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CW DECOMPOSITIONS OF EQUIVARIANT CW COMPLEXES

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We discuss conditions which ensure that a G-CW complex is G-homotopy equivalent to a CW complex with cellular action with respect to some CW decomposition of the compact Lie group G. For $G = SU(2)$, we prove that for every G-CW complex X, there exists a CW complex Y which is G-homotopy equivalent to X, such that the action $G \times Y \to Y$ is a cellular map.

1. INTRODUCTION

Let G be a compact Lie group. A *G-cell* of dimension n is a space of the form $G/H \times D^n$, where H is a closed subgroup of G and D^n is an n-cell. A *G-CW complex* X (or an *equivariant CW complex* in the terminology of [**9**]) is constructed by iterated attaching of G-cells. It is the union of G-spaces $X^{(n)}$ such that $X^{(0)}$ is a disjoint union of G-cells of dimension 0, that is, orbits G/H, and $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching G-cells of dimension $n+1$ along equivariant attaching maps $G/H \times \partial D^{n+1} \to X^{(n)}$. The space $X^{(n)}$, which is called the *n-skeleton* of X, is thus the union of all G-cells of dimension at most n (the topological dimension of $X^{(n)}$ is in general greater than n). For basic facts about G-complexes see the original papers [**5**] and [**3**] or the exposition in [**9**].

For discrete groups G it is well known that every G-CW complex is also a CW complex with a cellular action of G (this follows for example from [**9**, Proposition 1.16, p. 102]). For non-discrete groups, Illman [**4**] gave an example showing that a G-CW complex X does not always admit a CW decomposition, compatible with the given G-CW decomposition, and proved that there always exists a homotopy equivalent CW complex Y which is finite if X is a finite G-complex.

In this paper we consider the following problem. Given a G-CW complex X, does there exist a G-space Y, G-homotopy equivalent to X, with a CW decomposition such that the action $\rho\colon G \times Y \to Y$ is a cellular map with respect to some decomposition of G. The existence of such a Y is interesting from the point of view of equivariant homology and cohomology. For example, Greenlees and May showed that for some groups G the generalised Tate cohomology defined in [**1**] can be calculated from the CW decomposition

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of Y. Also, the Borel equivariant cohomology $H^*_G(X) = H^*(EG \times_G X)$ of a G-CW complex X can be computed using the cellular cohomology of the CW complex Y which is G-homotopy equivalent to $EG \times_G X$.

In general, it is not known to the authors if, for a given group G, every G-CW complex is G-homotopy equivalent to a CW complex Y with the required properties. For $G = S^1$, Greenlees and May [**1**, Lemma 14.1] gave a construction of Y for any X. In case of non-Abelian groups, the construction of Y is more difficult, since the fixed point sets $(G/H)^K$ of actions of subgroups $K < G$ on the orbits G/H are in general nontrivial. In [**7**], the two non-Abelian 1-dimensional compact Lie groups, the orthogonal group $O(2)$ and the continuous quaternionic group $N_{SU(2)}T$, are considered but the construction of Y rests on a property of these two groups which is satisfied only for a few particular groups G. In this paper we consider the 3-dimensional group $G = SU(2)$.

The paper is organised as follows. In Section 2, two conditions on the set of isotropy subgroups of a G-CW complex X which enable the construction of a G-homotopy equivalent CW complex Y by induction on the G-skeletons of X are stated. We show that the class of G-CW complexes with finitely many isotropy types satisfies these two conditions. We also show that if a group G has the property that the set of all closed subgroups satisfies these two conditions, then every G-CW complex has a G-homotopy equivalent CW complex with a cellular action of G. In Section 3, the actions of subgroups of the group $SU(2)$ on orbits of $SU(2)$ are analysed. We show that a set of closed subgroups of $SU(2)$ satisfies the two conditions of Section 2 and as a result obtain our main theorem.

Theorem 1. *Any $SU(2)$-CW complex X is G-homotopy equivalent to a CW complex Y which is an $SU(2)$-space with a cellular action of $SU(2)$.*

Finally, in Section 4 some other examples are discussed.

2. Representative families of subgroups

Let G be a compact Lie group, X a G-CW complex, and $\mathcal{H}$ the family of isotropy subgroups of the action.

For a G-CW complex X, a *representative family* $\mathcal{K}$ of isotropy subgroups is a family of closed subgroups of G such that each isotropy subgroup of X is conjugate to a member of $\mathcal{K}$. We shall call a representative family *good* with respect to a given CW decomposition of G if the following two conditions are satisfied.

(1) For each $H \in \mathcal{K}$, there is a CW decomposition of G/H with respect to which the action $\mu\colon G \times G/H \to G/H$ is cellular.

(2) For each $K \in \mathcal{K}$, the fixed point set $(G/H)^K$ is a subcomplex of the CW complex G/H.

Let us first prove that the existence of a good representative family suffices for the construction of a G-homotopy equivalent CW complex Y.

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Proposition 1. *Let G be a compact Lie group with a given CW decomposition, X a G-CW complex and $\mathcal{K}$ a good representative family of isotropy subgroups. Then there exists a CW complex Y with a cellular action of G and a G-homotopy equivalence $h\colon X \to Y$.*

Proof: Following [**1**, Lemma 14.1], we shall construct a CW complex Y and a G-homotopy equivalence $h\colon X \to Y$ by induction on the G-skeletons $X^{(i)}$ of X.

Since every isotropy subgroup of X is conjugate to a member of $\mathcal{K}$, the 0-skeleton $X^{(0)}$ is homeomorphic to a disjoint union of orbits G/H_i, where $H_i \in \mathcal{K}$. Since $\mathcal{K}$ is good, every orbit G/H has a CW decomposition satisfying conditions (1) and (2). Let Y_0 be $X^{(0)}$ with this CW decomposition on every G-cell G/H_i. Because of condition (1), the action $\mu\colon G \times Y_0 \to Y_0$ is cellular, and because of condition (2),

$$(Y_0)^K = \coprod (G/H_i)^K$$

is a subcomplex of Y_0 for every $K \in \mathcal{K}$. The G-homotopy equivalence on the 0-skeleton is the identity $h_0 = \mathrm{id}\colon X^{(0)} \to Y_0$.

By induction we assume that there exist a CW complex Y_{n-1} with a cellular action of G, such that for every $K \in \mathcal{K}$ the fixed point set $(Y_{n-1})^K$ is a subcomplex, and a G-homotopy equivalence

$$h_{n-1}\colon X^{(n-1)} \to Y_{n-1}.$$

For any G-cell $e^n_\nu \in X^{(n)}$, the attaching G-map $G/H_\nu \times S^{n-1} \to X^{(n-1)}$ is determined by its restriction

$$\varphi_\nu\colon S^{n-1} \to (X^{(n-1)})^{H_\nu}.$$

Let ψ_ν be a non-equivariant cellular approximation of the composition

$$h_{n-1} \circ \varphi_\nu\colon S^{n-1} \to (Y_{n-1})^{H_\nu}.$$

Since the action of G on Y_{n-1} is cellular, the natural G-extension

$$\widetilde{\psi}_\nu\colon G/H_\nu \times S^{n-1} \to Y_{n-1}$$

of ψ_ν is also cellular, and the space

$$Y_n = \coprod_{e^n_\nu \in X^{(n)}} (G/H_\nu \times D^n) \cup_{\coprod \widetilde{\psi}_\nu} Y_{n-1}$$

is a CW complex with a cellular action of G. For each $K \in \mathcal{K}$, the fixed point set $(Y_n)^K$ is obtained by gluing the subcomplexes $(G/H_\nu)^K \times D^n$, corresponding to the n-cells, and the subcomplex $(Y_{n-1})^K$ along a cellular map. So $(Y_n)^K$ is a subcomplex of Y_n. The G-homotopy h_n is obtained so that h_{n-1} is extended G-cell by G-cell over the whole space Y_n. In the direct limit, we obtain the desired CW complex Y and G-homotopy equivalence h. ∎

For example, for any compact Lie group G, every G-CW complex which consists of a free part and a part which is fixed by the action is G-homotopy equivalent to a CW complex with a cellular action of G. More generally,

PROPOSITION 2. *For every G-CW complex with a finite representative family of isotropy groups $\mathcal{K}$ which satisfies condition* (1) *there exists a G-homotopy equivalent CW complex Y with a cellular action of G.*

PROOF: By assumption, condition (1) is satisfied. The following lemma shows that condition (2) is also satisfied, so the family $\mathcal{K}$ is good. ∎

LEMMA 1. *If for a given $H \in \mathcal{K}$ the collection of fixed point sets $(G/H)^K, K \in \mathcal{K}$, is a finite family of subsets of G/H, then the orbit G/H has a CW-decomposition with respect to which every fixed point set $(G/H)^K$, $K \in \mathcal{K}$, is a subcomplex.*

PROOF: For every $K \in \mathcal{K}$ the orbit (G/H) is a smooth K-manifold, and the fixed point set $(G/H)^K$ is a submanifold [**9**, p. 42] which is nontrivial only if K is conjugate to a subgroup of H. The family $\{(G/H)^K, K \in \mathcal{K}\}$ is a finite family of smooth submanifolds of G/H which, by the differentiable slice theorem (compare for example [**2**, Theorem I.5]), intersect transversally. By [**6**, 10.11,10.14], this implies that there exists a CW decomposition of G/H such that each $(G/H)^K$, $K \in \mathcal{K}$, is a subcomplex. ∎

COROLLARY *If there exists a good representative family of all closed subgroups of a compact Lie group G, then every G-CW complex X has a G-homotopy equivalent CW complex Y with a cellular action of G.*

PROOF: This follows immediately from Proposition 1. ∎

3. A GOOD REPRESENTATIVE FAMILY FOR $SU(2)$

Let G be $SU(2) \cong Sp(1)$. An element $x \in G$ can be represented in the form

$$x = \begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{bmatrix}, \quad |z_1|^2 + |z_2|^2 = 1, \quad z_1, z_2 \in \mathbb{C}$$

or as the unit quaternion $q = z_1 + jz_2$. The centre Z_G is generated by $-I \in SU(2)$ (or $-1 \in Sp(1)$), the only element of order 2. The projection $G \to G/Z_G \cong SO(3)$ associates to a unit quaternion written in polar form as $q = (\cos\varphi, j\mathbf{e})$ the rotation with axis $\mathbf{e} \in \mathbb{R}^3$ through the angle φ.

The isomorphism classes of closed subgroups of $SU(2)$ are known. Since there are no non-Abelian 2-dimensional Lie groups, the dimension of a proper closed subgroup is at most 1. The 0- and 1-dimensional subgroups are ([**10**, p. 155]: [**8**, p. 404]):

1. the circle group $\mathbb{T}$, which is a maximal torus;
2. the normaliser $NT = N_{SU(2)}T$ of a maximal torus;

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3. a cyclic group $\mathbb{Z}/n$;
4. the quaternionic group

$$\langle x, y \mid x^2 = y^2, y^{-1}xy = x^{-1} \rangle$$

or a generalised quaternionic group

$$\langle x, y \mid x^n = y^2,\ y^{-1}xy = x^{-1} \rangle;$$

5. the special linear group $SL_2(\mathbb{F}_3)$, which is a lift of the tetrahedral subgroup of $SO(3)$;
6. the special linear group $SL_2(\mathbb{F}_5)$, which is a lift of the icosahedral subgroup of $SO(3)$;
7. a lift of the octahedral subgroup of $SO(3)$, which is an extension of the symmetric group S_4.

Let the representative family $\mathcal{K}$ consist of the following closed subgroups of $SU(2)$.

1. The conjugacy class of maximal tori is represented by the group of real rotations

$$SO(2) = \left\{ a_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in \mathbb{R} \right\}.$$

2. The conjugacy class of normalisers of maximal tori is represented by $NSO(2)$ which is generated by $SO(2)$ and the element

$$u = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

3. The cyclic groups $\mathbb{Z}/n$ are represented by subgroups $C_n < SO(2)$, generated by rotations $a_{2\pi/n}$. In a group of rank 1, a cyclic group of order n is completely determined by the maximal torus in which it lies, and so, since the maximal tori are all conjugate, every cyclic subgroup of order n is conjugate to C_n.

4. The generalised quaternionic groups are represented by subgroups $G_{2n} < NSO(2)$, where the generator x is the rotation $a_{\pi/n}$ and y is u. Let us show that there is only one conjugacy class of groups isomorphic to G_{2n} in $SU(2)$. Every subgroup $H = \langle x, y \rangle \cong G_{2n}$ of $SU(2)$ is contained in the normaliser NT of some maximal torus T, more precisely in $NT = T \cdot y$, where T is the maximal torus through x. Since all normalisers of maximal tori are conjugate, we can assume that $H < NSO(2)$. In this case, $x \in SO(2)$, and y is in the non-identity component of $NSO(2)$. All elements of the non-identity component of $NSO(2)$ are of the form

$$u(t) = \begin{bmatrix} i\cos t & i\sin t \\ i\sin t & -i\cos t \end{bmatrix}$$

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and are of order 4. For every t, the group $H(t) = \langle a_{2\pi/n}, u(t)\rangle \cong G_{2n}$, and is conjugate to G_{2n} by the element $u(t/2)$.

5. The remaining three groups have only one conjugacy class each in $SU(2)$, since their projections to $SO(3)$, the symmetry groups of the tetrahedron, octahedron or icosahedron, have only one conjugacy class each in $SO(3)$. Two copies, H_1 and H_2, of the same symmetry group in $SO(3)$ are conjugate by the matrix describing the change of basis which takes the polyhedron fixed by H_1 to the polyhedron fixed by H_2. An obvious choice for the representative of $SL_2(\mathbb{F}_3)$ is NG_4. For the remaining two groups, any choice of representatives is good.

PROPOSITION 3. *The family $\mathcal{K}$ is a good representative family of conjugacy classes of all closed subgroups of $SU(2)$ with respect to the standard decomposition of $SU(2)$ into one 0-cell and one 3-cell.*

PROOF: Let us first prove that the representative family $\mathcal{K}$ satisfies condition (1). □

LEMMA 2. *If $G = SU(2)$ is given the standard CW decomposition into one 0-cell and one 3-cell and $H < G$ is a closed subgroup, then for any CW decomposition of the orbit G/H, the action $\mu\colon G \times G/H \to G/H$ is cellular.*

PROOF: Choose $e^0 = I \in SU(2)$. For any closed subgroup H, the quotient G/H is a connected manifold of dimension 2 or 3. For any CW decomposition of G/H, the 0, 1 and 2 skeletons of $G \times G/H$ consist of cells of the form $e^0 \times f^j_\nu$, where f^j_ν is a j-cell of G/H, and $j = 0, 1$ or 2. Since multiplication by $e^0 = I$ is the identity,

$$\mu(e^0 \times f^j_\nu) = f^j_\nu \subset (G \times G/H)^{(j)}.$$

For $j \geqslant 3$, the j-skeleton of $G \times G/H$ is mapped to $G/H = (G/H)^{(3)}$. □

This implies that it suffices to find a CW decomposition for every orbit G/H, $H \in \mathcal{K}$, such that all fixed point sets $(G/H)^K$, $K \in \mathcal{K}$, are subcomplexes. In order to prove this we shall show that for every $H \in \mathcal{K}$ the family $\{(G/H)^K, K \in \mathcal{K}\}$ of fixed point sets is a finite family of subsets of G/H. By Lemma 1 it follows that the family $\mathcal{K}$ is good.

The fixed point set of the action of K on G/H can be described as

$$(G/H)^K = \{gH \mid g^{-1}Kg < H\}.$$

It is nontrivial only if K is subconjugate to H. So, for every $H \in \mathcal{K}$, it suffices to consider the subgroups $K \in \mathcal{K}$ which are subconjugate to H.

If H is a finite group, it has only finitely many subconjugate groups, so the family of fixed point sets $(G/H)^K$ is finite. It remains to consider the two 1-dimensional groups in $\mathcal{K}$.

The only nontrivial groups $K \in \mathcal{K}$ subconjugate to $SO(2)$ are $SO(2)$ and the cyclic groups C_n. A short computation shows that for every $n \neq 2$,

$$\big(G/SO(2)\big)^{C_n} = NSO(2)/SO(2) = \mathbb{Z}/2.$$

If $n = 2$, then C_2 is the centre Z_G, and $(G/SO(2))^{C_2} = G/SO(2)$. The family $\{(G/SO(2))^K, K \in \mathcal{K}\}$ therefore has two members: the whole space $G/SO(2)$ and $NSO(2)/SO(2)$.

A nontrivial group $K \in \mathcal{K}$ subconjugate to $NSO(2)$ is either a cyclic group C_n, a quaternionic group G_{2n}, $SO(2)$, or the whole group $NSO(2)$. For $K = C_n$, $n \neq 2, 4$, every subgroup of $NSO(2)$ conjugate to C_n must be contained in $SO(2)$, since every element $u(t)$ of the non-unit component of $NSO(2)$ has order 4. So $K = C_n$. Any conjugation $c_g : G \to G$ which maps C_n into $NSO(2)$ must therefore map the generator of C_n to an element of C_n. So

$$(G/NSO(2))^{C_n} = (G/NSO(2))^{SO(2)} = NSO(2)/NSO(2)$$

is a point. The group C_4 is conjugate to every cyclic subgroup of $NSO(2)$ generated by an element $u(t)$, and

$$(G/NSO(2))^{C_4} = \{gNSO(2) \mid g^{-1}a_{\pi/2}g = u(t) \text{ for some } t\}.$$

For $n = 2$, $(G/NSO(2))^{C_2} = G/NSO(2)$. The only subgroup of $NSO(2)$ conjugate to G_{2n} is G_{2n}, and any conjugation $c_g : G \to G$ which maps G_{2n} into $NSO(2)$ must preserve the subgroup C_n, and it must map u into some element $u(t)$. A simple computation shows that this is true for every $g \in NSO(2)$. On the other hand, it is not true if $g \notin NSO(2)$, since no such element preserves rotations. So, $(G/NSO(2))^{G_{2n}} = NSO(2)/NSO(2)$ is a point for all n. The remaining finite three subgroups in $\mathcal{K}$ are not isomorphic to any subgroup of $NSO(2)$. The family $(G/NSO(2))^K, K \in \mathcal{K}$, therefore has three members.

Proof of Theorem 1: Since we have found a representative family for the family of all closed subgroups of $SU(2)$ which is good with respect to the standard CW decomposition of the group $SU(2)$, the theorem follows from the Corollary. ∎

4. Some examples

In this section, we give several examples concerning a question posed in [7]. In [7] it is proved that for $G = O(2)$ or $G = N_{SU(2)}T$, a G-CW complex is G-homotopy equivalent to a CW complex with cellular action of G. The proof rests on the following property of these two 1-dimensional groups. The natural projection π from the set of all closed subgroups $\mathcal{S}(G)$ to the set of all conjugacy classes of closed subgroups $\mathcal{C}(G)$ has a section $\nu : \mathcal{C}(G) \to \mathcal{S}(G)$ such that if $(H) < (K)$ then $\nu((H)) < \nu((K))$, where the notation $(H) < (K)$ means that H is subcojugate to K. The question of necessary and sufficient conditions for the existence of such a ν is posed.

Let us first show that this condition is not satisfied in $SU(2)$. Let us pick any representative T for the maximal torus. A representative for the conjugacy class of G_4 must contain the representative $H_4 < T$ for $\mathbb{Z}/4$. Once a representative for G_4 is chosen, it determines the representative for $NG_4 \cong SL_2(\mathbb{F}_3)$, which also contains $H_4 < T$ but no cyclic groups $\mathbb{Z}/n \cong H_n < T$, $n > 4$. Specifically, it does not contain the representative for the conjugacy class $\mathbb{Z}/6$. On the other hand, $SL_2(\mathbb{F}_3)$ has elements of order 6, so it contains a copy of $\mathbb{Z}/6$. Therefore, $\mathbb{Z}/6$ is subconjugate to NG_4, but the representatives H_6 and NH_4 for the conjugacy classes of these two groups in $SU(2)$ cannot be chosen so that $H_6 < NH_4$.

A similar argument shows that, although the tetrahedral group contains a copy of $\mathbb{Z}/3$, we cannot choose a representative for the conjugacy class of $\mathbb{Z}/3$ which would be contained in the conjugacy class of the tetrahedral group in $SO(3)$.

It follows that no compact Lie group containing either $SU(2)$ or $SO(3)$ has a section with the required properties.

Here is an example of a finite group which does not have a section with the required properties. The authors would like to thank Aleš Vavpetič for pointing out this example. Let S_7 be the symmetric group on 7 letters. We have the following subgroups.

1. $H_2 = Z/2$, generated by one transposition,
2. $H_3 = Z/3$ generated by a 3-cycle,
3. $H_4 = Z/4$ generated by a 4-cycle,
4. $H_6 = Z/2 \times Z/3$ generated by a transposition and a disjoint 3-cycle,
5. $H_8 = Z/2 \times Z/4$ generated by a transposition and a disjoint 4-cycle,
6. $H_{12} = Z/3 \times Z/4$ generated by a 3-cycle and a disjoint 4-cycle.

Assume that $\langle(123),(4567)\rangle$ is the representative of H_{12}. Since $H_3 < H_{12}$, we must pick $\langle(123)\rangle$ for H_3, and for the same reason, $\langle(4567)\rangle$ for H_4. Since $H_3 < H_6$, the representative for H_6 must contain the 3-cycle (123), and the transposition generating H_2 must be some (ab), where $a, b \in \{4, \ldots, 7\}$. Since $H_2 < H_6$, where $\langle(ab)\rangle$ is the representative for H_2, then for H_8 the only possibility is $\langle(ab),(4567)\rangle$. But (ab) is not disjoint to (4567), and so $\langle(ab),(4567)\rangle$ is not isomorphic to H_8.

Furthermore, it obviously follows that no finite group containing S_7 has the required section.

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