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RESEARCH ARTICLE

Precise large deviations of the net loss process in a non-standard two-dimensional risk model

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Abstract
This paper investigates the precise large deviations of the net loss process in a two-dimensional risk model with consistently varying tails and dependence structures, and gives some asymptotic formulas which hold uniformly for all x varying in t-intervals. The study is among the initial efforts to analyze potential risk via large deviation results for the net loss process of the two-dimensional risk model, and can provide a novel insight to assess the operation risk in a long run by fully considering the premium income factors of the insurance company.

1. Introduction

Consider an insurance risk model in which an insurer simultaneously operates two types of claims sharing a common arrival process. When introducing this model, Chen et al. [5] made the following standard assumptions.

Assumption A_1. *The claim-size vectors $\vec{X}_i = (X_i^{(1)}, X_i^{(2)})^T$, $i \geq 1$, are a sequence of nonnegative, independent and identically distributed (i.i.d.) copies of $\vec{X} = (X^{(1)}, X^{(2)})^T$ with finite mean vector $\vec{\mu} = E\vec{X} = (EX^{(1)}, EX^{(2)})^T = (\mu_1, \mu_2)^T$ and common marginal distributions F_1 and F_2, respectively. And the claim-size vector $(X^{(1)}, X^{(2)})^T$ consists of two independent components, which means that $\{X_i^{(1)}, i \geq 1\}$ and $\{X_i^{(2)}, i \geq 1\}$ are mutually independent.*

Assumption A_2. *The claim inter-arrival times θ_i, $i \geq 1$, are a sequence of positive and i.i.d. random variables (r.v.s) with common distribution G and positive mean λ^{-1}. Then $\tau_i = \sum_{k=1}^{i} \theta_k$ are the common claim-arrival times of $\vec{X}_i$, $i \geq 1$, and constitute the common arrival process as:*

$$N(t) = \sup\{i \geq 1 : \tau_i \leq t\}, \quad t \geq 0,$$

which is the standard renewal process with a finite mean function $\lambda(t) = EN(t)$.

Assumption A_3. *The claim-size vectors $\{\vec{X}_i, i \geq 1\}$ and their inter-arrival times $\{\theta_i, i \geq 1\}$ are mutually independent.*

Let $\vec{x} = (x_1, x_2)^T$ be the insurer's initial capitals, and $\vec{c} = (c_1, c_2)^T$ be its premium income rates, then the risk process in the two-dimensional setting has the form:

$$\vec{U}(t) = \vec{x} + \vec{c}t - \sum_{i=1}^{N(t)} \vec{X}_i, \quad t \geq 0, \tag{1.1}$$

where $\vec{U}(t) = (U_1(t), U_2(t))^T$. And the processes of aggregate claims and net loss are expressed as, respectively,

$$\vec{S}(t) = \sum_{i=1}^{N(t)} \vec{X}_i = \left(\sum_{i=1}^{N(t)} X_i^{(1)}, \sum_{i=1}^{N(t)} X_i^{(2)} \right)^T, \quad t \geq 0, \tag{1.2}$$

and

$$\vec{L}(t) = \sum_{i=1}^{N(t)} \left(\vec{X}_i - \vec{c}\theta_i \right) = \left(\sum_{i=1}^{N(t)} \left(X_i^{(1)} - c_1\theta_i \right), \sum_{i=1}^{N(t)} \left(X_i^{(2)} - c_2\theta_i \right) \right)^T, \quad t \geq 0. \tag{1.3}$$

To avoid the certain ruin of the risk process (1.1), we assume that the safety load condition holds as:

$$\vec{c} > \lambda\vec{\mu}, \quad \text{or equivalently,} \quad c_i > \lambda\mu_i, i = 1, 2.$$

Taking in account the insurer's large initial capital and long operation time, the risk analysis is necessary to be implemented as the two prerequisites simultaneously tend to infinity, which matches well with the research goal of precise large deviations. The study of precise large deviations of aggregate claims in the one-dimensional risk model was initiated by Klüppelberg et al. [14], and revisited by many researchers afterwards. For example, Ng et al. [22] first extended the study of precise large deviations to the class $\mathscr{C}$, and Chen et al. [6] further extended the study to the continuous-time case and applied it to a non-standard renewal risk model. An increasing amount of scholarly attention is recently paid to the precise large deviations of aggregate claims (1.2) in the two-dimensional risk model. See, for example, [10, 11, 21, 23, 26, 27]. However, to our best knowledge, there is a dearth of large deviation results for the net loss process (1.3), which is more practical in insurance but much harder than that for the aggregate claim process. Hence in this paper, we study the asymptotic behaviors of precise large deviations for the net loss process for the two-dimensional case, which is among the initial efforts to analyze insurance risk via large deviation results for the net loss process involving the insurer's premium income factor.

Risk theory with dependence has been one of the major topics in actuarial science, and more contributions have imposed various dependence structures in investigating precise large deviations, including [12, 13, 19, 20, 24, 26, 28, 32]. Extensively used dependence structures were proposed by Wang et al. [25]. Say that r.v.s $\{\xi_i, i \geq 1\}$ are widely upper orthant dependent (WUOD), if for each $n \geq 1$, there is a positive number $g_U(n)$ such that for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n} \left\{ \xi_i > x_i \right\} \right) \leq g_U(n) \prod_{i=1}^{n} P(\xi_i > x_i).$$

Say that $\{\xi_i, i \geq 1\}$ are widely lower orthant dependent (WLOD), if for each $n \geq 1$, there is a positive number $g_L(n)$ such that for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n} \left\{ \xi_i \leq x_i \right\} \right) \leq g_L(n) \prod_{i=1}^{n} P(\xi_i \leq x_i).$$

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Further, say that $\{\xi_i, i \geq 1\}$ are widely orthant dependent (WOD), if they are both WUOD and WLOD.

Remark 1.1. The WUOD, WLOD, and WOD structures can be termed a joint name of "wide dependence", which is a more extended dependence so that it can cover the negative dependence, positive dependence and some others. See the examples of Wang et al. [25]. Recall that if $g_U(n) = g_L(n) = M$ for a constant $M > 0$ and all $n \geq 1$, then $\{\xi_i, i \geq 1\}$ are ENUOD, ENLOD, and ENOD, respectively, see [18]; while if $g_U(n) = g_L(n) = 1$ for all $n \geq 1$, then $\{\xi_i, i \geq 1\}$ are NUOD, NLOD, and NOD, respectively, see [2, 8].

For the two-dimensional case, since claim sizes $X^{(1)}$ and $X^{(2)}$ are both covered by an umbrella insurance policy, the complete independence between them was proposed mainly for the mathematical tractability rather than the practical relevance. Recently, Yang et al. [31] and Li [15] allowed $(X^{(1)}, X^{(2)})^T$ to follow the bivariate Farlie-Gumbel-Morgenstern distribution. Shen and Tian [23] imposed the dependence structure between $X^{(1)}$ and $X^{(2)}$, namely that there exists a constant $M > 0$ such that

$$\overline{F}_{1,2}(x_1, x_2) \leq M\overline{F}_1(x_1)\overline{F}_2(x_2), \tag{1.4}$$

where $\overline{F}_{1,2}(x_1, x_2) = P(X^{(1)} > x_1, X^{(2)} > x_2)$. Fu *et al.* [10] further extended the constant M to a finite positive function. Li [16, 17] introduced the strong asymptotic independence between $X^{(1)}$ and $X^{(2)}$.

Assume that in the paper the insurance claim sizes are heavy-tailed r.v.s, which can model large claims caused by severe accidents. For a proper distribution V, we denote its tail by $\overline{V}(x) = 1 - V(x)$, and its upper Matuszewska index by:

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y} \quad \text{with} \quad \overline{V}_*(y) = \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)}, \quad y > 1.$$

By definition, the following distribution classes

$$\mathscr{L} = \{V : \lim_{x\to\infty} \overline{V}(x+y)/\overline{V}(x) = 1 \ \text{ for any } y > 0\},$$

$$\mathscr{D} = \{V : \overline{V}_*(y) > 0 \ \text{ for any } y > 1\},$$

and

$$\mathscr{C} = \{V : L_V = \lim_{y\searrow 1} \overline{V}_*(y) = 1\},$$

are said to be the long-tailed class, dominatedly varying-tailed class, and consistently varying-tailed class, respectively.

More generally, we say that a distribution V on $(-\infty, \infty)$ belongs to a distribution class if $V(x)\mathbf{1}_{\{x\geq 0\}}$ belongs to the same class, where $\mathbf{1}_A$ is the indicator function of a set A. The inclusion relationship that $\mathscr{C} \subset \mathscr{L} \cap \mathscr{D}$ is proper. For more details on heavy-tailed distributions with their applications, we refer to [1, 9].

In this paper, we aim to study the asymptotic behaviors of precise large deviations of the net loss process (1.3) in the two-dimensional risk model with dependence structures. This study can provide a novel insight to analyze the potential risks by fully considering the premium income factors of the insurance company, and thus accurately assesses the insurance operation risk in a long run.

The rest part of this paper is organized as follows: we state our main results in Section 2, and prove them in Sections 3 and 3.1, respectively.

2. Main results

All limit relationships henceforth are for $t \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(t) \lesssim b(t)$ if $\limsup a(t)/b(t) \leq 1$, write $a(t) \gtrsim b(t)$ if $\liminf a(t)/b(t) \geq 1$, write $a(t) \sim b(t)$ if $a(t) \lesssim b(t)$ and $a(t) \gtrsim b(t)$, write $a(t) = o(1)b(t)$ if $\lim a(t)/b(t) = 0$, write $a(t) = O(1)b(t)$ if $\limsup a(t)/b(t) < \infty$, and write $a(t) \asymp b(t)$ if $a(t) = O(1)b(t)$ and $b(t) = O(1)a(t)$.

In the standard two-dimensional risk model, the independence assumptions among modeling components appear far too unrealistic in practice, and then considerably limits the usefulness of the existing results. Hence in the paper, we will extend or remove the involved independence assumptions, and consider a nonstandard two-dimensional model under the following dependence assumptions.

Assumption A_1^*. *The claim-size vectors $\vec{X}_i = (X_i^{(1)}, X_i^{(2)})^T$, $i \geq 1$, are a sequence of nonnegative and i.i.d. copies of $(X^{(1)}, X^{(2)})^T$ with marginal distributions F_1 and F_2, respectively, such that (1.4) holds.*

Assumption A_2^*. *The claim inter-arrival times θ_i, $i \geq 1$, are positive and WLOD r.v.s. such that for some $\epsilon > 0$,*

$$\lim_{n\to\infty} g_L(n) n^{-\epsilon} = 0. \tag{2.1}$$

Assumption A_2^{**}. *The claim inter-arrival times θ_i, $i \geq 1$, are positive and WOD r.v.s., and there exist a positive function $g(x)\bar{\downarrow}$ and some $m \geq 1$ and $\kappa \in (0,1)$ such that*

$$E\theta g(\theta) < \infty, \quad \frac{x^{m-1}}{g(x)}\bar{\downarrow}0, \quad \frac{g(x)}{x^{m-1+\kappa}}\bar{\downarrow}0, \quad \textit{and} \quad \max\{g_U(n), g_L(n)\} \leq g(n), n \geq 1, \tag{2.2}$$

where $g(x)\bar{\downarrow}$ means that the function $g(x)$ is almost decreasing (or nonincreasing), in the sense that there exists a constant $C > 0$ such that $g(x_1) \geq C g(x_2)$ for all $0 \leq x_1 < x_2 < \infty$.

Assumption A_3^*. *The claim-size vectors $\{\vec{X}_i, i \geq 1\}$ and their inter-arrival times $\{\theta_i, i \geq 1\}$ are arbitrarily dependent.*

Note that Assumptions A_1^*, A_2^*, and A_2^{**} impose dependence structures between/among the involved modeling components to relax the independence assumption, while Assumption A_3^* adopts arbitrary dependence to remove the independence assumption or certain dependence structures between claim-size vectors and their inter-arrival times. By Lemma 4.2 of [30], Assumption A_2^* gives an elementary renewal theorem for the nonstandard renewal process $\{N(t), t \geq 0\}$, namely that

$$\lambda(t) \sim \lambda t, \quad \text{as} \quad t \to \infty.$$

The main results of this paper are given below, among which the first theorem provides an asymptotic formula of precise large deviations for the net loss process (1.3) in our model with arbitrary dependence between claim-size vectors and their inter-arrival times.

Theorem 2.1. *Consider the net loss process (1.3) in the nonstandard two-dimensional risk model with Assumptions A_1^*, A_2^* and A_3^*. If $F_i \in \mathscr{C}$, $i = 1, 2$, and $\overline{G}(x) = o(1)\overline{F}_{12}(x, x)$, as $x \to \infty$, then for any $\lambda\vec{\mu} < \vec{\gamma} < \vec{\Gamma} < \vec{\infty}$, it holds uniformly for all $\vec{x} \in [\vec{\gamma}t, \vec{\Gamma}t]$ that*

$$P\left(\vec{L}(t) > \vec{x}\right) \sim (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t), \tag{2.3}$$

where $\vec{0} = (0,0)^T$, $\vec{\gamma} = (\gamma_1, \gamma_2)^T$, $\vec{\Gamma} = (\Gamma_1, \Gamma_2)^T$, and $\vec{\infty} = (\infty, \infty)^T$.

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The second theorem extends the uniformity of (2.3) to an infinite interval under slightly stronger conditions imposed on modeling components.

Theorem 2.2. *Consider the net loss process (1.3) in the nonstandard two-dimensional risk model with Assumptions* A_1^*, A_2^{**} *and* A_3. *If* $F_i \in \mathscr{C}$, $i = 1, 2$, *then for any* $\vec{\gamma} > \lambda\vec{\mu}$, *relation (2.3) still holds uniformly for all* $\vec{x} \geq \vec{\gamma}t$.

Following Theorems 2.1 and 2.2, we propose two corollaries for the one-dimensional setting, which cover the precise large deviation results for the net loss process of the nonstandard renewal risk model. Denote the net loss process of one-dimensional risk model by

$$L(t) = \sum_{i=1}^{N(t)} (X_i - c\theta_i), \quad t \geq 0, \tag{2.4}$$

where $c > 0$ is the premium income rate, and the claim sizes $\{X_i, i \geq 1\}$ are a sequence of nonnegative and i.i.d. r.v.s with finite μ and common distribution F.

Corollary 2.1. *Consider the net loss process (2.4) in the non-standard renewal risk model with Assumptions* A_2^* *and* A_3^*. *If* $F \in \mathscr{C}$, *and* $\overline{G}(x) = o(1)\overline{F}(x)$, *as* $x \to \infty$, *then for any* $\lambda\mu < \gamma < \Gamma < \infty$, *it holds uniformly for all* $x \in [\gamma t, \Gamma t]$ *that*

$$P\left(L(t) > x\right) \sim \lambda t \overline{F}(x + ct - \lambda\mu t). \tag{2.5}$$

Corollary 2.2. *Consider the net loss process (2.4) in the non-standard renewal risk model with Assumptions* A_2^{**} *and* A_3. *If* $F \in \mathscr{C}$, *then for any* $\gamma > \lambda\mu$, *relation (2.5) still holds uniformly for all* $x \geq \gamma t$.

Remark 2.1. Obviously, $c_i t$ and $\lambda\mu_i t$ in (2.3), $i = 1, 2$, (or, ct and $\lambda\mu t$ in (2.5)) capture the impacts of premium income and decentralization on the asymptotic behaviors of the precise large deviations, respectively.

3. Proofs of Theorem 2.1 and Corollary 2.1

3.1. Lemmas

The first lemma concerns the precise large deviations for sums of two-dimensional random vectors with dependent components, which results from Theorem 2.2 of [7], and will play a critical role to prove the main results. Denote the nth partial sums of random vector $\vec{\xi}_i = (\xi_i^{(1)}, \xi_i^{(2)})^T$, $i \geq 1$, by

$$\vec{S}_n = \sum_{i=1}^{n} \vec{\xi}_i = \left(\sum_{i=1}^{n} \xi_i^{(1)}, \sum_{i=1}^{n} \xi_i^{(2)} \right)^T, \quad n \geq 1.$$

Lemma 3.1. *Let* $\{\vec{\xi}_i, i \geq 1\}$ *be a sequence of real valued and independent random vectors with mean vector* $\vec{0}$ *and common marginal distributions* $V_k \in \mathscr{C}$, $k = 1, 2$. *If* $\xi_i^{(1)}$ *and* $\xi_i^{(2)}$ *satisfy (1.4) for every* $i \geq 1$, *then for any* $\vec{\gamma} > \vec{0}$, *it holds uniformly for all* $\vec{x} \geq \vec{\gamma}n$ *that*

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$$P(\vec{S}_n > \vec{x}) \sim \sum_{i=1}^{n}\sum_{j=1}^{n} P(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2)$$
$$\sim n^2 \overline{V}_1(x_1)\overline{V}_2(x_2), \quad as \quad n \to \infty.$$

The second lemma is due to Lemma 3.2(ii) of [11], which stems from Lemma 3.4 of [4], and gives an extended version with arbitrary dependence among the underlying r.v.s.

Lemma 3.2. *Let $\{\eta_i, i \geq 1\}$ be a sequence of real-valued and arbitrarily dependent r.v.s with generic r.v. η and mean 0. If $P(\eta > x) = o(1)\prod_{i=1}^{m}\overline{U}_i(x)$, as $x \to \infty$, for m distributions $U_i \in \mathscr{C}$, where m is any positive integer, then it holds uniformly for all $x \geq \gamma n$ that*

$$P\left(\sum_{i=1}^{n}\eta_i > x\right) = o(1)n\prod_{i=1}^{m}\overline{U}_i(x), \quad as \quad n \to \infty.$$

3.2. Proof of Theorem 2.1

Firstly, we establish the asymptotic lower bound of (2.3). For any, but small, $\delta \in (0, 1)$, we have

$$\begin{aligned}
&P(\vec{L}(t) > \vec{x})\\
&\geq P\left(\sum_{i=1}^{(1-\delta)\lambda t}\vec{X}_i - \vec{c}\sum_{i=1}^{(1+\delta)\lambda t}\theta_i > \vec{x}, (1-\delta)\lambda t \leq N(t) \leq (1+\delta)\lambda t, \sum_{i=1}^{(1+\delta)\lambda t}\theta_i \leq (1+\delta)^2 t\right)\\
&\geq P\left(\sum_{i=1}^{(1-\delta)\lambda t}\vec{X}_i > \vec{x} + \vec{c}(1+\delta)^2 t\right) - P\left(\left|\frac{N(t)}{\lambda t} - 1\right| > \delta\right) - P\left(\sum_{i=1}^{(1+\delta)\lambda t}\theta_i > (1+\delta)^2 t\right)\\
&= I_1(\vec{x}, t) - I_2(t) - I_3(t),
\end{aligned} \tag{3.1}$$

where the second step is due to an elementary inequality $P(ABC) \geq P(A) - P(\overline{B}) - P(\overline{C})$ for three random events A, B and C.

For $I_1(\vec{x}, t)$, it can be rewritten as

$$I_1(\vec{x}, t) \quad = \quad P\left(\sum_{i=1}^{(1-\delta)\lambda t}\vec{X}_i - (1-\delta)\lambda\vec{\mu}t > \vec{x} + \vec{c}(1+\delta)^2 t - (1-\delta)\lambda\vec{\mu}t\right).$$

Clearly, by the safety load condition that $\vec{c} > \lambda\vec{\mu}$, we know that $\vec{c}(1+\delta)^2 t - (1-\delta)\lambda\vec{\mu}t > \vec{0}$. Hence by Lemma 3.1, it holds uniformly for all $\vec{x} \geq \vec{\gamma}t$ that

$$I_1(\vec{x}, t) \sim ((1-\delta)\lambda t)^2 \prod_{i=1}^{2}\overline{F}_i(x_i + (1+\delta)^2 c_i t - (1-\delta)\lambda\mu_i t). \tag{3.2}$$

Since $\vec{x} \geq \vec{\gamma}t \geq \lambda\vec{\mu}t$ and $\vec{c} \geq \lambda\vec{\mu}$, we have that, for $i = 1, 2$,

$$\begin{aligned}
x_i + c_i(1+\delta)^2 t - \mu_i(1-\delta)\lambda t &= x_i + c_i t - \mu_i\lambda t + (2\delta+\delta^2)c_i t + \delta\lambda\mu_i t\\
&\leq \left(1 + \frac{(2\delta+\delta^2)c_i t + \delta\lambda\mu_i t}{x_i}\right)(x_i + c_i t - \mu_i\lambda t)\\
&\leq \left(1 + \frac{(2\delta+\delta^2)c_i + \delta\lambda\mu_i}{\gamma_i}\right)(x_i + c_i t - \mu_i\lambda t),
\end{aligned}$$

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which, along with $F_i \in \mathscr{C}$, $i = 1, 2$, implies that

$$\begin{aligned}
&\liminf_{\delta\to 0}\liminf_{t\to\infty}\inf_{x_i\geq\gamma_i t}\frac{\overline{F}_i(x_i+(1+\delta)^2c_it-(1-\delta)\lambda\mu_it)}{\overline{F}_i(x_i+c_it-\lambda\mu_it)}\\
&\geq \liminf_{\delta\to 0}\liminf_{t\to\infty}\inf_{x_i\geq\gamma_i t}\frac{\overline{F}_i\left(\left(1+\frac{(2\delta+\delta^2)c_i+\delta\lambda\mu_i}{\gamma_i}\right)(x_i+c_it-\mu_i\lambda t)\right)}{\overline{F}_i(x_i+c_it-\lambda\mu_it)}\\
&= 1. \qquad (3.3)
\end{aligned}$$

Similarly, it holds uniformly for all $\vec{x} \leq \vec{\Gamma}t$ that

$$\begin{aligned}
x_i+c_i(1+\delta)^2t-\mu_i(1-\delta)\lambda t &\geq \left(1+\frac{(2\delta+\delta^2)c_it+\delta\lambda\mu_it}{\Gamma_it+c_it-\mu_i\lambda t}\right)(x_i+c_it-\mu_i\lambda t)\\
&= \left(1+\frac{(2\delta+\delta^2)c_i+\delta\lambda\mu_i}{\Gamma_i+c_i-\mu_i\lambda}\right)(x_i+c_it-\mu_i\lambda t),
\end{aligned}$$

and then by $F_i \in \mathscr{C}$, $i = 1, 2$,

$$\begin{aligned}
&\limsup_{\delta\to 0}\limsup_{t\to\infty}\sup_{x_i\leq\Gamma_i t}\frac{\overline{F}_i(x_i+(1+\delta)^2c_it-(1-\delta)\lambda\mu_it)}{\overline{F}_i(x_i+c_it-\lambda\mu_it)}\\
&\leq \limsup_{\delta\to 0}\limsup_{t\to\infty}\sup_{x_i\leq\Gamma_i t}\frac{\overline{F}_i\left(\left(1+\frac{(2\delta+\delta^2)c_i+\delta\lambda\mu_i}{\Gamma_i+c_i-\mu_i\lambda}\right)(x_i+c_it-\mu_i\lambda t)\right)}{\overline{F}_i(x_i+c_it-\lambda\mu_it)}\\
&= 1. \qquad (3.4)
\end{aligned}$$

Thus, by (3.2)–(3.4) and the arbitrariness of $\delta \in (0, 1)$, we obtain that, uniformly for all $\vec{x} \in [\vec{\gamma}t, \vec{\Gamma}t]$,

$$I_1(\vec{x},t) \sim (\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i+c_it-\lambda\mu_it). \qquad (3.5)$$

For $I_2(t)$, by $\overline{G}(x) = o(1)\overline{F}_{12}(x, x)$ and (1.4), we get $\overline{G}(x) = o(1)\overline{F}_1(x)\overline{F}_2(x)$, as $x \to \infty$. Hence by Lemma 3.2, and $F_i \in \mathscr{C} \subset \mathscr{L} \cap \mathscr{D}$, $i = 1, 2$, we derive that, uniformly for all $\vec{x} \leq \vec{\Gamma}t$,

$$\begin{aligned}
I_2(t) &= P(N(t) > (1+\delta)\lambda t) + P(N(t) < (1-\delta)\lambda t)\\
&\leq P\left(\sum_{i=1}^{(1+\delta)\lambda t}\theta_i \leq t\right) + P\left(\sum_{i=1}^{(1-\delta)\lambda t+1}\theta_i > t\right)\\
&\leq P\left(\sum_{i=1}^{(1+\delta)\lambda t}((-\theta_i)-(-\lambda^{-1})) > \delta t-\lambda^{-1}\right) + P\left(\sum_{i=1}^{(1-\delta)\lambda t+1}(\theta_i-\lambda^{-1}) > \delta t-\lambda^{-1}\right)\\
&= o(1)\lambda t\overline{F}_1(\delta t-\lambda^{-1})\overline{F}_2(\delta t-\lambda^{-1})\\
&= o(1)\lambda t\prod_{i=1}^{2}\overline{F}_i\left(\frac{\delta x_i}{\Gamma_i}-\lambda^{-1}\right)\\
&= o(1)\lambda t\prod_{i=1}^{2}\overline{F}_i(x_i)\\
&= o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i+c_it-\lambda\mu_it), \qquad (3.6)
\end{aligned}$$

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where in the last step we used the fact that, for $i = 1, 2$,

$$\overline{F}_i(x_i + c_i t - \lambda\mu_i t) \asymp \overline{F}_i(x_i) \tag{3.7}$$

resulted from $F_i \in \mathscr{D}$, $\vec{c} > \lambda\vec{\mu}$, and $\overline{F}_i(x_i) \geq \overline{F}_i(x_i + c_i t - \lambda\mu_i t) \geq \overline{F}(x_i(1 + (c_i - \lambda\mu_i)/\gamma_i))$ for $\vec{x} \geq \vec{\gamma}t$.

For $I_3(t)$, by the similar derivation of (3.6), it holds uniformly for all $\vec{x} \leq \vec{\Gamma}t$ that

$$\begin{aligned} I_3(t) &= P\left(\sum_{i=1}^{(1+\delta)\lambda t} (\theta_i - \lambda^{-1}) > \delta(1+\delta)t\right) \\ &= o(1)\lambda t \prod_{i=1}^{2} \overline{F}_i(\delta(1+\delta)t) \\ &= o(1)\lambda t \prod_{i=1}^{2} \overline{F}_i\left(\frac{\delta(1+\delta)x_i}{\Gamma_i}\right) \\ &= o(1)\lambda t \prod_{i=1}^{2} \overline{F}_i(x_i) \\ &= o(1)(\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t). \end{aligned} \tag{3.8}$$

Therefore, by substituting (3.5), (3.6) and (3.8) into (3.1), we prove that, uniformly for all $\vec{x} \in [\vec{\gamma}t, \vec{\Gamma}t]$,

$$P(\vec{L}(t) > \vec{x}) \gtrsim (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t), \tag{3.9}$$

which gives the asymptotic lower bound of (2.3).

Subsequently, we deal with the asymptotic upper bound of (2.3). For any $\delta \in (0, 1)$ as above, we have

$$\begin{aligned} &P(\vec{L}(t) > \vec{x}) \\ &\quad \leq P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i - \vec{c}\sum_{i=1}^{(1-\delta)\lambda t} \theta_i > \vec{x}, (1-\delta)\lambda t \leq N(t) \leq (1+\delta)\lambda t, \sum_{i=1}^{(1-\delta)\lambda t} \theta_i \geq (1-\delta)^2 t\right) \\ &\qquad + P\left(\left|\frac{N(t)}{\lambda t} - 1\right| > \delta\right) + P\left(\sum_{i=1}^{(1-\delta)\lambda t} \theta_i < (1-\delta)^2 t\right) \\ &\quad \leq P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i > \vec{x} + \vec{c}(1-\delta)^2 t\right) + I_2(t) + I_5(t). \end{aligned} \tag{3.10}$$

Denote by

$$I_4(\vec{x}, t) = P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i > \vec{x} + \vec{c}(1-\delta)^2 t\right).$$

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By the safety load condition that $\vec{c} > \lambda\vec{\mu}$, one sees that $\vec{c}(1-\delta)^2 t - (1+\delta)\lambda\vec{\mu}t > \vec{0}$ for some sufficiently small $\delta > 0$. Hence by Lemma 3.1, it holds uniformly for all $\vec{x} \geq \vec{\gamma}t$ that

$$\begin{aligned} I_4(\vec{x}, t) &= P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i - (1+\delta)\lambda\vec{\mu}t > \vec{x} + \vec{c}(1-\delta)^2 t - (1+\delta)\lambda\vec{\mu}t\right) \\ &\sim ((1+\delta)\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + (1-\delta)^2 c_i t - (1+\delta)\lambda\mu_i t), \end{aligned}$$

which, along with the similar derivation of (3.5), implies that, uniformly for all $\vec{x} \in [\vec{\gamma}t, \vec{\Gamma}t]$,

$$I_4(\vec{x}, t) \sim (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t). \tag{3.11}$$

For $I_5(t)$, by the similar derivation of (3.8), it holds uniformly for all $\vec{x} \leq \vec{\Gamma}t$ that

$$\begin{aligned} I_5(t) &= P\left(\sum_{i=1}^{(1-\delta)\lambda t} (\theta_i - \lambda^{-1}) < \delta(\delta-1)t\right) \\ &= P\left(\sum_{i=1}^{(1-\delta)\lambda t} ((-\theta_i) - (-\lambda^{-1})) > \delta(1-\delta)t\right) \\ &= o(1)\lambda t \prod_{i=1}^{2} \overline{F}_i(\delta(1-\delta)t) \\ &= o(1)\lambda t \prod_{i=1}^{2} \overline{F}_i\left(\frac{\delta(1-\delta)x_i}{\Gamma_i}\right) \\ &= o(1)(\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i) \\ &= o(1)(\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t). \end{aligned} \tag{3.12}$$

Hence, by substituting (3.6), (3.11) and (3.12) into (3.10), we show that, uniformly for all $\vec{x} \in [\vec{\gamma}t, \vec{\Gamma}t]$,

$$P(\vec{L}(t) > \vec{x}) \lesssim (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t), \tag{3.13}$$

which is the claimed asymptotic upper bound of (2.3).

Finally, a combination of (3.9) and (3.13) proves that relation (2.3) holds uniformly for all $\vec{x} \in [\vec{\gamma}t, \vec{\Gamma}t]$.

3.3. *Proof of Corollary 2.1*

The proof can be given by going along the same lines of the proof of Theorem 2.1 with the following modifications.

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1. $\vec{L}(t)$, $\vec{x}$, $\vec{X}_i$, $\vec{c}$, $\vec{\mu}$, $\vec{\gamma}$, $\vec{\Gamma}$ and $\vec{0}$ are replaced by $L(t)$, x, X_i, c, μ, γ, Γ and 0, respectively.
2. For $i = 1, 2$, $\overline{F}_i$, x_i, c_i, γ_i and μ_i are replaced by $\overline{F}$, x, c, γ and μ, respectively.
3. Relation (3.2) is changed to

$$I_1(x,t) \sim (1-\delta)\lambda t\overline{F}(x + (1+\delta)^2 ct - (1-\delta)\lambda\mu t) \tag{3.14}$$

holding uniformly for all $x \geq \gamma t$, where we use Theorem 3.1 of [22] instead of Lemma 3.1.
4. By using (3.14), (3.3) and (3.4) with $\overline{F}_i$, x_i, c_i, γ_i and μ_i replaced by $\overline{F}$, x, c, γ and μ, respectively, and considering the arbitrariness of $\delta \in (0,1)$, relation (3.5) is changed to

$$I_1(x,t) \sim \lambda t\overline{F}(x + ct - \lambda\mu t), \tag{3.15}$$

which holds uniformly for all $x \in [\gamma t, \Gamma t]$.
5. By mimicking the proof of Theorem 1.1 of [4], relation (3.6) is changed to

$$I_2(t) = o(1)\lambda t\overline{F}(x) = o(1)\lambda t\overline{F}(x + ct - \lambda\mu t) \tag{3.16}$$

holding uniformly for all $x \leq \Gamma t$ under the conditions of Corollary 2.1, where the second step is due to the fact that

$$\overline{F}(x + ct - \lambda\mu t) \asymp \overline{F}(x), \tag{3.17}$$

resulted from $F \in \mathscr{D}$, $c > \lambda\mu$, and $\overline{F}(x) \geq \overline{F}(x + ct - \lambda\mu t) \geq \overline{F}(x(1 + (c - \lambda\mu)/\gamma))$ for $x \geq \gamma t$.
6. By the similar derivation of (3.16), relation (3.8) is changed to

$$I_3(t) = o(1)\lambda t\overline{F}(x + ct - \lambda\mu t) \tag{3.18}$$

holding uniformly for all $x \leq \Gamma t$.
7. By substituting (3.15), (3.16) and (3.18) into (3.1) with $\vec{L}(t)$, $\vec{x}$, $\vec{X}_i$ and $\vec{c}$ replaced by $L(t)$, x, X_i and c, respectively, relation (3.9) is changed to

$$P(L(t) > x) \gtrsim \lambda t\overline{F}(x + ct - \lambda\mu t), \tag{3.19}$$

holding uniformly for all $x \in [\gamma t, \Gamma t]$, which is the asymptotic lower bound of (2.5).
8. By the similar derivation of (3.15), relation (3.11) is changed to

$$I_4(x,t) \sim \lambda t\overline{F}(x + ct - \lambda\mu t), \tag{3.20}$$

holding uniformly for all $x \in [\gamma t, \Gamma t]$.
9. By the similar derivation of (3.18), relation (3.12) is changed to

$$I_5(t) = o(1)\lambda t\overline{F}(x + ct - \lambda\mu t) \tag{3.21}$$

holding uniformly for all $x \leq \Gamma t$.
10. By substituting (3.16), (3.20) and (3.21) into (3.10) with $\vec{L}(t)$, $\vec{x}$, $\vec{X}_i$ and $\vec{c}$ replaced by $L(t)$, x, X_i and c, respectively, relation (3.13) is changed to

$$P(L(t) > x) \lesssim \lambda t\overline{F}(x + ct - \lambda\mu t), \tag{3.22}$$

holding uniformly for all $x \in [\gamma t, \Gamma t]$, which establishes the asymptotic upper bound of (2.5).

Hence by (3.19) and (3.22), the uniformity of relation (2.5) for all $x \in [\gamma t, \Gamma t]$ is obtained.

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4. Proofs of Theorem 2.2 and Corollary 2.2

4.1. Lemmas

The following two lemmas are given for the nonstandard renewal process $\{N(t), t \geq 0\}$ with widely dependent inter-arrival times $\{\theta_i, i \geq 1\}$, among which the first one is due to Lemma 2.2 of [28].

Lemma 4.1. *Consider the nonstandard renewal process $\{N(t), t \geq 0\}$ with WLOD inter-arrival times $\{\theta_i, i \geq 1\}$ such that for some $\epsilon > 0$,*

$$\lim_{n\to\infty} g_L(n)e^{-\epsilon n} = 0. \tag{4.1}$$

Then for any $\delta > 0$, there exists some $r > 0$ such that

$$\lim_{t\to\infty} Ee^{rN(t)}\mathbf{1}_{\{N(t)>(1+\delta)\lambda t\}} = 0.$$

The second lemma gives the strong laws of large numbers for $\{\theta_i, i \geq 1\}$ and $\{N(t), t \geq 0\}$, see Theorem 2.4 of [29] or Theorem 4 of [3].

Lemma 4.2. *Consider the nonstandard renewal process $\{N(t), t \geq 0\}$ with Assumption A_2^{**}, then*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \theta_i}{n} = \lambda^{-1}, \quad \textit{and} \quad \lim_{t\to\infty} \frac{N(t)}{\lambda t} = 1, \quad a.s.$$

Remark that, among the conditions in Lemma 4.1, Assumptions A_2^* and A_2^{**}, the condition (4.1) is the most relaxed, the condition (2.1) takes second place, and the condition (2.2) is the strongest one.

4.2. Proof of Theorem 2.2

Firstly, we consider the uniform asymptotic lower bound of (2.3) for all $\vec{x} \geq \vec{\gamma}t$ under the conditions of Theorem 2.2. Let $\delta \in (0, 1)$ be fixed as above. Similarly to (3.1), we have

$$\begin{aligned} &P\left(\vec{L}(t) > \vec{x}\right) \\ &\geq P\left(\sum_{i=1}^{(1-\delta)\lambda t} \vec{X}_i > \vec{x} + \vec{c}(1+\delta)^2 t\right) P\left(\left|\frac{N(t)}{\lambda t} - 1\right| \leq \delta, \sum_{i=1}^{(1+\delta)\lambda t} \theta_i \leq (1+\delta)^2 t\right) \\ &= I_1(\vec{x}, t)(1 - I_2(t) - I_3(t)), \end{aligned} \tag{4.2}$$

where the first step is due to the independence in Assumption A_3, and the last step is due to an inequality $P(AB) \geq 1 - P(\overline{A}) - P(\overline{B})$ for two random events A and B.

By Lemma 4.2, it follows that

$$\lim_{t\to\infty} \sup_{\vec{x}\geq\vec{\gamma}t} I_2(t) = \lim_{t\to\infty} \sup_{\vec{x}\geq\vec{\gamma}t} P\left(\left|\frac{N(t)}{\lambda t} - 1\right| > \delta\right) = 0, \tag{4.3}$$

and

$$\lim_{t\to\infty} \sup_{\vec{x}\geq\vec{\gamma}t} I_3(t) = \lim_{t\to\infty} \sup_{\vec{x}\geq\vec{\gamma}t} P\left(\frac{\sum_{i=1}^{(1+\delta)\lambda t} \theta_i}{(1+\delta)t} - 1 > \delta\right) = 0. \tag{4.4}$$

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Then by substituting (3.3), (4.3), and (4.4) into (4.2), we obtain that relation (3.9) holds uniformly for all $\vec{x} \geq \vec{\gamma} t$.

Secondly, we turn to the asymptotic upper bound of (2.3), which holds uniformly for all $\vec{x} \geq \vec{\gamma} t$. For any $\delta \in (0, \min\{1, \gamma_1/\lambda\mu_1 - 1, \gamma_2/\lambda\mu_2 - 1\})$, we have

$$\begin{aligned} P\left(\vec{L}(t) > \vec{x}\right) &= P\left(\vec{L}(t) > \vec{x}, N(t) < (1-\delta)\lambda t\right) \\ &\quad + P\left(\vec{L}(t) > \vec{x}, (1-\delta)\lambda t \leq N(t) \leq (1+\delta)\lambda t\right) + P\left(\vec{L}(t) > \vec{x}, N(t) > (1+\delta)\lambda t\right) \\ &= I_6(\vec{x}, t) + I_7(\vec{x}, t) + I_8(\vec{x}, t). \end{aligned} \tag{4.5}$$

For $I_6(\vec{x}, t)$, by the independence in Assumption A_3, we obtain that

$$\begin{aligned} I_6(\vec{x}, t) &\leq P\left(\sum_{i=1}^{(1-\delta)\lambda t} \vec{X}_i > \vec{x}\right) P\left(\frac{N(t)}{\lambda t} - 1 < -\delta\right) \\ &= P\left(\sum_{i=1}^{(1-\delta)\lambda t} \vec{X}_i - (1-\delta)\lambda\vec{\mu} t > \vec{x} - (1-\delta)\lambda\vec{\mu} t\right) P\left(\frac{N(t)}{\lambda t} - 1 < -\delta\right). \end{aligned} \tag{4.6}$$

By $F_i \in \mathscr{C}$, $i = 1, 2$, and Lemma 3.1, we derive that, uniformly for all $\vec{x} \geq \vec{\gamma} t$,

$$\begin{aligned} &P\left(\sum_{i=1}^{(1-\delta)\lambda t} \vec{X}_i - (1-\delta)\lambda\vec{\mu} t > \vec{x} - (1-\delta)\lambda\vec{\mu} t\right) \\ &\sim ((1-\delta)\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i - (1-\delta)\lambda\mu_i t) \\ &\leq (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i\left(\left(1 - \frac{(1-\delta)\lambda\mu_i}{\gamma_i}\right) x_i\right), \end{aligned}$$

where the first step is from $\vec{x} - (1-\delta)\lambda\vec{\mu} t \geq \vec{\gamma} t - (1-\delta)\lambda\vec{\mu} t \geq \lambda\vec{\mu} t - (1-\delta)\lambda\vec{\mu} t = \delta\lambda\vec{\mu} t > \vec{0}$. Thus, by $F_i \in \mathscr{C} \subset \mathscr{D}$, $i = 1, 2$, Lemma 4.2, (3.7) and (4.6), it holds uniformly for all $\vec{x} \geq \vec{\gamma} t$ that

$$I_6(\vec{x}, t) = o(1)(\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i) = o(1)(\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t). \tag{4.7}$$

For $I_7(\vec{x}, t)$, it follows that

$$\begin{aligned} I_7(\vec{x}, t) &\leq P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i - \vec{c} \sum_{i=1}^{(1-\delta)\lambda t} \theta_i > \vec{x}\right) \\ &= P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i - \vec{c} \sum_{i=1}^{(1-\delta)\lambda t} \theta_i > \vec{x}, \left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| \leq \delta\right) \\ &\quad + P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i - \vec{c} \sum_{i=1}^{(1-\delta)\lambda t} \theta_i > \vec{x}, \left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| > \delta\right) \\ &= I_{71}(\vec{x}, t) + I_{72}(\vec{x}, t). \end{aligned} \tag{4.8}$$

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By (3.11) and Lemma 4.2, we get that, uniformly for all $\vec{x} \geq \vec{\gamma} t$,

$$\begin{aligned} I_{71}(\vec{x}, t) &\leq I_4(\vec{x}, t) P\left(\left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| \leq \delta\right) \\ &\sim (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda \mu_i t). \end{aligned} \tag{4.9}$$

Clearly, for $\vec{x} \geq \vec{\gamma} t$ and $\delta \in (0, \min\{1, \gamma_1/\lambda\mu_1 - 1, \gamma_2/\lambda\mu_2 - 1\})$, it holds that $\vec{x} - (1+\delta)\lambda\vec{\mu}t \geq \vec{\gamma}t - (1+\delta)\lambda\vec{\mu}t > \vec{0}$. Then by Lemmas 3.1 and 4.2, we prove that, uniformly for all $\vec{x} \geq \vec{\gamma} t$,

$$\begin{aligned} I_{72}(\vec{x}, t) &\leq P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i > \vec{x}\right) P\left(\left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| > \delta\right) \\ &= P\left(\sum_{i=1}^{(1+\delta)\lambda t} \vec{X}_i - (1+\delta)\lambda\vec{\mu}t > \vec{x} - (1+\delta)\lambda\vec{\mu}t\right) P\left(\left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| > \delta\right) \\ &= o(1)((1+\delta)\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i - (1+\delta)\lambda\mu_i t) \\ &= o(1)((1+\delta)\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i\left(\left(1 - \frac{(1+\delta)\lambda\mu_i}{\gamma_i}\right) x_i\right). \end{aligned}$$

Further, by $\overline{F}_i \in \mathscr{C} \subset \mathscr{D}$, $i = 1, 2$, the arbitrariness of $\delta \in (0, \min\{1, \gamma_1/\lambda\mu_1 - 1, \gamma_2/\lambda\mu_2 - 1\})$, and $1 - (1+\delta)\lambda\mu_i/\gamma_i > 0$, $i = 1, 2$, it holds uniformly for all $\vec{x} \geq \vec{\gamma} t$ that

$$I_{72}(\vec{x}, t) = o(1)(\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i). \tag{4.10}$$

Hence, by substituting (4.9) and (4.10) into (4.8), and using relation (3.7), we show that, uniformly for all $\vec{x} \geq \vec{\gamma} t$,

$$I_7(\vec{x}, t) \lesssim (\lambda t)^2 \prod_{i=1}^{2} \overline{F}_i(x_i + c_i t - \lambda\mu_i t). \tag{4.11}$$

For $I_8(\vec{x}, t)$, by Proposition 2.2 of Bingham et al. (1987), we know that if $F_i \in \mathscr{D}$, $i = 1, 2$, then for any $p_i > J^+_{F_i}$, there exist positive constants C_i and D_i, such that

$$\frac{\overline{F}_i(y_i)}{\overline{F}_i(x_i)} \leq C_i \left(\frac{x_i}{y_i}\right)^{p_i}, \tag{4.12}$$

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holds for all $x_i \ge y_i \ge D_i$, $i = 1, 2$. Then, we have

$$
\begin{aligned}
I_8(\vec{x}, t) &\le P\left(\sum_{i=1}^{N(t)} \vec{X}_i > \vec{x}, N(t) > (1+\delta)\lambda t\right) \\
&= \sum_{n>(1+\delta)\lambda t} P\left(\sum_{i=1}^{n} \vec{X}_i > \vec{x}\right) P(N(t) = n) \\
&= \left(\begin{array}{c} \sum_{(1+\delta)\lambda t<n<\min\{\frac{x_1}{D_1},\frac{x_2}{D_2}\}} + \sum_{(1+\delta)\lambda t<n<\frac{x_2}{D_2},n>\frac{x_1}{D_1}} \\ + \sum_{(1+\delta)\lambda t<n<\frac{x_1}{D_1},n>\frac{x_2}{D_2}} + \sum_{(1+\delta)\lambda t<n,n>\frac{x_1}{D_1},n>\frac{x_2}{D_2}} \end{array}\right) P\left(\sum_{i=1}^{n} \vec{X}_i > \vec{x}\right) P(N(t) = n) \\
&= \sum_{k=1}^{4} I_{8k}(\vec{x}, t).
\end{aligned}
\tag{4.13}
$$

By (1.4), (4.12), and Lemma 4.1, it holds uniformly for all $\vec{x} \ge \vec{\gamma} t$ that

$$
\begin{aligned}
I_{81}(\vec{x}, t) &\le \sum_{(1+\delta)\lambda t<n<\min\{\frac{x_1}{D_1},\frac{x_2}{D_2}\}} \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(X_i^{(1)} > \frac{x_1}{n}, X_j^{(2)} > \frac{x_2}{n}\right) P(N(t) = n) \\
&\le \sum_{(1+\delta)\lambda t<n<\min\{\frac{x_1}{D_1},\frac{x_2}{D_2}\}} (n^2 - n + Mn)\overline{F}_1\left(\frac{x_1}{n}\right)\overline{F}_2\left(\frac{x_2}{n}\right) P(N(t) = n) \\
&\le C_1C_2\overline{F}_1(x_1)\overline{F}_2(x_2) \sum_{(1+\delta)\lambda t<n<\min\{\frac{x_1}{D_1},\frac{x_2}{D_2}\}} \left(n^{p_1+p_2+2} + (M-1)n^{p_1+p_2+1}\right) P(N(t) = n) \\
&\le C_1C_2\overline{F}_1(x_1)\overline{F}_2(x_2) \left(EN^{p_1+p_2+2}(t)\mathbf{1}_{\{N(t)>(1+\delta)\lambda t\}} + (M-1)EN^{p_1+p_2+1}(t)\mathbf{1}_{\{N(t)>(1+\delta)\lambda t\}}\right) \\
&= o(1)\prod_{i=1}^{2}\overline{F}_i(x_i) \\
&= o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i).
\end{aligned}
\tag{4.14}
$$

Again by (4.12), fixing the variable y_i leads to

$$
x_i^{-p_i} \le \hat{C}_i\overline{F}_i(x_i), \qquad i = 1, 2,
\tag{4.15}
$$

where $\hat{C}_i = C_i y_i^{-p_i}(\overline{F}_i(y_i))^{-1}$, $i = 1, 2$. Thus, by (4.12), (4.15) and Lemma 4.1, we obtain that, uniformly for all $\vec{x} \ge \vec{\gamma} t$,

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$$
\begin{aligned}
I_{82}(\vec{x},t) &\le \sum_{(1+\delta)\lambda t<n<\frac{x_2}{D_2},n>\frac{x_1}{D_1}} \left(\frac{nD_1}{x_1}\right)^{p_1} P\left(\sum_{i=1}^{n} X_i^{(2)} > x_2\right) P(N(t)=n) \\
&\le D_1^{p_1} x_1^{-p_1} \sum_{(1+\delta)\lambda t<n<\frac{x_2}{D_2},n>\frac{x_1}{D_1}} n^{p_1+1}\overline{F}_2\left(\frac{x_2}{n}\right) P(N(t)=n) \\
&\le D_1^{p_1} C_2 x_1^{-p_1}\overline{F}_2(x_2) \sum_{(1+\delta)\lambda t<n<\frac{x_2}{D_2},n>\frac{x_1}{D_1}} n^{p_1+p_2+1} P(N(t)=n) \\
&\le D_1^{p_1} C_2\hat{C}_1\overline{F}_1(x_1)\overline{F}_2(x_2) EN^{p_1+p_2+1}(t)\mathbf{1}_{\{N(t)>(1+\delta)\lambda t\}} \\
&= o(1)\prod_{i=1}^{2}\overline{F}_i(x_i) \\
&= o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i). \qquad (4.16)
\end{aligned}
$$

Similarly, we still obtain that, uniformly for all $\vec{x} \ge \vec{\gamma}t$,

$$
I_{83}(\vec{x},t) = o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i). \qquad (4.17)
$$

By (4.15) and Lemma 4.1, it holds uniformly for all $\vec{x} \ge \vec{\gamma}t$ that

$$
\begin{aligned}
I_{84}(\vec{x},t) &\le \sum_{(1+\delta)\lambda t<n,n>\frac{x_1}{D_1},n>\frac{x_2}{D_2}} \left(\frac{nD_1}{x_1}\right)^{p_1}\left(\frac{nD_2}{x_2}\right)^{p_2} P(N(t)=n) \\
&\le D_1^{p_1}D_2^{p_2}x_1^{-p_1}x_2^{-p_2} \sum_{(1+\delta)\lambda t<n,n>\frac{x_1}{D_1},n>\frac{x_2}{D_2}} n^{p_1+p_2} P(N(t)=n) \\
&\le D_1^{p_1}D_2^{p_2}\hat{C}_1\hat{C}_2\overline{F}_1(x_1)\overline{F}_2(x_2) EN^{p_1+p_2}(t)\mathbf{1}_{\{N(t)>(1+\delta)\lambda t\}} \\
&= o(1)\prod_{i=1}^{2}\overline{F}_i(x_i) \\
&= o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i). \qquad (4.18)
\end{aligned}
$$

Consequently, we substitute (4.14), (4.16)–(4.18) into (4.13) to derive that, uniformly for all $\vec{x} \ge \vec{\gamma}t$,

$$
I_8(\vec{x},t) = o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i) = o(1)(\lambda t)^2\prod_{i=1}^{2}\overline{F}_i(x_i + c_i t - \lambda\mu_i t), \qquad (4.19)
$$

where the last step comes from (3.7). Further by (4.5), (4.7), (4.11), and (4.19), we prove that relation (3.13) holds uniformly for all $\vec{x} \ge \vec{\gamma}t$ under the conditions of Theorem 2.2.

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As a result, we show from the uniformity of (3.9) and (3.13) for all $\vec{x} \geq \vec{\gamma}t$ that relation (2.3) still holds uniformly for all $\vec{x} \geq \vec{\gamma}t$.

4.3. Proof of Corollary 2.2

Similarly to the proof of Corollary 2.1, we prove Corollary 2.2 by copying the proof of Theorem 2.2 with some modifications as follows:

1. $\vec{L}(t)$, $\vec{x}$, $\vec{X}_i$, $\vec{c}$, $\vec{\mu}$, $\vec{\gamma}$ and $\vec{0}$ are replaced by $L(t)$, x, X_i, c, μ, γ, and 0, respectively.
2. For $i = 1, 2$, $\overline{F}_i$, x_i, c_i, γ_i and μ_i are replaced by $\overline{F}$, x, c, γ, and μ, respectively.
3. The uniform asymptotic lower bound of (2.5) that relation (3.19) holds uniformly for all $x \geq \gamma t$ can be given by substituting (3.15), (4.3) and (4.4) into (4.2) with $\vec{L}(t)$, $\vec{x}$, $\vec{X}_i$ and $\vec{c}$ replaced by $L(t)$, x, X_i and c, respectively.
4. Relation (4.7) is changed to

$$I_6(x,t) = o(1)\lambda t\overline{F}(x) = o(1)\lambda t\overline{F}(x + ct - \lambda\mu t) \tag{4.20}$$

   holding uniformly for all $x \geq \gamma t$, which is due to $F \in \mathscr{C} \subset \mathscr{D}$, Lemma 4.2, (3.17), and the fact that

$$\begin{aligned} &P\left(\sum_{i=1}^{(1-\delta)\lambda t} X_i - (1-\delta)\lambda\mu t > x - (1-\delta)\lambda\mu t\right) \\ &\sim (1-\delta)\lambda t\overline{F}(x - (1-\delta)\lambda\mu t) \\ &\leq (\lambda t)^2\overline{F}\left(\left(1 - \frac{(1-\delta)\lambda\mu}{\gamma}\right)x\right), \end{aligned}$$

   holding uniformly for all $x \geq \gamma t$ resulted from Theorem 3.1 of [22] instead of Lemma 3.1.
5. Relation (4.9) is changed to

$$\begin{aligned} I_{71}(x,t) &\leq I_4(x,t)P\left(\left|\frac{\sum_{i=1}^{(1-\delta)\lambda t}\theta_i}{(1-\delta)t} - 1\right| \leq \delta\right) \\ &\sim \lambda t\overline{F}(x + ct - \lambda\mu t) \end{aligned} \tag{4.21}$$

   holding uniformly for all $x \geq \gamma t$, which is due to (3.20) and Lemma 4.2.
6. Relation (4.10) is changed to

$$I_{72}(x,t) = o(1)\lambda t\overline{F}(x). \tag{4.22}$$

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holding uniformly for all $x \geq \gamma t$, where we used $\overline{F} \in \mathscr{C} \subset \mathscr{D}$, the arbitrariness of $\delta \in (0, \min\{1, \gamma/\lambda\mu - 1\})$, $1 - (1+\delta)\lambda\mu/\gamma > 0$, and the fact that

$$
\begin{aligned}
I_{72}(x,t) &\leq P\left(\sum_{i=1}^{(1+\delta)\lambda t} X_i > x\right) P\left(\left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| > \delta\right) \\
&= P\left(\sum_{i=1}^{(1+\delta)\lambda t} X_i - (1+\delta)\lambda\mu t > x - (1+\delta)\lambda\mu t\right) P\left(\left|\frac{\sum_{i=1}^{(1-\delta)\lambda t} \theta_i}{(1-\delta)t} - 1\right| > \delta\right) \\
&= o(1)(1+\delta)\lambda t\overline{F}(x - (1+\delta)\lambda\mu t) \\
&= o(1)(1+\delta)\lambda t\overline{F}\left(\left(1 - \frac{(1+\delta)\lambda\mu}{\gamma}\right)x\right)
\end{aligned}
$$

holding uniformly for all $x \geq \gamma t$ resulted from Theorem 3.1 of [22] and Lemma 4.2.

7. By substituting (4.21) and (4.22) into (4.8) with $\vec{x}$, $\vec{X}_i$ and $\vec{c}$ replaced by x, X_i and c, respectively, relation (4.11) is changed to

$$
I_7(\vec{x}, t) \lesssim \lambda t\overline{F}(x + ct - \lambda\mu t). \tag{4.23}
$$

holding uniformly for all $x \geq \gamma t$, where we used (3.17) instead of (3.7).

8. Relation (4.12) is changed to that for any $p > J_F^+$, there exist positive constants C and D such that

$$
\frac{\overline{F}(y)}{\overline{F}(x)} \leq C\left(\frac{x}{y}\right)^p, \tag{4.24}
$$

holds for all $x \geq y \geq D$.

9. The derivation of $I_8(x,t)$ is reformulated as

$$
\begin{aligned}
I_8(x,t) &\leq P\left(\sum_{i=1}^{N(t)} X_i > x, N(t) > (1+\delta)\lambda t\right) \\
&= \sum_{n > (1+\delta)\lambda t} P\left(\sum_{i=1}^{n} X_i > x\right) P(N(t) = n) \\
&= \left(\sum_{(1+\delta)\lambda t < n < \frac{x}{D}} + \sum_{n > \frac{x}{D}}\right) P\left(\sum_{i=1}^{n} X_i > x\right) P(N(t) = n) \\
&= I_{81}(x,t) + I_{82}(x,t).
\end{aligned} \tag{4.25}
$$

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By (4.24) and Lemma 4.1, it holds uniformly for all $x \geq \gamma t$ that

$$\begin{aligned} I_{81}(x,t) &\leq \sum_{(1+\delta)\lambda t<n<\frac{x}{D}} n\overline{F}\left(\frac{x}{n}\right) P(N(t)=n) \\ &\leq C\overline{F}(x) \sum_{(1+\delta)\lambda t<n<\frac{x}{D}} n^{p+1} P(N(t)=n) \\ &\leq C\overline{F}(x) EN^{p+1}(t)\mathbf{1}_{\{N(t)>(1+\delta)\lambda t\}} \\ &= o(1)\overline{F}(x) \\ &= o(1)\lambda t\overline{F}(x). \end{aligned} \tag{4.26}$$

Fixing the variable y in (4.24) yields that

$$x^{-p} \leq \hat{C}\overline{F}(x), \tag{4.27}$$

where $\hat{C} = Cy^{-p}(\overline{F}(y))^{-1}$. Hence, by (4.27) and Lemma 4.1, it holds uniformly for all $x \geq \gamma t$ that

$$\begin{aligned} I_{82}(x,t) &\leq \sum_{n>\frac{x}{D}} \left(\frac{nD}{x}\right)^p P(N(t)=n) \\ &\leq D^p x^{-p} EN^p(t)\mathbf{1}_{\{N(t)>\frac{x}{D}\}} \\ &\leq D^p \hat{C}\overline{F}(x) EN^p(t)\mathbf{1}_{\{N(t)>\frac{x}{D}\}} \\ &= o(1)\overline{F}(x) \\ &= o(1)\lambda t\overline{F}(x). \end{aligned} \tag{4.28}$$

Therefore, by substituting (4.26) and (4.28) into (4.25), we obtain that, uniformly for all $x \geq \gamma t$,

$$I_8(x,t) = o(1)\lambda t\overline{F}(x) = o(1)\lambda t\overline{F}(x+ct-\lambda\mu t), \tag{4.29}$$

where the last step comes from (3.17).

Further by substituting (4.20), (4.23) and (4.29) into (4.5) with $\vec{L}(t)$ and $\vec{x}$ replaced by $L(t)$ and x, respectively, we show that relation (3.22) holds uniformly for all $x \geq \gamma t$, which is the uniform asymptotic upper bound of (2.5) under the conditions of Corollary 2.2, and thus this proof is completed.

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