IRREDUCIBILITY OF BERNOULLI POLYNOMIALS OF HIGHER ORDER

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The Bernoulli polynomials of order k, where k is a positive integer, are defined by

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{m=0}^{\infty} B_m^{(k)}(x) \frac{t^m}{m!}.$$

$B_m^{(k)}(x)$ is a polynomial of degree m with rational coefficients, and the constant term of $B_m^{(k)}(x)$ is the mth Bernoulli number of order k, $B_m^{(k)}$. In a previous paper **(3)** we obtained some conditions, in terms of k and m, which imply that $B_m^{(k)}(x)$ is irreducible (all references to irreducibility will be with respect to the field of rational numbers). In particular, we obtained the following two results.

THEOREM A. *Let p be an odd prime and let $k \leqslant p$ and $t > 0$. Then $B_{m(p-1)p^t}^{(k)}(x)$ is irreducible for $1 \leqslant m \leqslant p$.*

THEOREM B. *For any integer $k \geqslant 1$ there is an integer $T(k)$ such that for all $t \geqslant T(k)$, $B_{2^t}^{(k)}(x)$ is irreducible.*

In viewing Theorem A, one is led to wonder what the situation is when $k > p$. In this paper we shall give at least a partial answer to this question. We shall show that a result like Theorem B holds for all primes. Furthermore, we shall obtain an explicit bound for the $T(k)$ of Theorem B.

First, however, we must introduce some terminology and notation. Let p be any prime. A polynomial with rational coefficients will be called a p-Eisenstein polynomial if it satisfies the conditions of the Eisenstein irreducibility criterion with p as the prime involved in the conditions. Such a polynomial is irreducible.

If k is a positive integer and p is a prime, write

$$k = a_1 p^{j_1} + a_2 p^{j_2} + \ldots + a_r p^{j_r}, \quad 0 \leqslant j_1 < j_2 < \ldots < j_r,$$
$$0 \leqslant a_i \leqslant p - 1, \qquad i = 1, 2, \ldots, r,$$

and set $j_p(k) = j_1$ and $r_p(k) = r$. We shall make use of the following result of Carlitz **(2**, Theorem A**)**.

THEOREM C. *If k is a positive integer and p is a prime, then the denominator of $p^{r_p(k)} B_m^{(k)}$ is prime to p for all m.*

We shall now turn to the principal result of this paper.

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THEOREM 1. *Let p be a prime, let $1 \leqslant m < p$, and let $n = m(p-1)p^t$. Further, let $T^*_p(1) = 0$ and for $k \geqslant 2$,*

$$T^*_p(k) = \max[T^*_p(k-1), r_p(k-1)] + j_p(k-1).$$

Then, for all $t \geqslant T^*_p(k)$, $pB_n^{(k)}(x)$ is a p-Eisenstein polynomial.

Proof. For $k = 1$ the result follows from the work of Carlitz **(1**, §2**)**. We now assume that $k > 1$ and that $B_n^{(k-1)}(x)$ is irreducible for all $t \geqslant T^*_p(k-1)$. Let $t \geqslant T^*_p(k)$. We have **(4**, p. 145**)**

$$B_n^{(k)}(x) = \left(1 - \frac{n}{k-1}\right)B_n^{(k-1)}(x) + (x-k+1)\frac{n}{k-1}B_{n-1}^{(k-1)}(x).$$

Since $t \geqslant r_p(k-1) + j_p(k-1)$, it follows from Theorem C that both

$$\frac{n}{k-1}B_n^{(k-1)}(x) \text{ and } \frac{n}{k-1}B_{n-1}^{(k-1)}(x)$$

have coefficients whose denominators are prime to p. Hence

$$pB_n^{(k)}(x) \equiv pB_n^{(k-1)}(x) \pmod p.$$

Since $t \geqslant T^*_p(k-1)$, $pB_n^{(k-1)}(x)$ is a p-Eisenstein polynomial. Since the leading coefficient of $pB_n^{(k)}(x)$ is p, this polynomial is also a p-Eisenstein polynomial.

If we now define $T^{**}_p(k)$ to be the smallest non-negative integer such that for all $t \geqslant T^{**}_p(k)$, $pB_n^{(k)}(x)$ is a p-Eisenstein polynomial, then it is clear that if we redefine $T^*_p(k)$ by $T^*_p(1) = 0$ and for $k > 2$,

$$T^*_p(k) = \max[T^{**}_p(k-1), r_p(k-1)] + j_p(k-1),$$

the result of Theorem 1 continues to hold.

COROLLARY. *With p, m, and n as in Theorem* 1, *there is, for each positive integer k, a smallest non-negative integer $T_p(k)$ such that for all $t \geqslant T_p(k)$, $B_n^{(k)}(x)$ is irreducible.*

Suppose that $k < p$. Then $r_p(k-1) = 1$ and $j_p(k-1) = 0$. Hence $T_p(k) = 1$. Thus, Theorems A and B are corollaries of Theorem 1. We now consider in more detail the case $p = 2$.

THEOREM 2. *Let k be a positive integer. For all $t \geqslant k$, $B_{2^t}^{(k)}(x)$ is irreducible.*

Proof. Let $T'(1) = 1$ and for $k > 1$,

$$T'(k) = T'(k-1) + j_2(k-1).$$

We shall show that $T'(k) \geqslant T^*_2(k)$. This is true when $k = 1$. Suppose that $k > 1$ and that $T'(k-1) \geqslant T^*_2(k-1)$.

We first show that $T'(k) \geqslant r_2(k)$ for $k \geqslant 1$. This is true for $k = 1$, and we assume that $k > 1$ and that $T'(k-1) \geqslant r_2(k-1)$. If k is odd, then $j_2(k-1) > 0$, and so $T'(k) \geqslant r_2(k-1) + 1 = r_2(k)$. If k is even, then $j_2(k-1) = 0$, and so $T'(k) = T'(k-1) \geqslant r_2(k-1)$. But $r_2(k) \leqslant r_2(k-1)$, so $T'(k) \geqslant r_2(k)$ in this case also. Thus, for all $k > 1$, $T'(k) = T'(k-1) + j_2(k-1) \geqslant \max [T^*{}_2(k-1), r_2(k-1)] + j_2(k-1) = T_2(k)$.

We see, therefore, that $B_{2^t}^{(k)}(x)$ is irreducible for all $t \geqslant T'(k)$. But, $T'(k) = T'(1) + j_2(1) + \ldots + j_2(k-1) = 1 +$ (the number of times 2 divides $(k-1)!$) $\leqslant k$. This completes the proof of Theorem 2. This result is the best possible that this particular proof can yield, since for all $t \geqslant 1$, $T'(2^t) = 2^t$.

The estimate $T_2(k) \leqslant k$ given by Theorem 2 is a very rough one indeed. We have determined that $T_2(1) = T_2(2) = T_2(4) = 0$ and $T_2(3) = 2$. In determining $T_2(k)$, the following facts are helpful. First, $B_{k-1}^{(k)}(x)$ is reducible (**4**, p. 147). Second, $B_2^{(k)}(x)$ is irreducible if and only if $3k$ is not a perfect square (**3**, p. 317). Third, consulting the table on (**4**, p. 459), we find that $240B_4^{(k)}(x) = 240x^4 - 480kx^3 + 120k(3k-1)x^2 - 120k^2(k-1)x + k(15k^3 - 30k^2 + 5k + 2)$: this is a 5-Eisenstein polynomial if k is not divisible by 5, and a 3-Eisenstein polynomial if $k \equiv 1 \pmod 3$. It would not be surprising if $B_{2^t}^{(k)}(x)$ is irreducible except when $k = 2^t + 1$ and $t > 0$. This is in accord with the more general conjecture that $B_{2m}^{(k)}(x)$ is irreducible except when $k = 2m + 1$.

References

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