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A RESULT ON ITERATED CLIQUE GRAPHS

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Abstract

S. T. Hedetniemi and P. J. Slater have shown that if G is a triangle-free connected graph with at least three vertices, then

$$K^2(G) \cong G - \{x \in G \mid \deg(x, G) = 1\}$$

where $K(G)$ is the clique graph of G and $K^2(G) = K(K(G))$ is the first iterated clique graph.

In this paper, we generalize the above result to a wider class of graphs.

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1. Introduction

In this paper, all graphs (possibly infinite) will be undirected, without loops and without multiple edges. For any graph G, we shall use $x \in G$ to indicate that x is a vertex of G. The complete graph on n vertices is denoted by K_n.

A *clique* of a graph G is defined to be a complete subgraph of G, which is not contained in any larger complete subgraph of G. The *clique graph* $K(G)$ of G is a graph having the cliques of G as vertices, two vertices of $K(G)$ being adjacent if and only if the corresponding cliques have a nonempty intersection. By $K^2(G)$ we mean $K(K(G))$, and in general $K^n(G) = K(K^{n-1}(G))$.

Let G^* denote the graph obtained by contracting each component of G which is a complete graph to an isolated vertex. Then $K(G) \cong K(G^*)$. The degree of a vertex x in a graph G is denoted by $\deg(x, G)$.

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The graph G is said to have the *Helly property* if every set $\{C_i \mid i \in I\}$ of cliques of G, no two of which are disjoint (that is, $C_i \cap C_j \neq \varnothing$ for all $i, j \in I$), has nonempty total intersection (that is, $\bigcap_{i \in I} C_i \neq \varnothing$).

We say that G has the T_1 *property* if for any distinct vertices $x, y \in G^*$ with $\deg(x, G^*) \geqslant 2$, $\deg(y, G^*) \geqslant 2$, there exist $C, D \in K(G)$ with $x \in C, y \notin C$ and $x \notin D, y \in D$.

Hedetniemi and Slater (1972), show that if G is a triangle-free connected graph with at least three vertices, then $K^2(G) \cong G - \{x \in G \mid \deg(x, G) = 1\}$.

The main purpose of this paper is to generalize the above result to a wider class of graphs which satisfy both the Helly property and the T_1 property.

For other terms not defined here see Harary (1969).

2. The main theorem

The main purpose of this paper is to prove the following theorem.

THEOREM 2.1. *If G is a graph which satisfies the Helly property and the T_1 property, then*

$$K^2(G) \cong G^* - \{x \in G^* \mid \deg(x, G^*) = 1\}.$$

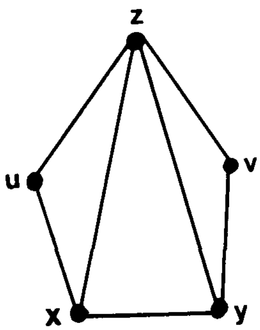
Given a graph G, for any $x \in G$, let $K(x)$ be the induced subgraph of $K(G)$ with vertex set $\{C \in K(G) \mid x \in C\}$.

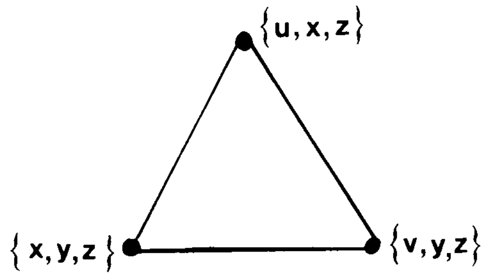
Before proving Theorem 2.1, we will first prove the following results.

LEMMA 2.2. *If G is a graph which satisfies the Helly property, then for any clique A of $K(G)$ there is an $x \in G^*$ with $\deg(x, G^*) \neq 1$ such that $A = K(x)$.*

PROOF. Let $\{C_i \in K(G) \mid i \in I\}$ be the vertex set of A. Since A is a clique of $K(G)$, $C_i \cap C_j \neq \varnothing$ for all $i, j \in I$. Now, G satisfies the Helly property, so we have $\bigcap_{i \in I} C_i \neq \varnothing$. Let $x \in \bigcap_{i \in I} C_i$. Then $\{C_i \mid i \in I\} \subseteq \{C \in K(G) \mid x \in C\}$, in fact, equality holds, for if $C \in K(G)$ and $x \in C$, then $C \cap C_i \neq \varnothing$ for every $i \in I$, and therefore C is adjacent in $K(G)$ to every vertex of the clique A, so must actually be one of its vertices. It remains to show that $\deg(x, G^*) \neq 1$. Suppose that $\deg(x, G^*) = 1$. Then x is adjacent to just one vertex $y \in G^*$, so it belongs to only one clique D of G^* and $D \cong K_2$. From the definition of G^*, we observe that the component of G^* induced by D has at least three vertices. Hence there is some clique D' of G^* such that $y \in D'$, $x \notin D'$. Thus D' is adjacent to D in $K(G^*)$, so no clique of $K(G^*)$ with D as a vertex can have only one vertex. But then the intersection of the vertices of such a clique cannot contain x, contrary to the choice of x. This contradiction implies $\deg(x, G^*) \neq 1$, completing the proof.

REMARK. Not every subgraph $K(x)$ need be a clique, even if G has the Helly property. This is illustrated by the following example.





The following lemma gives sufficient conditions for this to be true.

LEMMA 2.3. *Let G be a graph which satisfies the Helly property and the T_1 property. Then for any $x \in G^*$ with* $\deg(x, G^*) \neq 1$, *the subgraph $K(x)$ is a clique of $K(G)$.*

PROOF. Let $x \in G^*$ with $\deg(x, G^*) \neq 1$. Clearly the result is true if $\deg(x, G^*) = 0$. So we let $\deg(x, G^*) \geqslant 2$. Note that $K(x)$ is necessarily a complete subgraph of $K(G)$, so if it is not a clique there exists some $D \in K(G)$ such that $x \in D$ but $C \cap D \neq \varnothing$ for every $C \in K(x)$. Let $S = \bigcap \{C \in K(x)\}$. Now G has the Helly property, and $S \cap D$ is an intersection of pairwise nondisjoint cliques of G, so $S \cap D \neq \varnothing$. We shall prove that D does not exist, and therefore $K(x)$ is a clique of $K(G)$, by deriving the contradiction $S \cap D = \varnothing$.

Evidently $x \in S$, but $x \notin D$, so $x \in S \cap D$. Now consider $y \in G^*$ with $y \neq x$.

Case 1. $\deg(y, G^*) = 0$. Then y is not adjacent to x and hence $y \notin S$. Therefore $y \notin S \cap D$.

Case 2. $\deg(x, G^*) = 1$. Then y is adjacent to just one vertex in G^*, so it belongs to just one clique of G. But then $y \notin S \cap D$, since this is an intersection of at least two cliques of G.

Case 3. $\deg(y, G^*) > 1$. Since G has the T_1 property, there is some $C \in K(x)$ with $y \notin C$, so $y \notin S$. Hence $y \notin S \cap D$.

Thus $S \cap D = \varnothing$, as claimed, whence $K(x)$ is a clique of $K(G)$.

Combining Lemmas 2.2 and 2.3, we have the following theorem.

THEOREM 2.4. *Let G be a graph satisfying the Helly property and the T_1 property. Then A is a clique of $K(G)$ if and only if there is an $x \in G^*$ with* $\deg(x, G^*) \neq 1$ *such that* $A = K(x)$.

We are now in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Let φ: $\{x \in G^* \mid \deg(x, G^*) \neq 1\} \to \{A \in K^2(G)\}$ be a function defined by

$$\varphi(x) = K(x).$$

We shall show that φ is a graph isomorphism. By Lemma 2.3, φ is well-defined. To show that φ is one-one, let $x, y \in G^*$ be distinct vertices with $\deg(x, G^*) \neq 1$, $\deg(y, G^*) \neq 1$. If one of x, y is an isolated vertex, then clearly $K(x) \neq K(y)$ and hence $\varphi(x) \neq \varphi(y)$. So we let $\deg(x, G^*) \geqslant 2$, $\deg(y, G^*) \geqslant 2$. By the T_1 property, there exists $C \in K(G)$ with $x \in C$, $y \notin C$. Hence $C \in K(x)$ but $C \notin K(y)$, so $\varphi(x) \neq \varphi(y)$. Lemma 2.2 implies that φ is onto. It remains to show that if $x, y \in G^*$ are distinct vertices with $\deg(x, G^*) \neq 1$, $\deg(y, G^*) \neq 1$, then $\varphi(x)$ is adjacent to $\varphi(y)$ in $K^2(G)$ if and only if x is adjacent to y in G^*. If x and y are adjacent in G^*, they are vertices of some clique C of G^* (and hence of G), so $C \in \varphi(x)$ and $C \in \varphi(y)$. Hence $\varphi(x) \cap \varphi(y) \neq \varnothing$, so $\varphi(x)$ and $\varphi(y)$ are adjacent in $K^2(G)$. The converse argument readily follows.

Thus φ is an isomorphism, and the theorem is proved.

COROLLARY 2.5. *Let G and H be two graphs with no vertices of degree* 1, *and suppose both G and H have the Helly property and the T_1 property. Then*

$$K(G) \cong K(H) \quad \text{if and only if } G^* \cong H^*.$$

3. Special cases

We shall show that the following theorem is a special case of Theorem 2.1.

THEOREM 3.1 (Hedetniemi and Slater (1972)). *If G is a triangle-free connected graph with at least three vertices, then*

$$K^2(G) \cong G - \{x \in G \mid \deg(x, G) = 1\}.$$

To prove this result, we need the following lemmas, and the result will follow by applying Theorem 2.1.

LEMMA 3.2. *If G is a triangle-free connected graph, then G satisfies the Helly property.*

PROOF. If G has fewer than three vertices, G trivially satisfies the Helly property. Hence we may now suppose G has at least three vertices.

Let $\{C_i \mid i \in I\}$ be a family of pairwise nondisjoint cliques of G. We will show that $\bigcap_{i\in I} C_i \neq \varnothing$.

Observe that since G is connected and contains no triangles, $C_i \cong K_2$ for each $i \in I$. If C_i, C_j are two distinct cliques, with $i, j \in I$, nondisjointness ensures there is some $x \in C_i \cap C_j$. We claim that $x \in \bigcap_{i\in I} C_i$. Let $x, y, z \in G$ be distinct vertices and that $y \in C_i$, $z \in C_j$, and suppose $x \notin C_k$ for some $k \in I$. Then $C_i \cap C_k \neq \varnothing$ implies $y \in C_k$ since $C_i \cong K_2$, and similarly $z \in C_k$. But then y must be adjacent to z in G, so G contains a triangle on the vertices x, y, z. This contradicts the choice of G, so $x \in \bigcap_{i\in I} C_i$ follows. Thus G has the Helly property.

LEMMA 3.3. *If G is a triangle-free connected graph, then G has the T_1 property.*

PROOF. If G has fewer than three vertices, G trivially satisfies the T_1 property. Hence we may now suppose G has at least three vertices.

Let $x, y \in G^*$ be distinct vertices of degree at least 2. Then x is adjacent to some $z \in G^*$, $z \neq y$. Since G is triangle-free, $\{x, z\}$ is the vertex set of some $C \in K(G)$, so $x \in C$, $y \notin C$. Similarly there is some $D \in K(G)$ such that $x \notin D$, $y \in D$. Thus G has the T_1 property.

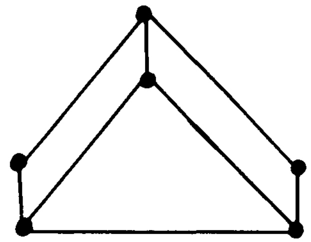
PROOF OF THEOREM 3.1. Let G be a triangle-free connected graph with at least three vertices. Then $G = G^*$. By Lemmas 3.2 and 3.3, G satisfies the Helly property and the T_1 property. By Theorem 2.1,

$$K^2(G) \cong G^* - \{x \in G^* \mid \deg(x, G^*) = 1\}.$$

Since $G = G^*$, the result follows.

REMARKS. 1. Theorem 3.1 is also true if connectedness of G is dropped (that is, G is a triangle-free graph with at least three vertices in each component). This is because G has the Helly property and the T_1 property if each of its components has the Helly property and the T_1 property.

2. Theorem 2.1 is a proper generalization of Theorem 3.1 as can be seen from the following graph G, which actually has the property $K(G) \cong G = G^*$.



$G \cong K^2(G)$

Let $A = (a_{ij})$ be a $(0, 1)$-matrix. The *row-column graph* $G(A)$ of A is a bipartite graph obtained as follows:

The vertices of $G(A)$ are the rows and the columns of A; a row and a column are adjacent if and only if the entry $a_{ij} = 1$. Observe that if every row and every column of A contains at least two ones, then $G(A)$ will have no vertices of degree 1. Thus

COROLLARY 3.4 (Cook (1970)). *If A is a $(0, 1)$-matrix and $G(A)$ is its row-column graph, then*

$$K^2(G(A)) \cong G(A)$$

provided that every row and every column contain at least two ones.

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