

## POSITIVE DEFINITE AND RELATED FUNCTIONS ON HYPERGROUPS

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**ABSTRACT.** In this paper we make use of semigroup methods on the space of compactly supported probability measures to obtain a complete Lévy-Khinchin representation for negative definite functions on a commutative hypergroup. In addition we obtain representation theorems for completely monotone and completely alternating functions. The techniques employed here also lead to considerable simplification of the proofs of known results on positive definite and negative definite functions on hypergroups.

**1. Positive definite and negative definite functions.** The analysis throughout will be carried out on a (locally compact) hypergroup  $X$  admitting a left Haar measure  $m$ . (For a definition and properties we refer to Jewett [6], whose notation we follow.) This includes those hypergroups that are compact (Jewett [6], Theorem 7.2A), discrete (Jewett [6], Theorem 7.1A) or commutative (Spector [10], Theorem III.4). The space of measures absolutely continuous with respect to left Haar measure will be denoted by  $M_a(X)$ , and the space of bounded Radon measures by  $M^b(X)$ . We reserve the symbols  $M^1(X)$ ,  $M_c^1(X)$  for the spaces of probability measures and those that have compact support respectively. We also denote the point measure at  $x \in X$  by  $\varepsilon_x$ , and the indicator function of a set  $A$  by  $1_A$ . Finally the adjoint  $\nu^*$  of a measure  $\nu$  is defined by  $\nu^*(A) = \overline{\nu(A^-)}$  for all measurable  $A \subset X$ .

For each  $x, y \in X$  write

$$f(x * y) := \int f d\varepsilon_x * \varepsilon_y, \quad \mu * f(x) := \int f(z^- * x) d\mu(z)$$

and

$$f * g(x) := \int f(x * y)g(y^-) dm(y) = \int f(y)g(y^- * x) dm(y)$$

Here  $f, g$  are measurable functions on  $X$  and  $\mu \in M^b(X)$ , and the latter equality holds whenever one of  $f, g$  is  $\sigma$ -finite (see Jewett [6], Theorem 5.1D).

The left  $x$ -translate of  $f$  is written  $f_x(y) = f(x * y)$ . In Bloom and Heyer [2], Definition 2.5 the concept of uniform continuity was introduced, in terms of these translates, and it was shown that continuous functions with compact support are indeed uniformly continuous. For the work that follows we need to extend this idea.

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DEFINITION 1.1. A locally bounded measurable function  $f$  is called *left locally uniformly continuous* at  $x_0 \in X$  if there exists a neighbourhood  $U$  of  $x_0$  such that for every  $\varepsilon > 0$  there exists a neighbourhood  $V$  of the identity  $e$  satisfying

$$|f(y * x) - f(x)| < \varepsilon$$

for all  $x \in U, y \in V$ .

Theorem 2.6 of Bloom and Heyer [2] shows that a continuous function is left locally uniformly continuous at every point in  $X$ . Here we show that every function that is left locally uniformly continuous at  $x_0$  is in fact continuous on a neighbourhood of  $x_0$ . However, we first need a preliminary result, extending Jewett [6], Theorem 6.2E. In the statement of Proposition 1.2,  $L^\infty_{loc}(X)$  denotes the set of locally bounded measurable functions on  $X$ , and  $L^1_c(X)$  the space of integrable functions on  $X$  with compact support.

PROPOSITION 1.2.  $L^1_c * L^\infty_{loc}(X) \subset C(X)$

PROOF. Let  $k \in L^1_c(X)$  and  $g \in L^\infty_{loc}(X)$ . Choose  $x_0 \in X, U$  a compact neighbourhood of  $x_0$ , and write  $C = \text{supp } k$ . Then by Jewett [6], Lemma 3.2B,  $C^- * U$  is compact, and for all  $x \in U$

$$\begin{aligned} k * g(x) &= \int \int g \, d\varepsilon_{y^-} * \varepsilon_x k(y) \, dm(y) \\ &= \int \int 1_{C^- * Ug} \, d\varepsilon_{y^-} * \varepsilon_x k(y) \, dm(y) \\ &= k * (1_{C^- * Ug})(x) \end{aligned}$$

where the middle equality uses the fact that  $\text{supp}(\varepsilon_{y^-} * \varepsilon_x) \subset C^- * U$  for  $y \in C$ . We now observe that  $k \in L^1(X)$  and  $1_{C^- * Ug} \in L^\infty(X)$  so that Jewett [6], Theorem 6.2E gives

$$k * (1_{C^- * Ug}) \in L^1 * L^\infty(X) \subset C(X).$$

Since  $k * g$  and  $k * (1_{C^- * Ug})$  have been shown to agree on a neighbourhood of  $x_0$  it follows that  $k * g$  is continuous at  $x_0$ . But  $x_0 \in X$  was chosen arbitrarily, and this gives the result. ■

COROLLARY 1.3. Every  $f \in L^\infty_{loc}(X)$  that is left locally uniformly continuous at  $x_0$  is continuous on a neighbourhood of  $x_0$ .

PROOF. Let  $U, \varepsilon$  and  $V$  be as in Definition 1.1 and write  $k = m(V)^{-1}1_V$ . We can assume without loss of generality that  $V$  is symmetric and compact. Now for  $x \in U$

$$\begin{aligned} |k * f(x) - f(x)| &= \left| \int f(y^- * x)k(y) \, dm(y) - f(x) \int k(y) \, dm(y) \right| \\ &\leq \int |f(y^- * x) - f(x)| m(V)^{-1}1_V(y) \, dm(y) \\ &< \varepsilon. \end{aligned}$$

But Proposition 1.2 gives that  $k * f \in C(X)$  from which the continuity of  $f$  on  $U$  follows. ■

DEFINITION 1.4. A locally bounded measurable function  $\chi$  on  $X$  is called *multiplicative* if

$$(i) \chi(e) = 1$$

$$(ii) \chi(x * y) = \chi(x)\chi(y)$$

for all  $x, y \in X$ . If in addition  $\chi(x^-) = \overline{\chi(x)}$  for all  $x \in X$  then  $\chi$  is called a *semicharacter*. A bounded semicharacter will be called a *character*. It is immediate that every semicharacter is real-valued on the set of hermitian elements of  $X$ . Write  $\hat{X}$  for the set of all continuous characters on  $X$ . Note that every character  $\chi$  satisfies  $|\chi(x)\chi(y)| \leq \|\chi\|_\infty$  for all  $x, y \in X$  so that  $\|\chi\|_\infty^2 \leq \|\chi\|_\infty$  and  $\|\chi\|_\infty \leq 1$ . Thus  $\|\chi\|_\infty = \chi(e) = 1$ . A character need not be continuous. Indeed consider any 2-fold absolutely continuous hypergroup  $X$  with trivial centre, that is,  $\varepsilon_x * \varepsilon_y \in M_a(X)$  for all  $x, y \in X \setminus \{e\}$ . An example of such a hypergroup is given in Jewett [6], Section 9.5 (see Example 3.4 below). Then

$$1_{\{e\}}(x * y) = \varepsilon_x * \varepsilon_y(\{e\}) = \begin{cases} 1, & x = y = e \\ 0, & \text{otherwise} \end{cases}$$

from which it follows that  $1_{\{e\}}$  is a character which, in the case  $X$  is nondiscrete, cannot be continuous. However we do have:

**PROPOSITION 1.5.** *Every multiplicative function that is not locally null is continuous.*

**PROOF.** Let  $\chi$  be a multiplicative function that is not locally null. This entails that  $\chi$  is bounded away from zero on a compact set  $C$  with  $m(C) > 0$ . Write  $g = m(C^-)^{-1}1_C\chi^{-1}$ . By the choice of  $C$  it is clear that  $g \in L_c^1(X)$ . Also  $\chi \in L_{loc}^\infty(X)$  and for each  $x \in X$

$$\begin{aligned} \chi * g(x) &= \int \chi(x * y)m(C^-)^{-1}1_C(y^-)\chi^{-1}(y^-) dm(y) \\ &= \int \chi(x)\chi(y)m(C^-)^{-1}1_C(y^-)\chi(y)^{-1} dm(y) \\ &= \chi(x). \end{aligned}$$

Now appeal to Proposition 1.2 to deduce that  $\chi$  is continuous. ■

Alternatively we can use Jewett [6], Theorem 6.3F to prove Proposition 1.5. However the above proof is more elementary.

**DEFINITION 1.6.** We call  $q \in L_{loc}^\infty(X)$  a *quadratic form* if it satisfies

$$q(x * y) + q(x * y^-) = 2q(x) + 2q(y)$$

for all  $x, y \in X$ .

Putting  $x = y = e$  gives  $q(e) = 0$ ; and  $x = y, x = y^-$  successively gives

$$q(x * x) + q(x * x^-) = 4q(x)$$

$$q(x * x^-) + q(x * x) = 2q(x) + 2q(x^-)$$

so that  $q(x^-) = q(x)$ .

**THEOREM 1.7.** *Every quadratic form  $q$  continuous at  $e$  is continuous everywhere.*

**PROOF.** Let  $(V_\alpha)$  be a base of symmetric compact neighbourhoods at  $e$ , and write  $k_\alpha = m(V_\alpha)^{-1}1_{V_\alpha}$ . For  $\varepsilon > 0$  choose  $\alpha_0$  such that for  $\alpha \geq \alpha_0$ ,  $|q(y)| < \varepsilon$  whenever  $y \in V_\alpha$ .

Now for each  $x \in X$ , using the symmetry of  $k_\alpha$ , we have

$$\begin{aligned} \left| \int (q(x * y) - q(x))k_\alpha(y) dm(y) \right| &= \frac{1}{2} \left| \int (q(x * y^-) - q(x))k_\alpha(y) dm(y) \right. \\ &\quad \left. + \int (q(x * y) - q(x))k_\alpha(y) dm(y) \right| \\ &= \left| \int q(y)k_\alpha(y) dm(y) \right| \\ &< \varepsilon. \end{aligned}$$

It follows that

$$\begin{aligned} |k_\alpha * q(x^-) - q(x^-)| &= \left| \int q(y^- * x^-)k_\alpha(y) dm(y) - q(x) \int k_\alpha(y) dm(y) \right| \\ &= \left| \int (q(x * y) - q(x))k_\alpha(y) dm(y) \right| < \varepsilon \end{aligned}$$

so that  $k_\alpha * q \rightarrow q$  uniformly on  $X$ . Appealing to Proposition 1.2 gives the continuity of  $q$ . ■

Jewett [6], Section 11.1 gives a definition of a positive definite function, which is assumed from the outset to be continuous. We note that continuity is not an essential part of the definition.

**DEFINITION 1.8.** We call  $\phi \in L^\infty_{\text{loc}}(X)$  *positive definite* if for all choices of  $n \in \mathbf{N}$ ,  $c_1, c_2, \dots, c_n \in \mathbf{C}$  and  $x_1, x_2, \dots, x_n \in X$

$$\sum \sum c_i \bar{c}_j \phi(x_i * x_j^-) \geq 0.$$

Standard arguments (see, for example, Berg, Christensen and Ressel [1], Chapter 3, Section 1) give that every positive definite function  $\phi$  satisfies

$$\phi(x * x^-) \geq 0, \phi(e) \geq 0, \phi(x^-) = \overline{\phi(x)}, |\phi(x)|^2 \leq \phi(e)\phi(x * x^-)$$

and, when  $\phi$  is bounded,  $|\phi(x)| \leq \phi(e)$  for all  $x \in X$ .

It turns out that the continuity of a positive definite function is determined by its continuity at  $e$ ; this will follow as a corollary to Theorem 1.10 below.

**DEFINITION 1.9.** We call  $\psi \in L^\infty_{\text{loc}}(X)$  *negative definite* if  $\psi(e) \geq 0$ ,  $\psi(x^-) = \overline{\psi(x)}$  for all  $x \in X$ , and for every  $n \in \mathbf{N}$ , every choice of  $c_1, c_2, \dots, c_n \in \mathbf{C}$  with  $\sum c_i = 0$ , and every choice of  $x_1, x_2, \dots, x_n \in X$

$$\sum \sum c_i \bar{c}_j \psi(x_i * x_j^-) \leq 0.$$

It follows from Berg, Christensen and Ressel [1], Lemma 3.2.1 that  $\psi \in L^\infty_{\text{loc}}(X)$  is negative definite if and only if for every choice of  $c_1, c_2, \dots, c_n \in \mathbf{C}$  and  $x_1, x_2, \dots, x_n \in X$

$$\sum \sum c_i \bar{c}_j (\psi(x_i) + \overline{\psi(x_j)} - \psi(x_i * x_j^-)) \geq 0.$$

Clearly  $\psi(x * x^-) \in \mathbf{R}$ ,  $\psi(x) + \psi(x^-) \geq \psi(x * x^-)$ ,  $\psi - \psi(e)$  is negative definite and, if  $\phi$  is positive definite,  $\phi(e) - \phi$  is negative definite (see Berg, Christensen and Ressel [1], Chapter 3, Section 1).

**THEOREM 1.10.** *A negative definite function continuous at  $e$  is continuous everywhere.*

**PROOF.** Since  $\psi - \psi(e)$  is negative definite we can without loss of generality assume that  $\psi(e) = 0$ . Taking  $x_1 = x$  and  $x_2 = y$  above we see that the matrix

$$\begin{pmatrix} \psi(x) + \overline{\psi(x)} - \psi(x * x^-) & \psi(x) + \overline{\psi(y)} - \psi(x * y^-) \\ \psi(y) + \overline{\psi(x)} - \psi(y * x^-) & \psi(y) + \overline{\psi(y)} - \psi(y * y^-) \end{pmatrix}$$

is positive definite, and hence has nonnegative determinant. This gives

$$|\psi(x) + \overline{\psi(y)} - \psi(x * y^-)|^2 \leq (\psi(x) + \overline{\psi(x)} - \psi(x * x^-))(\psi(y) + \overline{\psi(y)} - \psi(y * y^-))$$

for all  $x, y \in X$ . Choose  $x_0 \in X$ ,  $U$  a relatively compact neighbourhood of  $x_0$ ,  $\varepsilon > 0$ ,  $V$  a symmetric relatively compact neighbourhood of  $e$  such that  $|\psi(y)| < \varepsilon$  for all  $y \in V$ , and  $W$  a neighbourhood of  $e$  satisfying  $W * W^- \subset V$ .

Since  $\psi$  is locally bounded there is a constant  $K$  such that

$$|\psi(x) + \overline{\psi(x)} - \psi(x * x^-)| \leq K$$

for all  $x \in U$ . Hence for  $x \in U$ ,  $y \in W$

$$|\psi(x) - \psi(x * y^-)| - |\psi(y)| \leq |\psi(x) + \overline{\psi(y)} - \psi(x * y^-)| \leq K^{1/2}(3\varepsilon)^{1/2}$$

and

$$|\psi(x) - \psi(x * y^-)| \leq (3K\varepsilon)^{1/2} + \varepsilon$$

Thus  $\psi$  is locally uniformly continuous at  $x_0$ , and the result follows. ■

**COROLLARY 1.11.** *Every positive definite function that is continuous at  $e$  is continuous everywhere.*

**PROOF.** We need only observe that if  $\phi$  is positive definite then  $\phi(e) - \phi$  is negative definite, and then apply Theorem 1.10. ■

**2. The Lévy-Khinchin representation for negative definite functions on commutative hypergroups.** Negative definite functions allow in many cases a representation in terms of a *local part* and an integral term. This holds for instance on locally compact abelian groups (see Heyer [5], Theorem 5.6.19) and on discrete abelian semigroups (see Berg, Christensen and Ressel [1], Theorem 4.3.19). Here we establish such a representation for commutative hypergroups, giving two different versions; the first for real valued lower bounded continuous negative definite functions, and the second for complex negative definite functions with lower bounded real part in the case that the hypergroup is discrete.

Throughout this section assume  $X$  to be a commutative hypergroup and write  $S = M_c^1(X)$ . Then  $(S, *)$  is an abelian semigroup with neutral element  $\varepsilon_e$  and a natural involution  $s \rightarrow s^*$ . We exploit this by making use of some results from harmonic analysis on such semigroups.

We shall consider  $X$  as subset of  $S$  via the mapping  $x \rightarrow \varepsilon_x$ . A function  $F: S \rightarrow \mathbf{C}$  will be called *adapted* if its restriction  $f := F|_X$  is locally bounded and measurable, and  $F(s) = \int f ds$  for all  $s \in S$ . If for example  $f: X \rightarrow \mathbf{C}$  is continuous then  $F(s) := \int f ds$  is adapted.

A function  $F: S \rightarrow \mathbf{C}$  is called *locally weakly continuous* if for every net  $(s_\lambda)$  in  $S$  of measures having a common compact support such that  $s_\lambda \rightarrow s$  weakly (being equivalent in this case to vague convergence) it follows that  $F(s_\lambda) \rightarrow F(s)$ . It is easy to see that if  $F$  is locally weakly continuous and affine then it must be adapted. Indeed  $f := F|_X$  is continuous and  $F(s) = \int f ds$  for each finitely supported  $s \in S$ . For general  $s \in S$  there is a net  $(s_\lambda)$  in  $S$  of finitely supported measures converging to  $s$  with the property that  $\text{supp } s_\lambda \subset \text{supp } s$  for all  $\lambda$ . By the local weak continuity of  $F$  we have  $F(s) = \lim_\lambda F(s_\lambda) = \lim_\lambda \int f ds_\lambda = \int f ds$ .

LEMMA 2.1. *Let  $\phi$  (respectively  $\psi$ ) be a continuous positive (respectively negative) definite function on  $X$  and define  $\Phi, \Psi: S \rightarrow \mathbf{C}$  by  $\Phi(s) := \int \phi ds$ ,  $\Psi(s) := \int \psi ds$  respectively. Then  $\Phi$  is positive definite and  $\Psi$  is negative definite on  $S$ .*

PROOF. First consider  $\{s_1, s_2, \dots, s_n\} \subset S$  with finite support, that is  $s_i = \sum_k p_{i,k} \varepsilon_{x_{i,k}}$  with  $p_{i,k} \geq 0$  and  $\sum_k p_{i,k} = 1$  for all  $i$ . Then for  $\{c_1, c_2, \dots, c_n\} \subset \mathbf{C}$  we have

$$\sum_{i,j} c_i \bar{c}_j \Phi(s_i * s_j^*) = \sum_{(i,k),(j,\ell)} c_i p_{i,k} \bar{c}_j p_{j,\ell} \phi(x_{i,k} * x_{j,\ell}^-) \geq 0$$

using the positive definiteness of  $\phi$ .

If in addition  $\sum_i c_i = 0$  we also have  $\sum_{i,k} c_i p_{i,k} = 0$ , and  $\psi$  being negative definite guarantees that

$$\sum_{i,j} c_i \bar{c}_j \Psi(s_i * s_j^*) \leq 0.$$

The general case follows by approximation with measures of finite support. ■

As a first application we give a short proof of the following result, established in Voit [12], Corollary 3.6.

PROPOSITION 2.2. *For every bounded continuous negative definite function  $\psi: X \rightarrow \mathbf{C}$  there exists  $c \in \mathbb{R}$  such that  $c - \psi$  is positive definite.*

PROOF. The function  $\Psi: S \rightarrow \mathbf{C}$  introduced in Lemma 2.1 being likewise bounded we may apply Berg, Christensen and Ressel [1], Proposition 4.3.15 to find  $c \in \mathbb{R}$  and a bounded positive definite function  $\Phi: S \rightarrow \mathbf{C}$  such that  $\Psi = c - \Phi$ , and then  $c - \psi = \Phi|_X$  is positive definite (on the hypergroup). ■

In Section 1 we introduced the notion of a quadratic form on a hypergroup. These functions play a crucial role in the decomposition of negative definite functions.

PROPOSITION 2.3. *Nonnegative quadratic forms are negative definite.*

PROOF. Let  $q: X \rightarrow \mathbf{R}_+$  be a quadratic form and, for each  $s \in S$ , put  $Q(s) := \int q ds$ . Applying Jewett [6], Lemma 3.1E we obtain for  $s, t \in S$

$$\begin{aligned} Q(s * t) + Q(s * t^*) &= \int \int q(x * y) ds(x) dt(y) + \int \int q(x * y^-) ds(x) dt(y) \\ &= \int \int 2[q(x) + q(y)] ds(x) dt(y) = 2[Q(s) + Q(t)] \end{aligned}$$

showing that  $Q$  is a nonnegative quadratic form on  $S$ . By Berg, Christensen and Ressel [1], Theorem 4.3.9,  $Q$  is negative definite, and hence so is  $q = Q|_X$ . ■

A proof of the above result, assuming continuity of  $q$ , was given in Lasser [8], Proposition 1.10.

Our first Lévy-Khinchin-type decomposition reads as follows:

**THEOREM 2.4.** *Let  $\psi : X \rightarrow \mathbf{R}$  be a lower bounded continuous negative definite function on the commutative hypergroup  $X$ . Then there exist a nonnegative quadratic form  $q$  on  $X$  and a Radon measure  $\mu$  on  $\hat{X} \setminus \{1\}$  such that for all  $x \in X$*

$$\psi(x) = \psi(e) + q(x) + \int_{\hat{X}} (1 - \operatorname{Re} \chi(x)) d\mu(\chi).$$

Both  $q$  and the integral part  $\psi - \psi(e) - q$  are continuous negative definite functions. The pair  $(q, \mu)$  is uniquely determined by  $\psi$ , with  $q$  being given by

$$q(x) = \lim_{n \rightarrow \infty} \frac{\psi(x^{*n})}{n^2} + \lim_{n \rightarrow \infty} \frac{\psi((x * x^-)^{*n})}{2n}.$$

**PROOF.** Put  $\Psi(s) := \int \psi ds$  for all  $s \in S$ . Since  $\Psi$  is negative definite, real and bounded below it follows from Berg, Christensen and Ressel [1], Corollary 4.3.2 that  $\psi \geq \psi(e)$ . But  $\psi - \psi(e)$  is negative definite so, without loss of generality, we can assume  $\psi$  to be nonnegative with  $\psi(e) = 0$ .

We introduce  $\Delta_r \Psi$  defined by

$$(\Delta_r \Psi)(s) := \frac{1}{2} [\Psi(s * t) + \Psi(s * t^*)] - \Psi(s)$$

for every  $t \in S$ . It is easy to see that  $\Delta_r \Psi$  is locally weakly continuous and affine, hence adapted. Furthermore by Berg, Christensen and Ressel [1], Proposition 4.3.11,  $\Delta_r \Psi$  is bounded and positive definite on  $S$ . Therefore appealing to Bochner’s theorem for hypergroups (Jewett [6], Theorem 12.3B)

$$(\Delta_r \Psi)(s) = \int_{\hat{X}} \rho_\chi(s) d\sigma_r(\chi)$$

for some  $\sigma_r \in M_+^b(\hat{X})$ , where we denote the canonical extension of  $\chi \in \hat{X}$  to a function on  $S$  by  $\rho_\chi$  where  $\rho_\chi(s) := \int_X \chi ds$ , this extension being an affine character on  $S$ . A simple calculation yields

$$\begin{aligned} -\Delta_r \Delta_t \Psi(s) &= \int_{\hat{X}} \rho_\chi(s) [1 - \operatorname{Re} \rho_\chi(r)] d\sigma_r(\chi) \\ &= -\Delta_t \Delta_r \Psi(s) = \int_{\hat{X}} \rho_\chi(s) [1 - \operatorname{Re} \rho_\chi(t)] d\sigma_r(\chi) \end{aligned}$$

for  $r, t, s \in S$ , implying

$$[1 - \operatorname{Re} \rho_\chi(r)] d\sigma_r(\chi) = [1 - \operatorname{Re} \rho_\chi(t)] d\sigma_r(\chi)$$

by the uniqueness of the Fourier transform (Jewett [6], Theorem 12.2A). Noting that the  $\{\chi \in \hat{X} \mid \operatorname{Re} \chi(t) < 1\}$  are open sets in  $\hat{X}$  with union (over  $t$ ) given by  $\hat{X} \setminus \{1\}$ , we find a Radon measure  $\mu$  on  $\hat{X} \setminus \{1\}$  such that for each  $t \in S$

$$[1 - \operatorname{Re} \rho_\chi(t)] d\mu(\chi) = d\sigma_t(\chi) \text{ on } \hat{X} \setminus \{1\}.$$

The set  $\hat{S}$  of all (bounded semigroup) characters on the (discrete) semigroup  $S$  is a compact Hausdorff space with respect to the topology of pointwise convergence. The canonical mapping  $\zeta: \hat{X} \rightarrow \hat{S}, \zeta(\chi) := \rho_\chi$  is continuous, and obviously  $\Delta_t \Psi$  is the Laplace transform of  $\sigma_t^\zeta$  (the image measure of  $\sigma_t$  under  $\zeta$ ) for each  $t \in S$ , showing  $\mu^\zeta$  to be the Lévy measure for the negative definite function  $\Psi$ , cf. Berg, Christensen and Ressel [1], Lemma 4.3.12 and Definition 4.3.13. By Berg, Christensen and Ressel [1], Theorem 4.3.19 (Lévy functions always exist, see below), taking into account that  $\Psi$  is real valued, there exists a nonnegative quadratic form  $Q$  on  $S$  such that for all  $s \in S$

$$\begin{aligned} \Psi(s) &= Q(s) + \int_{\hat{S} \setminus \{1\}} (1 - \operatorname{Re} \rho(s)) d\mu^\zeta(\rho) \\ &= Q(s) + \int_{\hat{X} \setminus \{1\}} (1 - \operatorname{Re} \rho_\chi(s)) d\mu(\chi). \end{aligned}$$

The function  $\Psi_0(s) := \int_{\hat{X} \setminus \{1\}} (1 - \operatorname{Re} \rho_\chi(s)) d\mu(\chi)$  is adapted by Fubini’s theorem,  $(x, \chi) \rightarrow \chi(x)$  being continuous. Since  $\Psi$  is adapted it follows that  $Q$  is adapted as well. Approximating  $\mu$  by measures with compact support (or by applying Berg, Christensen and Ressel [1], Theorem 2.1.12),  $\psi_0 := \Psi_0|_X$  is lower semicontinuous. In particular it is measurable and hence so is  $q := Q|_X$ . We have thus reached the decomposition  $\psi = q + \psi_0$  in which all three functions are nonnegative and negative definite. Since  $\psi$  is continuous,  $q$  and  $\psi_0$  are continuous at  $e$ . An application of Theorem 1.10 finishes the main part of the proof.

The uniqueness of  $q$  and  $\mu$  as well as the formula for  $q$  follow immediately from the corresponding statement for semigroups; see Berg, Christensen and Ressel [1], Theorem 4.3.19. ■

REMARK 2.5. Under the assumption of a further technical condition on the hypergroup itself, called “property ( $F$ )”, this theorem was proved in Lasser [8], Theorem 3.9. The assumptions on  $\psi$  in that paper also differ in so far as the author assumes the Lévy measure to be symmetric, a property not easily expressible directly in terms of the given function  $\psi$ . However under this assumption the general Lévy formula for semigroups can be used as above to show that  $\operatorname{Im} \psi$  is then necessarily a continuous homomorphism, that is,  $\operatorname{Im} \psi(x * y^-) = \operatorname{Im} \psi(x) - \operatorname{Im} \psi(y)$  for  $x, y \in X$ . The real part has the representation as stated, and so property ( $F$ ) can be dispensed with.

In the remainder of this section we shall assume  $X$  to be a discrete commutative hypergroup. The semigroup  $S$  now consists of all probability measures on  $X$  with finite support. We can then prove a Lévy-type decomposition for arbitrary negative definite functions with lower bounded real part. The corresponding theorem for semigroups was shown in Berg, Christensen and Ressel [1], Theorem 4.3.19 under the condition that the

semigroup in question has a so called Lévy function. That this is always true was shown in 1986 by Buchwalter [3], who proved that there is a function  $L: S \times \hat{S} \rightarrow \mathbf{R}$  with the following properties:

- (i) For every  $\rho \in \hat{S}$  the function  $L(\cdot, \rho)$  is  $*$ -additive on  $S$ , that is,  $L(s * t^*, \rho) = L(s, \rho) - L(t, \rho)$  for  $s, t \in S$ .
- (ii) For every  $s \in S$  the function  $L(S, \cdot)$  is continuous on  $\hat{S}$  and  $L(s, \bar{\rho}) = -L(s, \rho)$ .
- (iii) For every measure  $\lambda \in M_+(\hat{S} \setminus \{1\})$  such that  $\int (1 - \text{Re } \rho(s)) d\lambda(\rho) < \infty$  for all  $s \in S$  it follows that

$$\int |1 - \rho(s) + iL(s, \rho)| d\lambda(\rho) < \infty.$$

(In fact there is even a fourth property of  $L$  but we do not need this here.)

Let  $\tilde{S} \subset \hat{S}$  denote the (closed) subset of all affine semigroup characters on  $S$ . We want to modify  $L$  to a function  $L_0: S \times \tilde{S} \rightarrow \mathbf{R}$  in such a way that  $L_0(\cdot, \rho)$  is affine for each  $\rho \in \tilde{S}$ , noting that  $S$  is in fact a so-called convex semigroup, cf. Ressel [9]. We define  $L_0$  on  $S \times \tilde{S}$  by

$$L_0(s, \rho) := \int L(\varepsilon_x, \rho) ds(x).$$

Obviously  $L_0(\cdot, \rho)$  is affine. Furthermore

$$\begin{aligned} L_0(s * t^*, \rho) &= \iint L(\varepsilon_x * \varepsilon_y, \rho) ds(x) dt(y) \\ &= \iint (L(\varepsilon_x, \rho) - L(\varepsilon_y, \rho)) ds(x) dt(y) \\ &= L_0(s, \rho) - L_0(t, \rho) \end{aligned}$$

and  $L_0(s, \cdot)$  is still continuous and satisfies  $L_0(s, \bar{\rho}) = -L_0(s, \rho)$ . Finally if  $\lambda \in M_+(\tilde{S} \setminus \{1\})$  satisfies  $\int (1 - \text{Re } \rho(s)) d\lambda(\rho) < \infty$  for all  $s \in S$  we see that

$$\int |1 - \rho(s) + iL_0(s, \rho)| d\lambda(\rho) \leq \iint |1 - \rho(\varepsilon_x) + iL(\varepsilon_x, \rho)| d\lambda(\rho) ds(x)$$

is finite since  $s$  has finite support.

**THEOREM 2.6.** *Let  $\psi: X \rightarrow \mathbf{C}$  be a negative definite function on the discrete commutative hypergroup  $X$  with a lower bounded real part. Then  $\psi$  has the representation*

$$\psi(x) = \psi(e) + ih(x) + q(x) + \int_{\hat{X} \setminus \{1\}} (1 - \chi(x) + iL_0(x, \chi)) d\mu(\chi)$$

for all  $x \in X$ , where  $h: X \rightarrow \mathbf{R}$  is a homomorphism,  $q: X \rightarrow \mathbf{R}_+$  is a nonnegative quadratic form, and  $\mu$  is a measure on  $\hat{X} \setminus \{1\}$  such that the integrals on the right hand side all exist. Given  $L_0$  the triple  $(h, q, \mu)$  is uniquely determined,  $q$  and  $\mu$  not depending on  $L_0$ .

**PROOF.** As before we define  $(\Delta_t \Psi)(s) := \frac{1}{2}[\Psi(s * t) + \Psi(s * t^*)] - \Psi(s)$  and observe that  $\Delta_t \Psi$  is affine. By Ressel [9], Theorem 1 the unique measure  $\sigma_t$  representing  $\Delta_t \Psi$  is concentrated on  $\tilde{S}$ , and  $\tilde{S}$  of course is in one-to-one correspondence with  $\hat{X}$  via  $\rho_\chi \cong \chi$ . The unique measure  $\mu$  on  $\hat{S} \setminus \{1\}$  such that for all  $t \in S$

$$[1 - \text{Re } \rho(t)] d\mu(\rho) = d\sigma_t(\rho) \text{ on } \hat{S} \setminus \{1\}$$

is therefore concentrated on  $\tilde{S} \setminus \{1\}$ , that is, on  $\hat{X} \setminus \{1\}$ . By Berg, Christensen and Ressel [1], Theorem 4.3.19,  $\Psi$  has the representation

$$\Psi(s) = \Psi(\varepsilon_e) + iH(s) + Q(s) + \int_{\tilde{S} \setminus \{1\}} (1 - \rho(s) + iL_0(s, \rho)) d\mu(\rho)$$

where  $H: S \rightarrow \mathbf{R}$  is  $*$ -additive, and  $Q$  is a nonnegative quadratic form on  $S$ . From  $\text{Re } \Psi(s) = \Psi(\varepsilon_e) + Q(s) + \int (1 - \text{Re } \rho(s)) d\mu(\rho)$  we see that  $Q$  is affine, and this in turn implies that  $H$  is affine. Putting  $h := H|_X$  and  $q := Q|_X$  we get the required representation of  $\psi$ . ■

**3. Completely monotone and completely alternating functions on hypergroups.** Let  $X$  denote a hermitian hypergroup which, by Jewett [6], Theorem 9.1A, must be commutative. For real valued  $\phi \in L^\infty_{\text{loc}}(X)$  and  $a \in X$  we define  $\nabla_a \phi: X \rightarrow \mathbf{R}$  by

$$(\nabla_a \phi)(x) := \phi(x) - \phi(x * a).$$

We call  $\phi$  *completely monotone* if  $\phi \geq 0$  and

$$\nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_n} \phi \geq 0$$

for all  $n \in \mathbf{N}$  and  $\{a_1, a_2, \dots, a_n\} \subset X$ . The function  $\phi$  is said to be *completely alternating* if

$$\nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_n} \phi \leq 0$$

for all  $n \in \mathbf{N}$  and  $\{a_1, a_2, \dots, a_n\} \subset X$ . With  $\Delta_a \psi := -\nabla_a \psi$  we see from

$$\nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_n} (\Delta_a \psi) = -\nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_n} \nabla_a \psi$$

that  $\psi \in L^\infty_{\text{loc}}(X)$  is completely alternating if and only if  $\Delta_a \psi$  is completely monotone for each  $a \in X$ .

In the following  $\hat{X}_+$  will denote the set of all nonnegative continuous characters on  $X$ , the elements of which of course take their values in  $[0, 1]$ .

LEMMA 3.1. *If  $\phi$  is completely monotone on  $X$  then  $\Phi: S \rightarrow \mathbf{R}$  defined by  $\Phi(s) := \int \phi ds$  is completely monotone on the semigroup  $S$ . A similar statement holds for completely alternating functions.*

PROOF. Applying Jewett [6], Lemma 3.1E we obtain for  $s, t \in S$

$$(\nabla_t \Phi)(s) := \Phi(s) - \Phi(s * t) = \int \int [\phi(x) - \phi(x * y)] ds(x) dt(y)$$

and by straightforward extension

$$\nabla_{t_1} \nabla_{t_2} \cdots \nabla_{t_n} \Phi(s) = \int \int \cdots \int \nabla_{y_1} \nabla_{y_2} \cdots \nabla_{y_n} \phi(x) dt_1(y_1) \cdots dt_n(y_n) ds(x)$$

implying immediately both assertions. ■

**THEOREM 3.2.**

- (i) A continuous function  $\phi : X \rightarrow \mathbf{R}$  is completely monotone if and only if there exists  $\mu \in M_+^b(\hat{X}_+)$  such that for all  $x \in X$ ,  $\phi(x) = \int \chi(x) d\mu(\chi)$ .
- (ii) A continuous function  $\psi : X \rightarrow \mathbf{R}$  is completely alternating if and only if there exists an additive continuous function  $h : X \rightarrow \mathbf{R}_+$  and  $\mu \in M_+(\hat{X}_+ \setminus \{1\})$  such that for all  $x \in X$

$$\psi(x) = \psi(e) + h(x) + \int (1 - \chi(x)) d\mu(\chi).$$

In both cases the representations are unique.

**PROOF.** Immediate inspection shows that it is sufficient to prove the necessity of the respective representations.

(i) By Berg, Christensen and Ressel [1], Theorem 4.6.5 the function  $\Phi$  has the unique decomposition

$$\Phi(s) = \int_{\hat{S}_+} \rho(s) d\tilde{\mu}(\rho)$$

where  $\tilde{\mu} \in M_+(\hat{S}_+)$ . In particular  $\Phi$  is positive definite, and so is therefore  $\phi$ , implying the existence of a unique measure  $\tilde{\mu} \in M_+^b(\hat{X})$  such that for all  $x \in X$

$$\phi(x) = \int \chi(x) d\mu(\chi).$$

Using again the canonical continuous embedding  $\zeta : \hat{X} \rightarrow \hat{S}$ ,  $\zeta(\chi) := \rho_\chi$ , and the continuity of  $(\chi, x) \rightarrow \chi(x)$ , we obtain

$$\begin{aligned} (\dagger) \quad \Phi(s) &= \int_X \int_{\hat{X}} \chi(x) d\mu(\chi) ds(x) = \int_{\hat{X}} \int_X \chi(x) ds(x) d\mu(\chi) \\ &= \int_{\hat{X}} \rho_\chi(s) d\mu(\chi) = \int_{\hat{S}} \rho(s) d\mu^\zeta(\rho) \end{aligned}$$

for all  $s \in S$ . Hence by uniqueness  $\mu^\zeta = \tilde{\mu}$  is concentrated on  $\hat{S}_+$ .

We now observe that because of the equivalence  $\chi \geq 0 \Leftrightarrow \zeta(\chi) \geq 0$  the canonical embedding  $\zeta : \hat{X} \rightarrow \hat{S}$  restricts to  $\zeta_+ : \hat{X}_+ \rightarrow \hat{S}_+$ . It follows from  $(\dagger)$  that  $\mu$  is concentrated on  $\hat{X}_+$ .

- (ii) It follows from the definition that  $\psi$  is lower bounded by  $\psi(e)$ . The function  $\Psi$ , being completely alternating by Lemma 3.1, is negative definite, cf. Berg, Christensen and Ressel [1], Theorem 4.6.7. By Theorem 2.4 we have the representation

$$\psi(x) = \psi(e) + h(x) + \int_{\hat{X} \setminus \{1\}} (1 - \chi(x)) d\mu(\chi)$$

where  $\mu \in M_+(\hat{X} \setminus \{1\})$  and  $h : X \rightarrow \mathbf{R}_+$  is a continuous quadratic form, the latter property reducing to  $h$  being additive since  $X$  is assumed to be hermitian. It remains to show that  $\mu$  is concentrated on  $\hat{X}_+ \setminus \{1\}$ . From

$$\Psi(s) = \Psi(\varepsilon_e) + H(s) + \int_{\hat{X} \setminus \{1\}} (1 - \rho_\chi(s)) d\mu(\chi)$$

we infer that  $\mu^\zeta$  is the representing measure of  $\Psi$  on the (discrete) semigroup  $S$ . As  $\Psi$  is completely alternating  $\mu^\zeta$  must be concentrated on  $\hat{S}_+ \setminus \{1\}$  implying as before that  $\mu$  is concentrated on  $\hat{X}_+ \setminus \{1\}$ . ■

In the locally compact abelian group case the absence of nontrivial nonnegative continuous characters means in view of Theorem 3.2 that the only completely monotone functions are those that are constant. In contrast there are (hermitian) hypergroups that admit a range of nonnegative continuous characters. We present here two such hypergroups that are discrete, nondiscrete respectively.

EXAMPLE 3.3. The following hypergroup structure on  $\mathbf{N}_0$  was introduced by Lasser [7], Section 2. Let  $(a_n), (b_n)$  and  $(c_n)$  be real sequences indexed by  $\mathbf{N}$  with  $a_n + b_n + c_n = 1, a_n, c_n > 0$  and  $b_n \geq 0$ . Define a sequence  $(P_n)$  of orthogonal polynomials by

$$P_0(x) = 1, P_1(x) = x, P_1(x)P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$$

for all  $x \in \mathbb{R}, n \in \mathbb{N}$ . If the coefficients  $g(m, n, k)$  in the linearisation formula

$$P_m P_n = \sum_{k=|m-n|}^{m+n} g(m, n, k) P_k$$

are nonnegative then we can define a convolution structure on  $\mathbf{N}_0$  via

$$\varepsilon_m * \varepsilon_n = \sum_{k=|m-n|}^{m+n} g(m, n, k) \varepsilon_k.$$

With this convolution,  $\varepsilon_0$  as unit and the identity involution,  $\mathbf{N}_0$  becomes a discrete hermitian hypergroup.

Lasser [7], Proposition 4 showed that the continuous semicharacters of  $\mathbf{N}_0$  are given by  $\alpha_x: n \rightarrow P_n(x)$  where  $x \in \mathbb{R}$ , and

$$\hat{\mathbf{N}}_0 = \{ \alpha_x: x \in \mathbb{R} \text{ and } \alpha_x \text{ is bounded} \}.$$

Under the homeomorphism between  $\mathbf{R}$  and the continuous semicharacters on  $\mathbf{N}_0$  given by  $x \rightarrow \alpha_x$ , the Plancherel measure  $\pi$  on  $\hat{\mathbf{N}}_0$  can be identified with the orthogonality measure of  $(P_n)$ , and  $\text{supp } \pi \subset [-1, 1]$ .

Write  $x_0 := \text{sup}(\text{supp } \pi)$ . Now the orthogonality measure of  $(P_n)$  is supported by the smallest closed interval containing all the zeros of  $(P_n)$ . Since the zeros of  $P_{n-1}$  separate those of  $P_n$  for  $n \geq 2$ , and  $P_n(1) = 1$  for all  $n \in \mathbf{N}_0$ , it is easily seen that  $\alpha_x > 0$  if and only if  $x \geq x_0$ . It is known that there are such hypergroup structures on  $\mathbf{N}_0$  for which  $x_0 < 1$ ; for example (cf. Voit [11], Example 2.18) take  $a > 2, c_n = 1/a, a_n = 1 - c_n$  and  $b_n = 0$  for all  $n \in \mathbf{N}$ . In this case  $x_0 = 2a^{-1}(a - 1)^{1/2}$ , and since  $\hat{\mathbf{N}}_0 \simeq [-1, 1]$  it follows that the positive characters in  $\hat{\mathbf{N}}_0$  can be identified with  $[x_0, 1]$ . In view of Theorem 3.2 such hypergroups admit a rich supply of completely monotone and completely alternating functions.

EXAMPLE 3.4. An early example of a hermitian hypergroup structure on  $\mathbf{R}_+$  was given by Naimark; see Jewett [6], Example 9.5. Here  $\varepsilon_0$  is the identity, and for  $x, y > 0$

$$\varepsilon_x * \varepsilon_y := (2 \sinh x \sinh y)^{-1} \int_{|x-y|}^{x+y} (\sinh t) \varepsilon_t dt.$$

Jewett showed that

$$(\mathbf{R}_+)^{\wedge} = \{ \chi_c : -1 \leq c < \infty \}$$

where  $\chi_c(x) := \sin bx / (b \sinh x)$  and  $c = b^2$  (the identity character is just  $\chi_{-1}$ ), and that

$$\text{supp } \pi = \{ \chi_c : 0 \leq c < \infty \}.$$

It is easy to see that  $\chi_0(x) = x / \sinh x$  and for  $-1 < c < 0$ ,  $\beta = \sqrt{|c|}$ ,

$$\chi_c(x) = \frac{\sinh \beta x}{\beta \sinh x}$$

so that the  $\chi_c$  are positive for all  $c \in [-1, 0]$ . All of these functions are completely monotone.

#### REFERENCES

1. Christian Berg, Jens Peter Reus Christensen and Paul Ressel, *Harmonic analysis on semigroups*. Graduate Texts in Mathematics, **100**, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1984.
2. Walter R. Bloom and Herbert Heyer, *Characterisation of potential kernels of transient convolution semigroups on a commutative hypergroup*. Probability measures on groups IX (Proc. Conf., Oberwolfach Math. Res. Inst., Oberwolfach 1988), Lecture Notes in Math., **1379**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1989, 21–35.
3. Henri Buchwalter, *Les fonctions de Lévy existent!*, Math. Ann. **274**(1986), 31–34.
4. Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis, vol II*. Die Grundlehren der mathematischen Wissenschaften, **152**, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
5. Herbert Heyer, *Probability measures on locally compact groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, **94**, Springer-Verlag, Berlin, Heidelberg, New York, 1977.
6. Robert I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math. **18**(1975), 1–101.
7. Rupert Lasser, *Orthogonal polynomials and hypergroups*, Rend. Mat. (Series VII)**3**(1983), 185–209.
8. ———, *Convolution semigroups on hypergroups*, Pacific J. Math. **127**(1987), 353–371.
9. Paul Ressel, *Integral representations on convex semigroups*, Math. Scand. **61**(1987), 93–111.
10. René Spector, *Mesures invariantes sur les hypergroupes*, Trans. Amer. Math. Soc. **239**(1978), 147–165.
11. Michael Voit, *Positive characters on commutative hypergroups and some applications*, Math. Z. **198**(1988), 405–421.
12. ———, *Negative definite functions on commutative hypergroups*. Probability measures on groups IX (Proc. Conf., Oberwolfach Math. Res. Inst., Oberwolfach 1988), Lecture Notes in Math., **1379**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1989, 376–388.

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