CLASSIFICATION OF p-GROUPS OF AUTOMORPHISMS OF RIEMANN SURFACES AND THEIR LOWER CENTRAL SERIES

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0.0. Introduction. In a previous paper **[7]**, I have made a study of the "nilpotent" analogue of Hurwitz theorem **[4]** by considering a particular family of signatures called "nilpotent admissible" **[5]**. We saw however, that if $\mu_N(g)$ represents the order of the largest nilpotent group of automorphisms of a surface of genus $g \geq 2$, then $\mu_N(g) \leq 16(g-1)$ and this upper bound occurs when the covering group is a triangle group having the signature $(0; 2, 4, 8)$ which is in its own 2-local form.

The restriction to the nilpotent groups enabled me to obtain much more precise information than was available in the general case. Moreover, all nilpotent groups attaining this maximum order turned out to be "2-groups". Since every finite nilpotent group is the direct product of its Sylow subgroups and the groups of automorphisms are factor groups of the Fuchsian groups, it is natural for us to study the Fuchsian groups having p-local signatures to obtain more precise information about the finite p-groups, and hence about the finite nilpotent groups.

This suggests a new problem of determining for each prime p, the "p-group" analogue of Hurwitz theorem. It turns out, as often happens in questions of this nature, that $p = 2$ and $p = 3$ are indeed quite exceptional and harder to deal with while computing their lower central series than other primes. Actually, $p = 3$ is the most difficult, but all the other primes $p \geq 5$ can be dealt with at once.

1.0. Notation and terminology. We adopt the notations of **[7]**, to which this paper is a sequel and to which the reader is referred for further details. We consider Fuchsian groups acting on the upper half of the complex plane. A Fuchsian group Γ has presentation

$$\left\langle x_j, a_k, b_k : x_j^{m_j}, x_1 \ldots x_r \,.\, \prod_k [a_k, b_k], j = 1, \ldots, r, k = 1, \ldots, g \right\rangle \tag{1.0.1}$$

where $[a, b] = aba^{-1}b^{-1}$; g is the genus. We call the symbol

$$S = (g; m_1, \ldots, m_r) \quad (r \geq 0, g \geq 0, m_j \geq 1) \tag{1.0.2}$$

the *signature* of the Fuchsian group Γ. If there are no periods m_i (or if all $m_i = 1$), Γ is called a *surface group*. Let $\Gamma = \Gamma(S)$ be the group of the signature S which acts on the upper half of the complex plane H^2; then Γ has the fundamental region F_Γ of hyperbolic area

$$\mu(F_\Gamma) = 2\pi\left[2g - 2 + \sum_1^r (1 - 1/m_i)\right], \tag{1.0.3}$$

and we call the rational number

$$\chi(S) = 2 - 2g + \sum_{1}^{r} (1/m_i - 1) \tag{1.0.4}$$

its *Euler characteristic*. We call the signature S *degenerate* if

(a) $g = 0$ and $r = 1$ or (b) $g = 0$ and $r = 2$, $m_1 \neq m_2$;

otherwise S will be called *nondegenerate*. If S is nondegenerate and Γ_1 is a subgroup of finite index in Γ, then there exists a signature S_1 such that $\Gamma_1 = \Gamma(S_1)$ and

$$[\Gamma : \Gamma_1] = \chi(S_1)/\chi(S). \tag{1.0.5}$$

A homomorphism Φ from a Fuchsian group $\Gamma(S)$ onto a finite group G will be called *smooth* if it preserves all the periods of Γ, i.e., if for every generator x_i, of order m_i, order of $\Phi(x_i)$ is also equal to m_i. If $\Phi : \Gamma(S) \to G$ is *smooth*, then $\ker \Phi$ is a Fuchsian surface group. A finite group G which has such a homomorphism onto it will be called a *smooth quotient group*. If p is any prime number, then Φ is called *p-smooth* if the order of $\Phi(x_i)$ is divisible by the highest power p^{α_i} of p which divides m_i.

If for each $i = 1, \ldots, r$, p^{α_i} is the highest power of the prime p which divides m_i, then we call the signature $S_p = (g; p^{\alpha_1}, \ldots, p^{\alpha_r})$ the *p-localization* of S. If every period of S is already some power of one fixed prime p, then we call the signature $S = S_p$ a *p-local* signature, and the group defined by S_p, i.e., $\Gamma(S_p)$, the *p-localized Fuchsian group*. We call a signature S *nilpotent-admissible* if every p-local signature S_p of S is nondegenerate.

1.1. Bounds for the order of the p-groups of automorphisms. In this section we study another family of Fuchsian groups, i.e., the Fuchsian groups with a p-local signature and obtain some results about their *nilpotent smooth quotient groups* [**7**].

THEOREM 1.1.1. *Let Γ be a finitely generated cocompact Fuchsian group with a nilpotent-admissible p-local signature $S = (g; p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_r})$, where p is any prime number. Let, as in* [**7**], *$\mu(F_\Gamma) = \mu$ denote the hyperbolic measure of the fundamental region for Γ. Then (i) if $p = 2$, $\mu \geq \pi/4$ (ii) if $p = 3$, $\mu \geq 4\pi/9$ and (iii) if p is any other prime, $\mu \geq 2\pi(p-3)/p$, and equalities occur only when Γ is the* (2, 4, 8), (3, 3, 9) *and* (p, p, p) *triangle group, respectively.*

Proof. If Γ has the above signature, then by (1.0.3) of the previous section

$$\mu = 2\pi\left[2g - 2 + \sum_{j=1}^{r} (1 - 1/p^{\alpha_j})\right]. \tag{1.1.1}$$

Since S is nilpotent-admissible by the first three cases in the proof of theorem (1.8.3) of [**7**], the right hand side of the above is minimal only when $g = 0$ and $r = 3$, i.e., among the triangle groups.

Thus we have the following inequality:

$$\mu \geq 2\pi(1 - 1/p^{\alpha_1} - 1/p^{\alpha_2} - 1/p^{\alpha_3}). \tag{1.1.2}$$

Next we consider three cases.

Case 1. $p = 2$, $S = S_2$ is a 2-local signature; an upper bound for $\sum_1^3 1/2^{\alpha_j}$ is obtained as follows.

(i) $\alpha_1 = \alpha_2 = \alpha_3 = 1$; $\mu < 0$ and Γ cannot be a Fuchsian group.

(ii) $\alpha_1 = \alpha_2 = 1$, $\alpha_3 \geq 2$; again $\mu < 0$ and Γ is not Fuchsian.

(iii) $\alpha_j \geq 2$, $j = 1, 2, 3$; $\mu \geq \pi/2$.

Thus the only possible choice is $\alpha_1 = 1$, $\alpha_2 \geq 2$, $\alpha_3 \geq 2$; $\mu = 2\pi(1/2 - 1/2^{\alpha_2} - 1/2^{\alpha_3})$.

Subcase 1. $\alpha_2 = \alpha_3 = 2$; $\mu = 0$ and Γ is not Fuchsian.

Subcase 2. Only $\alpha_2 = 2$ and $\alpha_3 \geq 3$, which is the only possible choice that remains. Thus $\mu \geq 2\pi(1/2 - 1/4 - 1/8) = \pi/4$, and equality occurs when Γ is the $(2, 4, 8)$ triangle group. This also shows that the bound for the 2-local signatures is the same as the bound for the nilpotent-admissible signatures obtained in [**7**].

Case 2. $p = 3$, $S = S_3$ is a 3-local signature; $\mu = 2\pi(1 - 1/3^{\alpha_1} - 1/3^{\alpha_2} - 1/3^{\alpha_3})$. A minimum cannot occur when $\alpha_1 = \alpha_2 = \alpha_3 = 1$, for this gives $\mu = 0$ and Γ is not Fuchsian. Thus the only possibility is $\alpha_1 = \alpha_2 = 1$, $\alpha_3 \geq 2$: $\mu \geq 2\pi(1 - 1/3 - 1/3 - 1/9) = 4\pi/9$ and equality occurs when Γ is the $(3, 3, 9)$ triangle group.

Case 3. Finally $p \geq 5$, $S = S_p$ is a p-local signature, and a minimum occurs only when $\alpha_1 = \alpha_2 = \alpha_3 = 1$. $\mu \geq 2\pi(1 - 3/p) = 2\pi(p - 3)/p$, and equality occurs when Γ is the (p, p, p) triangle group. This completes the proof.

This leads immediately to the main result. Let p be any prime number.

THEOREM 1.1.2. *Let G_p be any finite p-group acting on some compact Riemann surface X of genus $g \geq 2$. Then if $p = 2$, G_2 has order $|G_2| \leq 16(g - 1)$, if $p = 3$, G_3 has order $|G_3| \leq 9(g - 1)$, and if $p \geq 5$, G_p has order $|G_p| \leq 2p(g - 1)/(p - 3)$. Equalities occur if and only if $X = H^2/\Gamma$, where Γ is a proper normal subgroup of finite index in the group of the signatures* $(0; 2, 4, 8)$, $(0; 3, 3, 9)$ *and* $(0; p, p, p)$, *respectively.*

Proof. By a standard argument (see [**7**, p. 252]) there is a p-local signature S_p and a smooth homomorphism $\phi : \Gamma(S_p) \to G_p$ such that

$$|G_p| = \frac{4\pi(g - 1)}{\mu}.$$

By Theorem 1.1.1, if $p = 2$, $\mu \geq \pi/4$, if $p = 3$, $\mu \geq 4\pi/9$, and if $p \geq 5$, $\mu \geq 2\pi(p - 3)/p$, with equalities if and only if $S_2 = (0; 2, 4, 8)$, $S_3 = (0; 3, 3, 9)$ and $S_p = (0; p, p, p)$ for $p \geq 5$, respectively. The results now follow.

The following theorem is a special case of Theorem 2.1.1 of [**7**, p. 252].

THEOREM 1.1.3. *Let $S_p = (0; p^{\alpha_1}, \ldots, p^{\alpha_r})$ be any p-local signature with genus zero. Then every nilpotent automorphism group covered by $\Gamma(S_p)$ is a p-group.*

1.2. 2-groups of automorphisms of Riemann surfaces. In view of Theorem 1.1.2 of the last section, which shows that the upper bound for the order of the 2-groups of automorphisms of Riemann surfaces is the same as the upper bound for the nilpotent groups of automorphisms, it is natural to say that every nilpotent automorphism group covered by the group of the signature $(0; 2, 4, 8)$ must be a 2-group. Thus if a surface of genus $g \geq 2$ admits a nilpotent automorphism group of order $16(g-1)$, then $g-1$ must be a power of 2.

The structure of the above group has been studied in detail in a previous paper by this author and for a proof to the following theorem the reader is referred to [**7**, p. 254].

THEOREM 1.2.1. *For any integer $n \geq 4$, there exists a 2-group G of order 2^n acting on a compact Riemann surface X of genus $g = 2^{n-4} + 1$.*

1.3. 3-groups of automorphisms of Riemann surfaces. In the second part of Theorem 1.1.2 we discovered that if G_3 is any finite 3-group acting on some compact Riemann surface X of genus $g \geq 2$, then the order of G_3 cannot exceed $9(g-1)$. We now consider the case of equality in more detail.

The argument in Theorem 1.1.2 shows that if $|G_3| = 9(g-1)$, then $X = H^2/\Gamma$ where Γ is a proper normal subgroup of finite index in the group of $(0; 3, 3, 9)$. Let $S_3 = (0; 3, 3, 9)$, then the Euler characteristic of S_3, i.e., $\chi(S_3) = -2/9$, which is negative. Thus by [**5**, Corollary 6.6, p. 307], $\Gamma(S_3)$ can cover infinitely many Riemann surface automorphism groups which are finite 3-groups. Therefore, if a surface of genus g admits a nilpotent automorphism group G of order $9(g-1)$, then $g-1$ must be a power of 3, i.e., $|G| = 9(g-1) = 3^n$, which implies that the genus g is given by $g = 3^{n-2} + 1$, where $n \geq 2$.

Next in the above if $n = 2$, then $|G| = 9$, and $g = 2$, and hence $\Gamma(S_3)$ must have the presentation $\langle P, Q \mid P^9 = Q^3 = (PQ)^3 = 1 \rangle$. But then G must have 2 distinct subgroups of order 3, i.e., $G = Z_3 \oplus Z_3$, hence it cannot have an element of order 9. This implies that $\Gamma(0; 3, 3, 9)$ cannot cover a group of order 9 with genus $g = 2$.

Also if $n = 3$, then $|G| = 27$, and $g = 4$. But there is only one non-abelian group of order p^3 (p any odd prime) which contains an operation of order p^2 (see W. Burnside [**1**]) and this group is given by $G = \langle \alpha, \beta \mid \alpha^9 = \beta^3 = 1, \beta^{-1}\alpha\beta = \alpha^4 \rangle$. Now since all elements of order 3 must lie in the subgroup $H = \langle \alpha^3 \rangle \oplus \langle \beta \rangle \simeq Z_3 \oplus Z_3$ of G, it follows that the product of any two elements of order 3 in G is again an element of order 3 or the identity element. Therefore, G cannot be a $(3, 3, 9)$-group. This implies that the first possible value of n is 4. But if $n = 4$, then $|G| = 81$ and $g = 10$. Burnside [**1**] proves that there are fifteen types of group of order p^4 where p is any odd prime. It will be sufficient for our purpose in this paper to choose from the Burnside's list for the case $p = 3$ the group with the following presentation:

$$G = \langle a, b, c \mid a^9 = b^3 = c^3 = 1, ab = ba, ac = cab, bc = ca^{-3}b \rangle. \tag{1.3.1}$$

And it can be checked easily that ac has order 3, and $b = a^{-1}c^{-1}ac$, hence G is a $(3, 3, 9)$ group. Next let Γ the group of the signature $(0; 3, 3, 9)$ be generated by P and Q where $P^9 = Q^3 = (PQ)^3 = 1$. Let $\Psi: \Gamma \to G$ be a homomorphism defined by $\Psi(P) = a$, $\Psi(Q) = c$, and $\Psi(PQ) = ac$. Since every element of finite order belonging to $\ker \Psi$ must be conjugate to some power of P, Q or PQ, we have that Ψ is *smooth*. Therefore ker Ψ is a Fuchsian surface group of genus $g = 10$, and G is a *smooth quotient group* of order 81 for this Fuchsian group Γ.

We denote the above kernel by N_1 and use it as the first step in the induction argument of the lemma (2.1.1) given in [**7**, p. 253].

Therefore as in the case of 2-groups of automorphisms we now have the following Existence Theorem on 3-groups of automorphisms of compact Riemann surfaces.

THEOREM 1.3.1 (Existence). *For any integer $n \geq 4$, there exists a nilpotent 3-group G of order 3^n acting on a compact Riemann surface X of genus $g = 3^{n-2} + 1$.*

Proof. Similar procedure as in the proof of the theorem (2.1.2) given in [**7**, p. 254].

1.4. The descending central series of the groups $(0; 2, 4, 8)$ **and** $(0; 3, 3, 9)$. In this section we investigate the structure of the $(2, 4, 8)$ and $(3, 3, 9)$ groups and to do this, we closely examine the first few terms in the lower central series of these groups. Let Γ be the group of the signature $(0; 2, 4, 8)$ or that of $(0; 3, 3, 9)$; then the descending central series of Γ is defined by

$$\Gamma = \gamma_1(\Gamma) \supset \gamma_2(\Gamma) \supset \gamma_3(\Gamma) \supset \ldots \supset \gamma_k(\Gamma) \supset \ldots \qquad (1.4.1)$$

where

$$\gamma_{k+1}(\Gamma) = [\gamma_k(\Gamma), \Gamma] \qquad (k = 1, 2, \ldots),$$

and where all the subgroups are normal, indeed characteristic, in Γ and we set

$$G_{i,i+1} = \gamma_i(\Gamma)/\gamma_{i+1}(\Gamma), \quad \text{and} \quad \phi_{i,i+1} : \gamma_i(\Gamma) \to G_{i,i+1} \qquad (1.4.2)$$

where $\phi_{i,i+1}$ is the canonical homomorphism for all $i = 1, 2, \ldots$, and $G_{i,i+1}$ is the quotient group of $\gamma_i(\Gamma)$ by $\gamma_{i+1}(\Gamma)$. Since the p-local signatures are nondegenerate with $\chi(S_p) \leq 0$, the terms in the lower central series are strictly decreasing (see [**5**, p. 310]). There is more than one way of determining these terms. The main technique used here to obtain the results in the following two theorems is the algorithm of Reidermeister–Schreier by which one can obtain generators and relators for subgroups of given groups (see for instance [**2**]). We apply several Tietze transformations to eliminate the number of generators in some of the subgroups and use a lemma by A. H. M. Hoare et al [**3**] on some long words being a product of a finite number of some commutators. Since the calculations of the results obtained in the next two theorems are very involved and tedious, the details are suppressed.

THEOREM 1.4.1. *The terms $\gamma_r(\Gamma)$ of the lower central series of the Fuchsian group $\Gamma = \Gamma(0; 2, 4, 8)$ satisfy the following*:

$\gamma_2(\Gamma)$ *has the signature* $(1; 2, 2)$, *and the quotient group* $\Gamma/\gamma_2(\Gamma)$ *has order* 8;

$\gamma_3(\Gamma)$ *has signature* $(1; 2, 2, 2, 2)$, *and the quotient group* $\Gamma/\gamma_3(\Gamma)$ *has order* 16;

$\gamma_4(\Gamma)$ *has signature* $(5; \ldots)$, *i.e., the first surface group in the lower central series of this group and the first smooth quotient group* $\Gamma/\gamma_4(\Gamma)$ *has order* 64, *and if we assume the*

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generators for this smooth quotient group are P and Q it has the presentation

$$\Gamma/\gamma_4(\Gamma) \simeq \langle P^2, Q^8, (PQ)^4, [PQ^2P, Q] \rangle,$$

where $[A, B] = ABA^{-1}B^{-1}$. $\gamma_n(\Gamma)$ *has signature* $(2^{K_n}+1; \ldots)$, *and the quotient group* $\Gamma/\gamma_n(\Gamma)$ *has order* 2^{K_n+4} *where* K_n *is a certain strictly increasing function of n* $(K_4 = 2)$.

COROLLARY 1.4.1. *If G is a 2-group of automorphisms of a Riemann surface covered by the group of the signature* $(0; 2, 4, 8)$, *then the nilpotency class of G is* ≥ 3.

THEOREM 1.4.2. *The terms* $\gamma_r(\Gamma)$ *of the lower central series of the group* $\Gamma = \Gamma(0; 3, 3, 9)$ *satisfy the following*:

$\gamma_2(\Gamma)$ *has signature* $(1; 3, 3, 3)$, *and the quotient group* $\Gamma/\gamma_2(\Gamma)$ *has order* 9;

$\gamma_3(\Gamma)$ *has signature* $(1; 3, 3, 3, 3, 3, 3, 3, 3, 3)$, *and the group* $\Gamma/\gamma_3(\Gamma)$ *has order* 27;

$\gamma_4(\Gamma)$ *has signature* $(28; \ldots)$ *i.e., the first surface group in the lower central series of this group and the first smooth quotient group* $\Gamma/\gamma_4(\Gamma)$ *has order* 243.

$\gamma_n(\Gamma)$ *has signature* $(3^{K_n}+1; \ldots)$, *where* K_n *is a certain strictly increasing function of n* $(K_4 = 3)$. *The quotient group* $\Gamma/\gamma_n(\Gamma)$ *has order* 3^{K_n+2}.

COROLLARY 1.4.2. *If G is a 3-group of automorphisms of a Riemann surface covered by the group of the signature* $(0; 3, 3, 9)$, *then the nilpotency class of G is* ≥ 3.

REMARK. From the results obtained in the above two theorems one should observe that in both cases, the 2-groups and the 3-groups of automorphisms of Riemann surfaces, the first surfce group occurs after 3 steps. Moreover there exists a 2-group of order 2^6 with the presentation

$$\langle P, Q \mid P^2 = Q^8 = (PQ)^4 = [PQ^2P, Q] = 1 \rangle$$

acting as a group of automorphisms of S of genus $g = 5$, and a 3-group of order 3^5 with the presentation

$$\langle P, Q \mid P^9 = Q^3 = (PQ)^3 = [P, Q]^3 = [P, [P, Q]]^3 = [Q, [P, Q]]^3 = 1 \rangle$$

which can act as a group of automorphisms of a Riemann surface of genus $g = 28$.

But as we shall see in the next section all other p-groups of automorphisms of Riemann surfaces can be dealt with at once.

2.0. p-Groups of automorphisms of compact Riemann surfaces when $p \geq 5$. In this section, our object is to investigate other p-groups of automorphisms of a compact Riemann surface by considering primes different from 2 and 3, and complete our theory of p-groups of automorphisms.

We shall observe, however, that the primes 2 and 3 have been indeed quite exceptional in this classification of p-groups. We recall the last part of Theorem 1.1.2 of §1.1, which shows that if p is any prime other than 2 and 3, and G_p is any p-group acting on a surface X of genus $g \geq 2$, then the order of G_p cannot exceed $2p(g-1)/(p-3)$. And equality occurs if and only if $X \simeq H^2/\Gamma$, where H^2 is the upper half complex plane, and Γ is a proper normal subgroup of finite index in the (p, p, p)-triangle group. In the following we consider the case of equality in more detail. But first of all from Theorem 1.1.3 of §1.1 we can conclude that every nilpotent automorphism group covered by the group of the signature $(0; p, p, p)$ must be a p-group. Thus if a surface of genus $g \geq 2$ admits a nilpotent automorphism group G of order $2p(g-1)/(p-3)$, then $(2g-2)/(p-3)$ must be a power of p, i.e., we need to have the equation

$$|G| = 2p(g-1)/(p-3) = p^n,$$

which implies that the genus g is given by $g = p^{n-1}(p-3)/2 + 1$, for all prime $p \geq 5$. Next we shall show that the integer n can take all positive values. But first let us examine the case $n = 1$. If $n = 1$, then $|G| = p$, i.e., $G \simeq Z_p$ which is the unique cyclic group of order p, and the genus $g = (p-1)/2$.

But since in this case $p \geq 5$, the group G contains more than 3 elements of order p. Suppose Γ is a (p, p, p)-group generated by x_1 and x_2, such that

$$x_1^p = x_2^p = (x_1x_2)^p = 1. \tag{2.0.1}$$

Let $\lambda : \Gamma \to Z_p$ be a homomorphism defined by $\lambda(x_1) \equiv 1 \pmod p$, $\lambda(x_2) \equiv 2 \pmod p$, and $\lambda(x_1x_2) \equiv 3 \pmod p$. Thus as in the previous section λ is smooth, and so $\ker \lambda$ is torsion free. Hence $\ker \lambda$ is a Fuchsian surface group of genus $g = (p-1)/2$, and Z_p is a *smooth quotient group* for $\Gamma(0; p, p, p)$. As in the cases of 2-groups and 3-groups we may now denote this kernel by N_1 and use it in the first step in the induction argument of the lemma (2.1.1) in [**7**, p. 253]. Thus we have the following result.

THEOREM 2.0.1. *Let Γ be a (p, p, p)-triangle group. Then for any positive integer n there exists a p-group of order p^n acting on some compact Riemann surface of genus $g = p^{n-1}(p-3)/2 + 1$.*

2.1. The structure of (p, p, p)-triangle groups for any prime $p \geq 5$. In this final section we examine the lower central series of the (p, p, p)-groups for the sake of completeness. We use a general theorem to prove that if p is any prime other than 2 and 3, then beginning with the derived group $\Gamma' = \gamma_2(\Gamma)$ all the terms in the lower central series of the group $\Gamma(0; p, p, p)$ are surface groups.

We require the following theorem by C. Maclachlan [**6**, p. 701].

THEOREM 2.1.1. *The derived group Γ' of a Fuchsian group $\Gamma(g; m_1, m_2, \ldots, m_r)$ is a surface group if and only if the m_i satisfy the* L.C.M. *condition, i.e., for each $i = 1, 2, \ldots, r$, $m_i \mid$* l.c.m. *of $\{m_1, m_2, \ldots, \hat{m}_i, \ldots, m_r\}$ where the roof indicates omission of the symbol underneath it.*

THEOREM 2.1.2. *Let $p \geq 5$ be any prime number. Let Γ be the group of the signature $S_p = (0; p, p, p)$; then all the terms $\gamma_r(\Gamma)$, $r \geq 2$ in the lower central series of Γ are surface groups and satisfy the following*:

$\gamma_2(\Gamma) = \Gamma'$ has signature $((p-1)(p-2)/2; \ldots)$, and the first smooth quotient group $\Gamma/\gamma_2(\Gamma)$ has order p^2 and the presentation $\langle x_1, x_2 \mid x_1^p, x_2^p, [x_1, x_2] \rangle$;

$\gamma_n(\Gamma)$ for all $n > 2$ has signature $(p^{K_n}(p-3)/2 + 1; \ldots)$ and the smooth quotient

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group $\Gamma/\gamma_n(\Gamma)$ *has order* p^{K_n+1}, *where* K_n *is a certain strictly increasing function of* n $(K_2=1)$. *These factor groups are of course all nilpotent smooth* p*-groups.*

Proof. Since the periods of Γ satisfy the L.C.M. condition by Theorem 2.1.1 of Maclachlan Γ' is a surface group, i.e., $\Gamma'=\gamma_2(\Gamma)\simeq\Gamma(g';\ )$. Next using the notation of §1.4, let $\phi_{12}:\Gamma\to\Gamma/\Gamma'$ be the canonical homomorphism where $\Gamma'=[\Gamma,\Gamma]$. To find the presentation for Γ/Γ' we simply abelianize the relators (2.0.1) and we obtain the relators

$$p\phi_{12}(x_1)=p\phi_{12}(x_2)=p\phi_{12}(x_1)+p\phi_{12}(x_2)=0.$$

But these relators yield $p\phi_{12}(x_1)=p\phi_{12}(x_2)=0$, which implies $\Gamma/\Gamma'\simeq Z_p\oplus Z_p$. Next using the identity (1.0.5) of §1.0, we obtain

$$[\Gamma:\Gamma']=p^2=\chi(\Gamma')/\chi(\Gamma)=(2-2g')/(3/p-1)$$

which implies the genus $g'=(p-1)(p-2)/2$. Thus $\gamma_2(\Gamma)$ has signature $((p-1)(p-2)/2;\ldots)$, and the smooth quotient group Γ/Γ' has the claimed presentation. It follows finally that all the terms $\gamma_n(\Gamma)$ for $n>2$ in the series, being subgroups of Γ', are also Fuchsian surface groups, and by (1.0.5) have signatures $(p^{K_n}(p-3)/2+1;\ldots)$, and the quotient groups $\Gamma/\gamma_n(\Gamma)$ are nilpotent smooth p-groups of order p^{K_n+1} where K_n is a certain function of n which is strictly increasing $(K_2=1)$.

REFERENCES

1. W. Burnside, *Theory of groups of finite order* (New York, 1955).

2. D. L. Johnson, *Presentation of groups,* London Mathematical Society Lecture Note Series No. 22.

3. A. H. M. Hoare, A. Karrass and D. Solitar, Subgroups of infinite index in Fuchsian groups, *Math. Z.* **125** (1972), 59–69.

4. A. Hurwitz, Ueber algebraische Gebilde mit eindeutigen Transformationen in sich, *Math. Ann.* **41** (1892), 403–42.

5. A. M. Macbeath, Residual nilpotency of Fuchsian groups, *Illinois J. Math.* (2) **28** (1984), 299–311.

6. C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces, *Proc. London Math. Soc.* (3) **15** (1965), 699–712.

7. R. Zomorrodian, Nilpotent automorphism groups of Riemann surfaces, *Trans. Amer. Math. Soc.* **288** (1985), 241–255.

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