variable for example. Then, for each t, $\Phi(\eta_1(t))$ has the uniform distribution and $\tau_t = F^{-1}(\Phi(\eta_1(t)))$ has the distribution of F. When τ_1 has finite variance, a normed $T_t = \sum_{i=1}^{t} \tau_i$ converges in finite-dimensional distribution to fractional Brownian motion, while in the RΓ case, for τ_1 with infinite variance but finite mean, a normed T_t converges to a Lévy-stable process.

From Taqqu and Levy (1986) (see also Liu (2000)), set

$$W(t) = \sum_{k=0}^{\infty} W_k I(S_{k-1} < t \le S_k) = W_{N(t)},$$

where $S_k = S_0 + \sum_{j=1}^{k} U_j$ is a renewal sequence with positive integer-valued interarrival times U_j, and $N(t)$ is the associated counting process. Therefore, $W(t)$ takes the random value W_k for the duration of the kth interarrival time. Also assume that the $\{U_k\}$ are i.i.d. with $\mathrm{P}(U_1 \ge u) \sim u^{-a}h(u)$ for $1 < a < 2$ and $h(\cdot)$ is slowly varying with $\mathrm{E}\, U_1 = \mu$, that the $\{W_k\}$ are i.i.d. with $\mathrm{E}\, W_1 = 0$ and $\mathrm{E}\, W_1^2 < \infty$, and that the $\{U_k\}$ and $\{W_k\}$ are independent. So that $\{S_k\}$ is stationary choose $\mathrm{P}(S_0 = u) = \mu^{-1}\,\mathrm{P}(U_i \ge u+1)$, $u = 0, 1, \dots$, so that, by Karamata's theorem,

$$\mathrm{P}(S_0 \ge u) = \sum_{k=u}^{\infty} \mathrm{P}(S_0 = k) \sim \mu^{-1}(a-1)^{-1}u^{-(a-1)}h(u),$$

which implies that $\mathrm{E}\, S_0 = \infty$. Then we obtain

$$\operatorname{cov}(W(t), W(t+s)) = \mathrm{E}\, W_k^2 \sum_{k=0}^{\infty} \mathrm{P}(S_{k-1} < t < t+s \le S_k) = \mathrm{E}\, W_k^2\, \mathrm{P}(S_0 \ge s),$$

so that

$$\sum_{s=0}^{\infty} \operatorname{cov}(W(t), W(t+s)) = \infty,$$

giving LRD. In fact, the quantity

$$\zeta_k^{\mathrm{TL}}(t) = \frac{\sum_{i=1}^{\lfloor kt \rfloor} W(i)}{k^{1/a}L(k)}$$

for $L(\cdot)$ slowly varying and $t \in [0, 1]$, converges in finite-dimensional distribution as k tends to ∞ to a self-similar Lévy-stable process with parameter a. Using the notation from Section 2, we could set $\tau_t = W(t) + 1$ and $T_t = \sum_{i=1}^{t} \tau_i$, taking W_k, $k = 1, 2, \dots$, to be mean-corrected i.i.d. Γ or RΓ random variables for example.

However, there is no distribution of U_i which gives $\mathrm{P}(S_0 \ge s) = (1 + \omega|s|^a)^{(H-1)/a}$, where $a = 2$ and $\omega = 1$ for $0.648 < H < 1$.

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