AN ASYMPTOTIC EXPANSION FOR A CLASS OF MULTIVARIATE NORMAL INTEGRALS*

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1. Introductory Discussion and Summary.

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ be a normal random vector with zero expectation vector and with a variance-covariance matrix which has 1 for its diagonal elements and ρ for its off-diagonal elements. Consider the quantity

$$(1.1)\quad I_n(h;\rho) = (2\pi)^{-n/2}\{1+(n-1)\rho\}^{-1/2}(1-\rho)^{-(n-1)/2}\int_h^\infty \cdots \int_h^\infty e^{-Q(\boldsymbol{x})/2}dx_1\cdots dx_n,$$

where

$$(1.2)\quad \begin{aligned} Q(\boldsymbol{x}) &= [\{1+(n-1)\rho\}(1-\rho)]^{-1}[\{1+(n-2)\rho\}\sum x_i^2 - 2\rho \sum_{j>i} x_i x_j] \\ &= (1-\rho)^{-1}[\sum x_i^2 - \rho\{1+(n-1)\rho\}^{-1}(\sum x_i)^2]. \end{aligned}$$

Thus $I_n(h;\rho)$ is the probability that each of n normally distributed, equally correlated and standardized random variables with common correlation ρ shall not fall short of h. Clearly $1 - I_n(h;\rho)$ is also the distribution function of the random variable $\max_i \mathrm{x}_i$, and this supplies one application (cf. [3]) of $I_n(h;\rho)$. A second application relates to the familiar one-factor model in factor analysis for the special case of equal weights [8]. Another situation in which knowledge of $I_n(h;\rho)$ is important is in some models of test design in psychology. Other applications will arise or probably exist at present.

In a previous paper [8] (see also [8] for further references), $I_n(h;\rho)$ was expressed as the product of the density funcion of $\boldsymbol{x}$ at the cut-off point $\boldsymbol{h} = (h, h, \cdots, h)$ and an infinite power series in h. In this paper it will be shown for $h > 0$ that $I_n(h;\rho)$ can be expressed asymptotically as the product of the density function at $\boldsymbol{h}$ and an infinite series in negative powers of h. This result can be regarded as the generalization for $n > 1$ of the well-known asymptotic expansion of Mill's ratio

$$(1.3)\quad \int_x^\infty e^{-t^2/2}dt/e^{-x^2/2} \sim x^{-1}(1 - x^{-2} + 1.3x^{-4} - 1.3.5x^{-6} + \cdots) \qquad (x > 0).$$

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2. The Asymptotic Development of $I_n(h;\rho)$

Under the transformation

$$(2.1)\qquad \begin{aligned} y_1 &= [1+(n-1)\rho]^{-1/2}\sum_{j=1}^{n} b_{1j}x_j, \\ y_i &= (1-\rho)^{-1/2}\sum_{j=1}^{n} b_{ij}x_j \qquad (i=2,3,\cdots,n), \end{aligned}$$

where $((b_{ij}))$, $i, j = 1, 2, \cdots, n$, is orthogonal with $b_{1j} = n^{-1/2}$, (1.1) reduces to

$$(2.2)\qquad I_n(h;\rho) = (2\pi)^{-n/2}\int\underset{R}{\cdots}\int e^{-\Sigma y_i^2/2}dy_1\cdots dy_n$$

with R defined by

$$(2.3)\quad R\colon [1+(n-1)\rho]^{1/2}[n(1-\rho)]^{-1/2}y_1+\sum_{j=2}^{n} b_{ji}y_j \geqq (1-\rho)^{-1/2}h \qquad (i=1,2,\cdots,n)$$

[8]. R is a polyhedral half-cone in y-space with vertex at the point $(r_0, 0, 0, \cdots, 0)$, where

$$(2.4)\qquad r_0 = [n/\{1+(n-1)\rho\}]^{1/2}h,$$

such that the angle between any two faces of the cone is arc cos $(-\rho)$; further, the axis of the cone passes through the origin in y-space. $I_n(h;\rho)$ is, then, the probability measure, under an n-dimensional spherical normal distribution with unit standard deviation in any direction, of a regular, symmetrically oriented polyhedral half-cone with common dihedral angle arc cos $(-\rho)$, and with vertex at a distance r_0 from the centre of the distribution. Let P be any point within the cone distant r from the centre of the distribution, η from the axis of the cone and x from the vertex of the cone in a direction parallel to the axis. The probability-mass of an *infinitesimal* element of volume $d\tau$ at P is

$$(2.5)\qquad (2\pi)^{-n/2}e^{-r^2/2}d\tau = (2\pi)^{-1/2}e^{-(r_0+x)^2/2}dx.\ (2\pi)^{-(n-1)/2}e^{-\eta^2/2}dS,$$

where dS is the measure of an infinitesimal element in the $(n-1)$-flat orthogonal to the axis of the cone and distant x from the vertex (cf. [5]). Consider the probability-mass in that portion of the cone (an infinitesimal "slab") demarcated by two adjoining $(n-1)$-flats orthogonal to the axis of the cone and distant x and $x+dx$ from the vertex of the cone. It is easily shown that the intersection of the first of these two flats with the cone is a regular $(n-1)$-dimensional simplex with centroid at the foot of the perpendicular from P to the axis of the cone and with edges of length

$$[2n\{1+(n-1)\rho\}/(1-\rho)]^{1/2}x.$$

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Let $K_N(l)$ denote the probability measure, under an N-dimensional spherical normal distribution with unit standard deviation in any direction, of a regular N-dimensional simplex with centroid at the centre of the distribution and with edges of length l. Then according to (2.5) the probability measure of the infinitesimal slab is

$$(2.6)\qquad (2\pi)^{-1/2}e^{-(r_0+x)^2/2}dx\cdot K_{n-1}\left[\left(\frac{2n\{1+(n-1)\rho\}}{1-\rho}\right)^{1/2}x\right].$$

Consequently, the probability measure of the cone is

$$(2.7)\qquad \begin{aligned} I_n(h;\rho) &= \int_0^\infty (2\pi)^{-1/2}e^{-(r_0+x)^2/2}K_{n-1}\left[\left(\frac{2n\{1+(n-1)\rho\}}{1-\rho}\right)^{1/2}x\right]dx \\ &= (2\pi)^{-1/2}e^{-r_0^2/2}\int_0^\infty e^{-r_0x}\cdot e^{-x^2/2}K_{n-1}(\lambda x)dx, \end{aligned}$$

where

$$(2.8)\qquad \begin{aligned} \lambda &\equiv \lambda_n(\rho) \\ &= [2n\{1+(n-1)\rho\}/(1-\rho)]^{1/2} \end{aligned}$$

and r_0 is given by (2.4). Formula (2.7) which is of considerable intrinsic interest may be used also to develop the required asymptotic expansion of $I_n(h;\rho)$ for $h>0$. From here on we shall then assume that $h>0$ [1].

The K-functions are closely related to Godwin's G-function [1], [2] introduced in connection with the distribution of the absolute mean deviation in normal samples, and some further statistical applications of the functions have been discussed in [4] and [5]. Clearly, $K_N(x)$ is bounded by 1. Again, it has been shown elsewhere [7] that $K_N(x)$ has a power series expansion with infinite radius of convergence. Consequently, Watson's lemma [10] (p. 236) may be used to obtain a valid asymptotic expansion for the integral in (2.7) by expanding $\exp(-x^2/2)K_{n-1}(\lambda x)$ in its Taylor series at $x=0$ and integrating term by term. In fact, let

$$(2.9)\qquad \psi_{n-1}(x)\equiv\psi_{n-1}(x;\lambda)\equiv e^{-x^2/2}K_{n-1}(\lambda x)=\sum_{i=0}^\infty c_{n-1,i}x^i/i!,$$

where the $c_{n-1,i}$ are functions of λ (and therefore of ρ). Then (2.7) gives with the aid of Watson's lemma,

[1] The centre of the distribution is interior or exterior to the halfcone according as to whether $h<0$ or $h>0$. The integral formula for $I_n(h;\rho)$ in (2.7) is valid for all h, but for the asymptotic expansion developed subsequently (equ. (2.22)) $h>0$. ($I_n(0;\rho)$ is known to be equal to the normed measure of a regular $(n-1)$-dimensional spherical simplex with common dihedral angle arc cos $(-\rho)$. The reader is referred to [9] where tables of such normed measures are provided for $n=1(1)\,51-i$ and $\rho=1/i$, $i=1(1)\,12$.)

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$$I_n(h;\rho) \sim (2\pi)^{-1/2} e^{-r_0^2/2} \sum_{i=0}^{\infty} c_{n-1,i} r_0^{-(i+1)}. \tag{2.10}$$

This is the required formula. It should be noted that the probability density in the original distribution at the point $(h, h, \cdots, h)$ is

$$(2\pi)^{-n/2}\{1+(n-1)\rho\}^{-1/2}(1-\rho)^{-(n-1)/2}e^{-r_0^2/2}, \tag{2.11}$$

thereby justifying the assertion at the end of the introductory Section.

It now remains to determine the coefficients $c_{n-1,i}$ in (2.10) $(c_{n-1,i} = \psi_{n-1}^{(i)}(0))$. On differentiating (2.9) j times at $x = 0$ we obtain after some simplification

$$\begin{aligned} c_{n-1,n-1+2k} &= \sum_{s=0}^{k} (-\tfrac{1}{2})^{k-s} \frac{(n-1+2k)!}{(k-s)!} \lambda^{n-1+2s} a_{n-1,n-1+2s} && (k=0,1,\cdots), \\ c_{n-1,m} &= 0 && (m = 0, 1, 2, \cdots, n-2), \\ c_{n-1,n+2r} &= 0 && (r = 0, 1, \cdots), \end{aligned} \tag{2.12}$$

where the a's are defined by

$$K_N(x) = \sum_{j=0}^{\infty} a_{N,j} x^j \qquad (N = 0, 1, 2, \cdots)$$

$(a_{N,j} = K_N^{(j)}(0)/j!)$. In the derivation of (2.12) use has been made of the fact that

$$\begin{aligned} a_{N,j} &= 0 && (j = 0, 1, 2, \cdots, N-1), \\ a_{N,N+2r+1} &= 0 && (r = 0, 1, 2, \cdots). \end{aligned} \tag{2.13}$$

Formula (2.13) in its turn derives by induction from the following recursion relationship between the a's proved elsewhere [7]:

$$a_{N,s} = (2s)^{-1}\{(N+1)/(N\pi)\}^{1/2} \sum_{q=0}^{[(s-1)/2]} \{-4N(N+1)\}^{-q} a_{N-1,s-1-2q}/q! \qquad (s = 1, 2, \cdots), \tag{2.14}$$

$[(s-1)/2]$ denoting, as usual, the integral part of $(s-1)/2$. Though (2.14) may be exploited to derive explicit expressions for the non-zero a's these are more easily obtained recursively by repeated application of (2.14) on noting that, trivially,

$$\begin{aligned} a_{0,j} &= 0 && (j = 1, 2, \cdots), \\ &= 1 && (j = 0). \end{aligned} \tag{2.15}$$

This yields for the first there non-zero $a_{n-1,j}$, (2.16) $$a_{n-1,n-1} = \frac{n^{1/2}}{2^{n-1}\pi^{(n-1)/2}} \frac{1}{(n-1)!},$$

(2.17) $$a_{n-1,n+1} = -\frac{n^{1/2}}{2^{n-1}\pi^{(n-1)/2}} \frac{n-1}{4} \frac{1}{(n+1)!},$$

(2.18) $$a_{n-1,n+3} = \frac{n^{1/2}}{2^{n-1}\pi^{(n-1)/2}} \frac{(n-1)(n^2+7n-6)}{32n} \frac{1}{(n+3)!}$$

((2.14) shows that the non-zero a's oscillate in sign).

On applying (2.16), (2.17) and (2.18) in (2.12), the first there non-zero c's are obtained:

(2.19) $$\begin{aligned} c_{n-1,n-1} &= (n-1)!\lambda^{n-1}a_{n-1,n-1} \\ &= n^{1/2}2^{-(n-1)}\pi^{-(n-1)/2}\lambda^{n-1}, \end{aligned}$$

(2.20) $$\begin{aligned} c_{n-1,n+1} &= (n+1)!\{-\tfrac{1}{2}\lambda^{n-1}a_{n-1,n-1} + \lambda^{n+1}a_{n-1,n+1}\} \\ &= -n^{1/2}2^{-(n-1)}\pi^{-(n-1)/2}\{\tfrac{1}{2}n(n+1)\lambda^{n-1} + \tfrac{1}{4}(n-1)\lambda^{n+1}\}, \end{aligned}$$

(2.21) $$\begin{aligned} c_{n-1,n+3} &= (n+3)!\{\tfrac{1}{8}\lambda^{n-1}a_{n-1,n-1} - \tfrac{1}{2}\lambda^{n+1}a_{n-1,n+1} + \lambda^{n+3}a_{n-1,n+3}\} \\ &= n^{1/2}2^{-(n-1)}\pi^{-(n-1)/2}\{\tfrac{1}{8}n(n+1)(n+2)(n+3)\lambda^{n-1} \\ &\qquad + \tfrac{1}{8}(n-1)(n+2)(n+3)\lambda^{n+1} \\ &\qquad + \tfrac{1}{32}\frac{(n-1)(n^2+7n-6)}{n}\lambda^{n+3}\}. \end{aligned}$$

Thus from (2.10),

(2.22) $$\begin{aligned} I_n(h;\rho) \sim (2\pi)^{-1/2}e^{-r_0^2/2}\{&c_{n-1,n-1}r_0^{-n} + c_{n-1,n+1}r_0^{-(n+2)} \\ &+ c_{n-1,n+3}r_0^{-(n+4)} + \cdots\}, \end{aligned}$$

where the first three coefficients in the asymptotic expansion are given by (2.19), (2.20) and (2.21) (further coefficients may be obtained in the manner shown). A slightly more convenient form of (2.22) is

(2.23) $$\begin{aligned} I_n(h;\rho) \sim\ & (\tfrac{1}{2}n)^{1/2}\pi^{-n/2}e^{-r_0^2/2}(t/r_0)^{n-1}r_0^{-1} \\ &\times [1 - \{\tfrac{1}{2}(n)_2 + (n-1)t^2\}r_0^{-2} \\ &+ \{\tfrac{1}{8}(n)_4 + \tfrac{1}{2}(n+2)_2(n-1)t^2 + \tfrac{1}{2}(n-1)(n^2+7n-6)n^{-1}t^4\}r_0^{-4} - \cdots], \end{aligned}$$

where

(2.24) $$\begin{aligned} t \equiv t_n(\rho) &= \lambda/2 \\ &= [n\{1+(n-1)\rho\}/2(1-\rho)]^{1/2} \end{aligned}$$

and $(n)_m$ denotes $n(n+1)\cdots(n+m-1)$. It will be noted that the present asymptotic expansion is particularly suitable for large h (i.e., the cut-off point is not near the centre of the distribution) and algebraically small ρ.

Finally, observe that for $n = 1$ (2.22) reduces to (1.3), since $\psi_0(x) = \exp(-x^2/2)$ and

(2.25) $$c_{0,2j} = (-\tfrac{1}{2})^j (2j)!/j!.$$

(The polyhedral half-cone is here the interval $[h, \infty)$.) For $n = 2$, (2.22) reduces to

(2.26) $$I_2(h;\rho) \sim \pi^{-1} e^{-r_0^2/2}(t/r_0^2)[1-(3+t^2)r_0^{-2} + (15+10t^2+3t^4)r_0^{-4} - \cdots].$$

This agrees with a formula obtained previously [6] for the probability measure, $W(r_0; \alpha)$, $r_0 > 0$, under a standardized circular normal distribution, of a sector of angle α, vertex at a distance r_0 from the centre of the distribution and with one arm of the sector passing through the latter point. The relationship between I_2 and W is

(2.27) $$I_2(h; \rho) = 2W(r_0; \theta/2)$$

where $\theta = 2 \arctan t_2(\rho) = 2 \arctan\{(1+\rho)/(1-\rho)\}^{1/2}$. It has been shown in [6] that the bivariate normal integral for arbitrary cut-off point may be expressed in terms of the difference of two W-functions (and therefore of two I_2-functions).

3. The Accuracy of the Asymptotic Expansion

In this section we obtain upper bounds to the error induced by taking the first m terms of the asymptotic expansion as an approximation to $I_n(h; \rho)$.

Let ϕ be the angle between the axis of the half-cone and the line joining P and the vertex of the cone, and let ξ be the distance of P from this vertex. Then (using the notation of Section 2)

$$r^2 = r_0^2 + \xi^2 + 2r_0\xi\cos\phi,$$

and the probability-mass of an infinitesimal volume-element of content $d\tau$ at P is

(3.1) $$\begin{aligned}&(2\pi)^{-n/2}\exp[-\tfrac{1}{2}r^2]d\tau = \\ &(2\pi)^{-n/2}\exp[-\tfrac{1}{2}(r_0^2+\xi^2+2r_0\xi\cos\phi)]\xi^{n-1}d\xi\, d\omega,\end{aligned}$$

where $d\omega$ is the solid angle subtended at the vertex of the cone by the volume-element (or, equivalently, the surface-content of an infinitesimal element on the surface of a unit sphere whose centre coincides with the vertex of the cone). Thus the probability-mass of the half-cone is

(3.2) $$I_n(h; \rho) = (2\pi)^{-n/2} e^{-r_0^2/2} \int_0^\infty \int_\Omega e^{-(r_0\cos\phi)\xi}\xi^{n-1}e^{-\xi^2/2}d\xi\, d\omega,$$

where Ω is the $(n-1)$-dimensional regular spherical simplex (with common dihedral angle arc cos $(-\rho)$ formed by the intersection of the half-cone and the surface of the unit sphere. Again, if

$$G_{n-1}(\xi) = \xi^{n-1}e^{-\xi^2/2},$$

then the derivatives of $G_{n-1}(\xi)$ at the origin, $G_{n-1}^{(q)}(0)$, are given by

$$G_{n-1}^{(n-1+2i)}(0) = (-1)^i \frac{(n-1+2i)!}{2^i i!} \qquad (i = 0, 1, 2, \cdots)$$

with all other derivatives vanishing. Therefore, repeated integration by parts yields

$$\text{(3.3)} \quad \int_0^\infty e^{-(r_0\cos\phi)\xi} G_{n-1}(\xi)d\xi = \sum_{i=0}^{m-1} (-1)^i \frac{(n-1+2i)!}{2^i i!} \frac{1}{(r_0 \cos\phi)^{n+2i}} + R_m(r_0 \cos\phi),$$

where

$$\text{(3.4)} \quad \begin{aligned} R_m(r_0\cos\phi) &= (r_0\cos\phi)^{-(n+2m-2)} \int_0^\infty e^{-(r_0\cos\phi)\xi} G_{n-1}^{(n+2m-2)}(\xi)d\xi \\ &= (r_0\cos\phi)^{-(n+2m-1)} \int_0^\infty e^{-(r_0\cos\phi)\xi} G_{n-1}^{(n+2m-1)}(\xi)d\xi, \end{aligned}$$

after a further single integration by parts. On using (3.3) and (3.4) in (3.2),

$$\text{(3.5)} \quad I_n(h;\rho) = (2\pi)^{-n/2}e^{-r_0^2/2}\left\{\sum_{i=0}^{m-1}(-1)^i \frac{(n-1+2i)!}{2^i i!}\alpha_{n,i} r_0^{-(n+2i)} + \int_\Omega R_m(r_0\cos\phi)d\omega,\right.$$

where

$$\text{(3.6)} \quad \alpha_{n,i} = \int_\Omega \sec^{n+2i}\phi\, d\omega.$$

In (3.5), the remainder after m terms is

$$\text{(3.7)} \quad E_m = (2\pi)^{-n/2}e^{-r_0^2/2}\int_\Omega R_m(r_0\cos\phi)d\omega.$$

An upper bound to $|E_m|$ can be obtained from an upper bound to $R_m(r_0\cos\phi)$ in (3.4). The latter upper bound is itself obtained by deriving first an upper bound to $G_{n-1}^{(n+2m-1)}(\xi)$ for $\xi \geqq 0$. If, then,

$$\text{(3.8)} \quad |G_{n-1}^{(n+2m-1)}(\xi)| \leqq A_{n-1,2m},$$

(3.4) gives

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(3.9) $$|R_m(r_0 \cos \phi)| < A_{n-1,2m}(r_0 \cos \phi)^{-(n+2m)},$$

whence by (3.7)

(3.10) $$\begin{aligned}|E_m| &< (2\pi)^{-n/2}e^{-r_0^2/2}A_{n-1,2m}\int_\Omega (r_0 \cos \phi)^{-(n+2m)}d\omega \\ &= A_{n-1,2m}(2\pi)^{-n/2}e^{-r_0^2/2}\alpha_{n,m}r_0^{-(n+2m)},\end{aligned}$$

which is proportional to the $(m+1)$th term of the series

(3.11) $$(2\pi)^{-n/2}e^{-r_0^2/2}\cdot\sum_{i=0}^{\infty}(-1)^i\frac{(n-1+2i)!}{2^i i!}\alpha_{n,i}r_0^{-(n+2i)}.$$

Consequently, (3.11) is a valid asymptotic expansion when $r_0 > 0$ of $I_n(h;\rho)$. Moreover, the series (3.11) must be identical with the series (2.22), since a given function determines uniquely (if at all) a series of the form $\sum c_p/r_0^p$, so that (3.10) provides an upper bound to the error in using (2.22).

We now proceed to determine a value[2] for $A_{n-1,2m}$. Let

(3.12) $$\xi^{n-1} = \beta_{n-1,0}H_0(\xi) + \beta_{n-1,1}H_1(\xi) + \cdots + \beta_{n-1,n-1}H_{n-1}(\xi),$$

where $H_j(\xi)$ are the Tchebycheff-Hemite polynomials orthogonal to the weight function $\exp(-\xi^2/2)$ and normalized so that the coefficient of ξ^j in $H_j(\xi)$ is 1. On multiplying (3.12) by $H_j(\xi)\exp(-\xi^2/2)$, and integrating over the real line, we find

(3.13) $$\beta_{n-1,j} = \int_{-\infty}^{\infty}\xi^{n-1}H_j(\xi)e^{-\xi^2/2}d\xi \Big/ \int_{-\infty}^{\infty}H_j^2(\xi)e^{-\xi^2/2}d\xi.$$

The value of the denominator in (3.13) is well-known to be $\sqrt{(2\pi)}j!$. In order to evaluate the numerator, define

$$\gamma_{n-1,j} = \int_{-\infty}^{\infty}\xi^{n-1}H_j(\xi)e^{-\xi^2/2}d\xi \qquad (j = 0, 1, \cdots, n-1).$$

Integration by parts gives the recursion relationship

(3.14) $$\gamma_{n-1,j} = (n-1)\gamma_{n-2,j-1},$$

and on successive application of (3.14)

$$\begin{aligned}\gamma_{n-1,j} &= (n-1)(n-2)\cdots(n-j)\gamma_{n-1-j,0} \\ &= (n-1)(n-2)\cdots(n-j)\int_{-\infty}^{\infty}\xi^{n-1-j}e^{-\xi^2/2}d\xi,\end{aligned}$$

whence

[2] That $A_{n-1,2m} < \infty$ is evident from the fact that all derivatives of $G_{n-1}(\xi)$ are products of polynomials in ξ and $\exp(-\xi^2/2)$.

$$\gamma_{n-1,j} = (n-1)(n-2)\cdots(n-j)2^{(n-j)/2}\Gamma(\tfrac{1}{2}(n-j)) \quad (n-1-j \text{ even}),$$
(3.15)
$$= 0 \quad (n-1-j \text{ odd}).$$

On substituting (3.15) and (3.13) and using the duplication formula for the gamma function, we obtain

(3.16)
$$\beta_{n-1,j} = \frac{(n-1)!}{2^{(n-1-j)/2}(\frac{1}{2}(n-1-j))!j!} \quad (n-1-j \text{ even}),$$
$$= 0 \quad (n-1-j \text{ odd}).$$

Reverting to (3.12),

$$G_{n-1}(\xi) = \xi^{n-1}e^{-\xi^2/2} = \sum_{j=0}^{n-1} \beta_{n-1,j}H_j(\xi)e^{-\xi^2/2},$$

and therefore

(3.17)
$$G_{n-1}^{(n-1+2m)}(\xi) = (-1)^{n-1}\sum_{j=0}^{n-1}\beta_{n-1,j}H_{n-1+j+2m}(\xi)e^{-\xi^2/2}$$

on recalling that

(3.18)
$$\frac{dp}{d\xi^p}e^{-\xi^2/2} = (-1)^p H_p(\xi)e^{-\xi^2/2}.$$

An upper bound to $|H_{n-1+j+2m}(\xi)|\exp(-\xi^2/2)$ in (3.17) is readily deduced from the well-known identity

$$e^{-\xi^2/2} = \int_{-\infty}^{\infty} e^{i\xi x}\cdot(2\pi)^{-1/2}e^{-x^2/2}dx.$$

Hence, on applying (3.18),

$$(-1)^p H_p(\xi)e^{-\xi^2/2} = \int_{-\infty}^{\infty}(ix)^p e^{i\xi x}\cdot(2\pi)^{-1/2}e^{-x^2/2}dx,$$

from which we obtain (for ξ real),

(3.19)
$$|H_p(\xi)|e^{-\xi^2/2} \leqq (2\pi)^{-1/2}\int_{-\infty}^{\infty}|x|^p e^{-x^2/2}dx = \pi^{-1/2}2^{p/2}\Gamma(\tfrac{1}{2}(p+1)).$$

Thus, from (3.16), (3.17) and (3.19),

(3.20)
$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \pi^{-1/2}\sum_j{}' \frac{(n-1)!}{2^{(n-1-j)/2}(\frac{1}{2}(n-1-j))!j!} \cdot 2^{m+(j+n-1)/2}\Gamma(m+\tfrac{1}{2}(j+n\delta)),$$

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$\sum_j'$ denoting summation over all non-negative integral $j \leqq n-1$ such that $n-1-j$ is even.

If n is odd, set $j = 2i$ in (3.20). Then

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \pi^{-1/2} \sum_{i=0}^{(n-1)/2} \frac{(n-1)!}{2^{(n-1-2i)/2}(\frac{1}{2}(n-1)-i)!(2i)!} \cdot 2^{m+i+(n-1)/2}\Gamma(m+i+\tfrac{1}{2}n) \qquad (n = 1, 3, \cdots),$$

and, on using the gamma duplication formula in the form

$$\pi^{-1/2}\Gamma(m+i+\tfrac{1}{2}n) = (2m+n-1+2i)!/\{(m+\tfrac{1}{2}(n-1)+i)!2^{2m+n-1+2i}\},$$

the latter inequality simplifies to

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(n-1)!}{2^{m+n-1}} \sum_{i=0}^{(n-1/2)} \frac{(2m+n-1+2i)!}{(m+\frac{1}{2}(n-1)+i)!(2i)!(\frac{1}{2}(n-1)-i)!} \qquad (n = 1, 3, \cdots). \tag{3.21}$$

Similarly if n is even, set $j = 2i+1$ in (3.20). Then

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \pi^{-1/2} \sum_{i=0}^{(n-2)/2} \frac{(n-1)!}{2^{(n-2-2i)/2}(\frac{1}{2}(n-2)-2i)!(2i+1)!} \cdot 2^{m+i+n/2}\Gamma(n+i+\tfrac{1}{2}(n+1)) \qquad (n = 2, 4, \cdots), \tag{3.22}$$

and, on using the gamma duplication formula in the form

$$\pi^{-1/2}\Gamma(m+i+\tfrac{1}{2}(n+1)) = (2m+n+2i)!/\{(m+\tfrac{1}{2}n+i)!2^{2m+n+2i}\},$$

the last inequality reduces to

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(n-1)!}{2^{m+n-1}} \sum_{i=0}^{(n-2)/2} \frac{(2m+n+2i)!}{(m+\frac{1}{2}n+i)!(2i+1)!(\frac{1}{2}(n-2)-i)!} \qquad (n = 2, 4, \cdots). \tag{3.23}$$

Formula (3.21) and (3.23) provide the required inequalities in the sense that their right-hand members (refer to (3.8)) may be substituted for $A_{n-1,2m}$ in (3.10) to supply the desired upper bound for the error after m terms. A weaker (but at the same time simpler) upper bound may be obtained by noting that in (3.21)

$$(n-1)!(2m+n-1+2i)!/(2i)! = (n-1)!(2i+1)(2i+2)\cdots(2m+n-1+2i) \leqq (2n-2+2m)!,$$

whence

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$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(2n-2+2m)!}{2^{m+n-1}} \sum_{i=0}^{(n-1)/2} \frac{1}{(m+\frac{1}{2}(n-1)+i)!(\frac{1}{2}(n-1)-i)!}$$

$$= \frac{(2n-2+2m)!}{2^{m+n-1}} \sum_{s=0}^{(n-1)/2} \frac{1}{(m+n-1-s)!s!}\ .$$

$$(3.24) \qquad = \frac{(2n-2+2m)!}{(n-1+m)!} \cdot \left(\frac{1}{2}\right)^{m+n-1} \sum_{s=0}^{(n-1)/2} \binom{n-1+m}{s} \qquad (n = 1, 3, \cdots).$$

Similarly, for n even, observe that in (3.23)

$$(n-1)!(2m+n+2i)!/(2i+1)! = (n-1)!(2i+2)(2i+3)\cdots(2m+n+2i) \leqq (2n-2+2m)!,$$

whence

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(2n-2+2m)!}{2^{m+n-1}} \sum_{i=0}^{(n-2)/2} \frac{1}{(m+\frac{1}{2}n+i)!(\frac{1}{2}(n-2)-i)!}$$

$$(3.25) \qquad = \frac{(2n-2+2m)!}{2^{m+n-1}} \sum_{s=0}^{(n-2)/2} \frac{1}{(m+n-1-s)!s!}$$

$$= \frac{(2n-2+2m)!}{(n-1+m)!} \cdot \left(\frac{1}{2}\right)^{m+n-1} \sum_{s=0}^{(n-2)/2} \binom{n-1+m}{s} \qquad (n = 2, 4, \cdots).$$

The inequalities (3.24) and (3.25) may be combined in the following single inequality valid for all n (odd or even):

$$(3.26) \qquad |G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(2n-2+2m)!}{(n-1+m)!} \cdot \left(\frac{1}{2}\right)^{m+n-1} \sum_{s=0}^{[(n-1)/2]} \binom{n-1+m}{s} \qquad (n = 1, 2, \cdots).$$

Thus an upper bound to the $(n-1+2m)$th derivative of $G_{n-1}(\xi)$ is provided by the product of $(2n-2+2m)!/(n-1+m)!$ and the cumulative sum of the first (or last) $[(n+1)/2]$ probabilities in a binomial distribution with index $n-1+m$ and parameter 1/2. (The latter cumulative sum is, of course, readily available from various statistical tables.) This upper bound may now be substituted for $A_{n-1,2m}$ in (3.10) to give the desired simplified upper bound to the error after m terms in the asymptotic expansion as a multiple of the $(m+1)$th term.

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