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VERTICALITY OF (−1)-LINES IN SCROLLS OVER SMOOTH SURFACES

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ABSTRACT. Let S be a smooth surface contained as an ample divisor in a smooth complex projective threefold X, which is a $\mathbf{P}^1$-bundle, and assume that $\mathfrak{L} = O_X(S)$ induces $O_{\mathbf{P}^1}(1)$ on the fibres of X. The following fact is proven. The restriction to S of the bundle projection of X is exactly the reduction morphism of the pair $(S, \mathfrak{L}_S)$, provided that this one is not a conic bundle. The proof is very simple and does not involve any consideration on the nefness of the adjoint bundle $\mathbf{K}_X \otimes \mathfrak{L}^2$. Some applications of the proof are given.

Statement of the results Let $p : X \longrightarrow S^{\sim}$ be a $\mathbf{P}^1$-bundle over a smooth complex projective surface $S^{\sim}$ and let $\mathfrak{L}$ be an ample line bundle on X satisfying the following conditions:

i) $\mathfrak{L}$ induces $O_{\mathbf{P}^1}(1)$ on every fibre f of p, and

ii) the complete linear system $|\mathfrak{L}|$ contains a smooth surface S.

If i) holds, the polarized threefold $(X, \mathfrak{L})$ is said to be a scroll over $S^{\sim}$. If also ii) holds, it is known (e.g. see [6, Lemma I-A], or [5, (6.5)]) that S is a meromorphic non-holomorphic section of p, i.e. S meets the general fibre of p at a single point, but must contain some fibres. This means that the morphism $p_{|S}: S \longrightarrow S^{\sim}$ contracts some (-1)-curves of S, which, due to i), are (-1)-lines of the polarized surface (S, L), where $L = \mathfrak{L}_S$. Assume that (S, L) is not a conic bundle. Then, as is known, all the (-1)-lines of (S, L) are disjoint and the birational morphism $r: S \longrightarrow S'$ contracting all of them gives rise to a smooth surface S' containing an ample line bundle L' such that $K_S \otimes L = r^*(K_{S'} \otimes L')$. The pair (S', L') is usually referred to as the reduction of (S, L). The above description implies that the reduction morphism r factors through $p_{|S}$. The main result I prove in this note is that in fact $r = p_{|S}$. In other words,

THEOREM 1. *Let X, $\mathfrak{L}$, S and L be as before. If (S, L) is not a conic bundle, then $p_{|S}$ contracts all the (-1)-lines of (S, L).*

This result is well known at least when $\mathfrak{L}$ is very ample and perhaps also when $\mathfrak{L}$ is ample and spanned by its global sections. Actually in these cases, it could be proved by using general results in adjunction theory [8, secs. 2, 3]. When $\mathfrak{L}$ is merely ample and condition ii) is fulfilled (e.g. it can happen that S is the unique member of $|\mathfrak{L}|$) it could be proved again by using the general theory. Actually, by [4], [1], we know that $K_X \otimes \mathfrak{L}^2$

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is nef apart from some simple exceptions. Continuing the proof from this point on is not difficult, though it will take about as much space as the proof I give here, which is very direct, elementary and does not involve any consideration on the nefness of $K_X \otimes \mathfrak{L}^2$.

Note that Th. 1 immediately extends to higher dimensions in the following form.

THEOREM 1′. *Let $p\colon Y \longrightarrow S^{\sim}$ be a $\mathbf{P}^{n-2}$-bundle over a smooth projective surface $S^{\sim}$ and let $\mathfrak{L}$ be an ample line bundle on Y such that: j) $\mathfrak{L}$ induces $O_{\mathbf{P}^{n-2}}(1)$ on every fibre of p and jj) there is a smooth surface S, which is the transverse intersection of $n-2$ smooth elements of $|\mathfrak{L}|$. Then $p_{|S}$ is just the reduction morphism of $(S, \mathfrak{L}_S)$.*

A slight modification of the argument proving Theorem 1, allows me to prove also

THEOREM 2. *Let X, $\mathfrak{L}$, S and L be as before and assume that (S, L) is a conic bundle over some smooth curve $B \neq \mathbf{P}^1$. Then $(X, \mathfrak{L})$ has the structure of a quadric bundle over B.*

As a consequence of Theorem 2 and of [7, Thm. IV] we have the following fact (in the special setting of very ample line bundles e.g. see [3, sec. 1]).

COROLLARY. *Let X be a smooth complex projective threefold and let $\mathfrak{L}$ be an ample line bundle on X satisfying ii). Then $(X, \mathfrak{L})$ is a quadric bundle over a smooth curve $B \neq \mathbf{P}^1$ iff $(S, \mathfrak{L}_S)$ is a conic bundle over B.*

Note again that the sufficient condition for $(X, \mathfrak{L})$ to be a quadric bundle given by the above statement does not involve any consideration on the nefness of $K_X \otimes \mathfrak{L}^2$. This fact can be useful in some instances.

PROOFS

(1.1) PROOF OF THEOREM 1. By contradiction, assume that $p_{|S}$ does not contract a (-1)-line E. Then, since all (-1)-lines of (S, L) are disjoint, $E^{\sim} = p(E)$ is a (-1)-line of $(S^{\sim}, L^{\sim})$, where $L^{\sim}$ is the ample line bundle on $S^{\sim}$ defined by

$$K_S \otimes L = p_{|S}^{*}(K_{S^{\sim}} \otimes L^{\sim}). \tag{1.1.1}$$

Consider the rational $\mathbf{P}^1$-bundle $F = p^*E^{\sim}$ and note that E is a section of F. Moreover

$$\mathfrak{L}_F = O_F(E). \tag{1.1.2}$$

In particular, due to the ampleness of $\mathfrak{L}$, E is an ample, hence a very ample, divisor in F [**2**, p. 380]. Now note that

$$K_X = \mathfrak{L}^{-2} \otimes p^*(K_{S^{\sim}} \otimes L^{\sim}). \tag{1.1.3}$$

Actually, by adjunction, $K_{X_f} = K_f = O_{P^1}(-2)$, so that $K_X \otimes \mathfrak{L}^2$ restricts trivially to the fibres of X. Hence $K_X \otimes \mathfrak{L}^2 = p^*D$, for some line bundle D on $S^{\sim}$. On the other hand, by restricting $K_X \otimes \mathfrak{L}^2$ to S, adjunction gives $K_S \otimes L = p_{|S}^* D$, and by comparing this with (1.1.1) we get (1.1.3). Note also that

$$(p^*(K_{S^\sim} \otimes L^\sim))_F = \mathcal{O}_F \quad \text{and} \quad \mathcal{O}_X(F) = \mathcal{O}_F(-f),$$

since $E^\sim$ is a (-1)-line of $(S^\sim, L^\sim)$. Then, by restricting (1.1.3) to F, by adjunction and by (1.1.2) we get

$$K_F = (K_X \otimes \mathcal{O}_X(F))_F = (\mathfrak{L}^{-2})_F \otimes \mathcal{O}_F(-f) = \mathcal{O}_F(-2E - f).$$

Therefore $c_1(F)^2 = 4E^2 + 4$. On the other hand it must be $c_1(F)^2 = 8$, since F is a rational $\mathbf{P}^1$-bundle, and this implies $E^2 = 1$, contradicting the very ampleness of E on F. ∎

(1.2) Proof of Theorem 2. Let $\pi: S \longrightarrow B$ be the ruling projection of the conic bundle (S, L). Note that all irreducible components of the reducible fibres of π are (-1)-lines, but S should also contain some (-1)-line transversal to the fibres of π. In this case however it would be $B = \mathbf{P}^1$, contradiction. Let $L^\sim$ be the ample line bundle on $S^\sim$ defined by (1.1.1). Since $p_{|S}$ contracts only (-1)-lines contained in the fibres of π, we have that $(S^\sim, L^\sim)$ is again a conic bundle over B and π factors through $p_{|S}$ and the ruling projection $\pi^\sim : S^\sim \longrightarrow B$ of $(S^\sim, L^\sim)$. Moreover the composite map $\pi^\sim \circ p : X \longrightarrow B$ fibres X over B with general fibre a rational $\mathbf{P}^1$-bundle F. We have to show that (F, L_F) is a quadric of $\mathbf{P}^3$ polarized by its hyperplane bundle. To see this note that (1.1.3) does not depend on the assumption of Theorem 1. Furthermore we have $\mathcal{O}_X(F)_F = \mathcal{O}_F$, since F is a fibre of $\pi^\sim \circ p$; in addition $(p^*(K_{S^\sim} \otimes L^\sim))_F = \mathcal{O}_F$, since $K_{S^\sim} \otimes L^\sim$ restricts trivially to the fibres of $\pi^\sim$. Then, by adjunction, we get from (1.1.3)

$$K_F = (K_X \otimes \mathcal{O}_X(F))_F = (K_X)_F = (\mathfrak{L}^{-2})_F.$$

Since $\mathfrak{L}_F$ is ample, and thus very ample [2, p. 380], this shows that F is a Del Pezzo surface of index 2, and this means exactly that (F, L_F) is a quadric of $\mathbf{P}^3$ polarized by its hyperplane bundle. This completes the proof. ∎

(1.3) Proof of Corollary. We only sketch the proof of the if part, the converse being trivial. Since S is a ruled surface it follows from [7, Thm. IV] that either $(X, \mathfrak{L})$ admits a reduction $(X', \mathfrak{L}')$ belonging to a precise list, or $(X, \mathfrak{L})$ is a scroll over a smooth surface. In the former case one immediately checks that $(X', \mathfrak{L}')$ and hence $(X, \mathfrak{L})$ is a quadric bundle. In the latter case the assertion follows from Theorem 2. ∎

References

1. T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive*. Adv. Studies in Pure Math. **10**, 1987. Algebraic Geometry, Sendai, 1985, 167–178.
2. R. Hartshorne, *Algebraic Geometry*. Springer Verlag, New York, 1977.
3. P. Ionescu, *Embedded projective varieties of small invariants*. Algebraic Geometry, Proc. Bucharest 1982, 142–186. Lect. Notes Math. 1056. Springer Verlag, New York, 1984.
4. P. Ionescu, *Generalized adjunction and applications*. Math. Proc. Cambridge Phil. Soc. **99** (1986), 457–472.

5. A. Lanteri, D. Struppa, *Projective manifolds whose topology is strongly reflected in their hyperplane sections*. Geometriae Dedicata, **21** (1986) 357–374.

6. A. J. Sommese, *On manifolds that cannot be ample divisors*. Math. Ann., **221** (1976), 55–72.

7. A. J. Sommese, *Ample divisors on threefolds*. Algebraic Threefolds, Proc. Varenna 1981. Lect. Notes Math. **947**, 229–240. Springer Verlag, New York, 1982.

8. A. J. Sommese, *On the adjunction theoretic structure of projective varieties*, Complex Analysis and Algebraic Geometry, Proc. Göttingen 1985. Lect. Notes Math **1194**, 175–213. Springer Verlag, New York, 1986.

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