Math. Proc. Camb. Phil. Soc. (2021), **171**, 421–448

doi:10.1017/S0305004120000304
First published online 10 December 2020

The factorisation property of $\ell^\infty(X_k)$

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Abstract

In this paper we consider the following problem: let X_k, be a Banach space with a normalised basis $(e_{(k,j)})_j$, whose biorthogonals are denoted by $(e^*_{(k,j)})_j$, for $k\in\mathbb{N}$, let $Z=\ell^\infty(X_k:k\in\mathbb{N})$ be their ℓ^∞-sum, and let $T:Z\to Z$ be a bounded linear operator with a large diagonal, *i.e.*,

$$\inf_{k,j}\left|e^*_{(k,j)}(T(e_{(k,j)})\right|>0.$$

Under which condition does the identity on Z factor through T? The purpose of this paper is to formulate general conditions for which the answer is positive.

2020 Mathematics Subject Classification: Primary 46B25, 46B26, 47A68, 30H10.

† Supported by the Austrian Science Foundation (FWF) under Grant Number Pr.Nr. P28352, P32728 and by the 2019 workshop in Analysis and Probability at Texas A&M University.
‡ Supported by the National Science Foundation under Grant Number DMS-1912897.
§ Supported by the Austrian Science Foundation (FWF) under Grant Number Pr.Nr. P28352 and by the 2019 workshop in Analysis and Probability at Texas A&M University.
‖ Supported by the National Science Foundation under Grant Numbers DMS-1464713 and DMS-1711076.

1. *Introduction*

Throughout this paper, we assume X_k is for each $k \in \mathbb{N}$ a Banach space which has a normalized basis $(e_{(k,j)})_j$ and let $(e^*_{(k,j)})_j \subset X^*_k$ be the coordinate functionals. Let Z be the space

$$Z = \ell^\infty(X_k : k \in \mathbb{N}) = \Big\{(x_k) : x_k \in X_k,\ k\in\mathbb{N},\ \|(x_k)\| = \sup_{k\in\mathbb{N}} \|x_k\|_{X_k} < \infty\Big\}. \qquad (1\cdot1)$$

We say a bounded linear operator $T: Z \to Z$ has *large diagonal*, if

$$\inf_{k,j} \left|e^*_{(k,j)}\big(T e_{(k,j)}\big)\right| > 0.$$

The main focus of this work is the following problem concerning operators on Z.

Problem 1·1. Does the identity operator I_Z on Z factor through every bounded linear operator $T: Z \to Z$ with a large diagonal, *i.e.*, do there exist bounded linear operators $A, B: Z \to Z$ such that $I_Z = ATB$?

If Problem 1·1 has a positive answer, we say that Z has the *factorisation property* (with respect to the array $(e_{(k,j)})$).

In our previous work [13, theorem 7·6] we solved the factorisation problem for unconditional sums of Banach spaces with bases (e.g. ℓ^p or c_0 sums). In that case, an appropriate linear ordering of the array $(e_{(k,j)})$ is a basis of the unconditional sum. Since Z is a non-separable Banach space, the array $(e_{(k,j)})$ *cannot* be reordered into a basis of Z. In particular, we lose the norm convergence of the series expansion of vectors in Z which are not in the c_0-sum of the X_k. Consequently, the arguments given in [13] are not applicable to the space Z.

Historically, the first factorisation problem of that type appeared in the 1967 paper [14] by Lindenstrauss, in which he proved that the space ℓ^∞ is prime. Later in 1982, Capon [4] actually showed that whenever X has a symmetric basis, $\ell^\infty(X)$ has the factorisation property. Bourgain proved in his 1983 work [2] that H^∞ is primary, by solving a factorisation problem of ℓ^∞-sums of finite dimensional spaces (Bourgain's localisation method). The first applications of Bourgain's localisation method appear shortly thereafter in works by the third named author [17] and by Blower [1]. The cases $X_k = L^p$, $1 < p < \infty$, $k \in \mathbb{N}$ and $X_k = H^1$, $k \in \mathbb{N}$, were treated by Wark [26] in 2007 and the third named author [19] in 2012, respectively. The ℓ^∞-sum of mixed-norm Hardy and BMO spaces and the ℓ^∞-sum of non-separable Banach spaces with a subsymmetric weak* Schauder bases were recently treated by the first named author in [11, 12].

In our previous paper [13], we developed an approach to factorisation problems based on two player games; the type of games we are referring to were first considered by Maurey, Milman, Tomczak–Jaegermann in [16] and further developed by Odell–Schlumprecht [21] and Rosendal [25] who coined the term *infinite asymptotic games* (see also [6, 22, 23]). Thereby, we were able to unify the proofs of several known factorisation results as well as provide new ones. We exploited those infinite asymptotic games to define the concept of *strategically reproducible bases* in Banach spaces.

In the present paper, we develop a two player game approach to solve the factorisation Problem 1·1 on Z if the array $(e_{(k,j)})_{k,j}$ is *uniformly asymptotically curved*; that is, if for

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every bounded array $(x_{(k,j)})_{k,j}$ for which the kth row, $(x_{(k,j)})_j$, is a block basis of $(e_{(k,j)})_j$, for every $k \in \mathbb{N}$, we have

$$\lim_n \sup_k \left\| \frac{1}{n} \sum_{j=1}^{n} x_{(k,j)} \right\|_{X_k} = 0. \qquad (1\cdot2)$$

Our first main Theorem 3·9 isolates conditions on the array $(e_{(k,j)})$ which guarantee that Problem 1·1 has a positive solution. Moreover, if we drop the restriction that the array $(e_{(k,j)})_{k,j}$ is uniformly asymptotically curved, then we were able to successfully treat the following

Problem 1·2. Does for every $T: Z \to Z$ with large diagonal with respect to $(e_{(k,j)})$ exist an infinite $\Gamma \subset \mathbb{N}$ such that the identity on $Z_\Gamma := \ell^\infty(X_k : k \in \Gamma)$ factor through T.

In the special case that $X_k = X$, $k \in \mathbb{N}$, our solution to Problem 1·2 implies a positive solution to Problem 1·1.

2. *Preliminaries*

In this section, we introduce the necessary notation and concepts.

2·1. *Review of strategically reproducible bases*

Let X denote a Banach space and $S \subset X$. We define $[S]$ as the norm-closure of span S, where span S denotes the linear span of S. Given sequences (x_i) in X and $(\widetilde{x}_i)$ in possibly another Banach space $\widetilde{X}$, we say that (x_i) and $(\widetilde{x}_i)$ are *impartially C-equivalent* if for any finite choice of scalars $(a_i) \in c_{00}$ we have

$$\frac{1}{\sqrt{C}} \left\| \sum_{i=1}^{\infty} a_i \widetilde{x}_i \right\| \le \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \le \sqrt{C} \left\| \sum_{i=1}^{\infty} a_i \widetilde{x}_i \right\|.$$

For a Banach space X we denote by $\mathrm{cof}(X)$ the set of cofinite-dimensional subspaces of X, while $\mathrm{cof}_{w^*}(X^*)$ denotes the set of cofinite-dimensionl w^*-closed subspaces of X^*.

Let $C > 0$. Given an operator $T: X \to X$, we say that *the identity C-factors through T* if there are bounded linear operators $A, B: X \to X$ with $\|A\|\|B\| \le C$ and $I = ATB$; moreover, we say that *the identity almost C-factors through T* if it $(C+\varepsilon)$-factors through T for all $\varepsilon > 0$. If (e_j) is a basis for X and (e_j^*) denotes its biorthogonal sequence and an operator T on X satisfies $\inf_j \left|e_j^*(Te_j)\right| > 0$, then we say that T has *large diagonal (with respect to (e_j))*. An operator T on X satisfying $e_m^*(Te_j) = 0$ whenever $j \neq m$, is called a *diagonal operator*.

We recall some definitions from [13].

Definition 2·1. Let X be a Banach space with a normalised Schauder basis (e_j) and its biorthogonals $(e_j^*) \subset X^*$.

(i) We say that (e_j) has the factorisation property if whenever $T: X \to X$ is a bounded linear operator with $\inf_j |e_j^*(Te_j)| > 0$ then the identity of X factors through T. More precisely, for a map $K: (0,\infty) \to \mathbb{R}^+$ we say that (e_j) has the *$K(\cdot)$-factorisation property in X* if for every $\delta > 0$ and bounded linear operator $T: X \to X$, with $\inf_j |e_j^*(Te_j)| \ge \delta$ the identity I_X on X almost $K(\delta)$-factors through T, *i.e.*, for every $\varepsilon > 0$ there are operators $A, B: X \to X$, with $\|A\|\|B\| \le K(\delta) + \varepsilon$ and $I_X = BTA$.

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(ii) We say that the basis (e_j) has the *uniform diagonal factorisation property in X* if for every $\delta > 0$ there exists $K(\delta) > 0$ so that for every bounded diagonal operator $T : X \to X$ with $\inf_j \left|e_j^*(Te_j)\right| \geq \delta$ the identity almost $K(\delta)$-factors through T. If we wish to be more specific we shall say that (e_j) has the $K(\delta)$-diagonal factorisation property.

Remark 2·2. First, we remark that if (e_j) is unconditional, then it satisfies Definition 2·1 (ii). Secondly, recall that by [**13**, remark 3·11] we have $1/\delta \leq K(\delta) \leq K(1)/\delta$.

Also recall the following definition of *strategic reproducibility* [**13**] of a Banach space X with a basis (e_j).

Definition 2·3. Let X be a Banach space with a normalised Schauder basis (e_j) and fix positive constants $C \geq 1$, and $\eta > 0$.

Consider the following two-player game between Player I and Player II:

Pregame. Before the first turn Player I is allowed to choose at the beginning of the game a partition of $\mathbb{N} = N_1 \cup N_2$.

Turn n, Step 1. Player I chooses $\eta_n > 0$, $W_n \in \mathrm{cof}(X)$, and $G_n \in \mathrm{cof}_{w^*}(X^*)$,

Turn n, Step 2. Player II chooses $i_n \in \{1, 2\}$, a finite subset E_n of N_{i_n} and sequences of non-negative real numbers $(\lambda_i^{(n)})_{i\in E_n}$, $(\mu_i^{(n)})_{i\in E_n}$ satisfying

$$1 - \eta < \sum_{i\in E_n} \lambda_i^{(n)} \mu_i^{(n)} < 1 + \eta.$$

Turn n, Step 3. Player I chooses $(\varepsilon_i^{(n)})_{i\in E_n}$ in $\{-1, 1\}^{E_n}$.

We say that Player II has a winning strategy in the game $\mathrm{Rep}_{(X,(e_i))}(C, \eta)$ if he can force the following properties on the result:

For all $j \in \mathbb{N}$ we set

$$x_j = \sum_{i\in E_j} \varepsilon_i^{(j)} \lambda_i^{(j)} e_i \text{ and } x_j^* = \sum_{i\in E_j} \varepsilon_i^{(j)} \mu_i^{(j)} e_i^*$$

and demand:

(i) the sequences (x_j) and (e_j) are impartially $(C + \eta)$-equivalent;
(ii) the sequences (x_j^*) and (e_j^*) are impartially $(C + \eta)$-equivalent;
(iii) for all $j \in \mathbb{N}$ we have $\mathrm{dist}(x_j, W_j) < \eta_j$; and
(iv) for all $j \in \mathbb{N}$ we have $\mathrm{dist}(x_j^*, G_j) < \eta_j$.

We say that (e_j) is *C-strategically reproducible in X* if for every $\eta > 0$ Player II has a winning strategy in the game $\mathrm{Rep}_{(X,(e_j))}(C, \eta)$.

It was shown in [**13**, remark 3·5] that in the case that (e_j) is shrinking and unconditional, then Definition 2·3 is equivalent to a considerably simpler formulation.

Definition 2·3 was used in [**13**] to prove the following factorisation result:

THEOREM 2·4 ([**13**, theorem 3·12]). *Let X be a Banach space with a normalized Schauder basis (e_j) that has a basis constant λ. Assume also that:*

(i) *the basis (e_j) has the $K(\delta)$-diagonal factorisation property and*
(ii) *the basis (e_j) is C-strategically reproducible in X.*

Then (e_j) has the $\lambda C^2 K(\delta)$-factorisation property.

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2·2. *Dyadic Hardy spaces and BMO*

We now turn to defining the dyadic Hardy spaces, BMO and VMO.

For a more in depth discussion of the biparameter Hardy spaces, we refer to [10]; see also [13]. Let $\mathcal{D}$ denote the collection of dyadic intervals given by

$$\mathcal{D} = \{[k2^{-n}, (k+1)2^{-n}) : n, k \in \mathbb{N}_0, 0 \le k \le 2^n - 1\}.$$

For $I \in \mathcal{D}$ we let $|I|$ denote the length of the dyadic interval I. Let h_I be the L^∞-normalised Haar function supported on $I \in \mathcal{D}$; that is, for $I = [a, b)$ and $c = (a+b)/2$, we have $h_I(x) = 1$ if $a \le x < c$, $h_I(x) = -1$ if $c \le x < b$, and $h_I(x) = 0$ otherwise. For $1 \le p < \infty$, the *Hardy space* H^p is the completion of

$$\operatorname{span}\{h_I : I \in \mathcal{D}\}$$

under the square function norm

$$\Big\| \sum_{I\in\mathcal{D}} a_I h_I \Big\|_{H^p} = \left(\int_0^1 \Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2(x) \Big)^{p/2} dx \right)^{1/p}. \tag{2·1}$$

The Haar system $(h_I)_{I\in\mathcal{D}}$ is a 1-unconditional basis of H^p, and thus gives rise to a canonical lattice structure. Finally, we define VMO as the norm closure of $(h_I)_{I\in\mathcal{D}}$ inside BMO, the dual of H^1, where we canonically identify h_I with the linear functional $f \mapsto \int h_I(x) f(x) dx$.

Next, let X denote any Banach space. We will now define the vector-valued Banach spaces $H^p[X]$, $1 \le p < \infty$, BMO$[X]$ and VMO$[X]$. Put $\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\}$, $\mathcal{D}^n = \{I \in \mathcal{D} : |I| \ge 2^{-n}\}$, $n \ge 0$ and let (r_I) denote a sequence of independent Rademacher functions. We define

$$H^p[X] = \left\{ f \in L^1(X) : \|f\|_{H^p[X]} < \infty \right\}, \tag{2·2}$$

where for every $f = \sum_{I\in\mathcal{D}} f_I h_I \in L^1(X)$, $f_I \in X$, $I \in \mathcal{D}$, the norm is given by

$$\|f\|_{H^p[X]} = \int_0^1 \Big\| \sum_{I\in\mathcal{D}} r_I(t) f_I h_I \Big\|_{L^p(X)} dt.$$

Of special interest for us is the case $p = 1$. Müller and Schechtman observed that Davis' inequality holds for Banach spaces with the UMD property [20, theorem 6], *i.e.*, there exists a constant $C > 0$ depending only on the UMD-constant of the Banach space X such that

$$C^{-1}\|f\|_{H^1[X]} \le \int_0^1 \sup_n \|\mathbb{E}_n(f)\|_X dt \le C\|f\|_{H^1[X]},$$

where $\mathbb{E}_n$ denotes the conditional expectation with respect to $\mathcal{D}_n$. For a detailed presentation of UMD spaces, we refer to Pisier's recent monograph [24].

We now define BMO$[X]$:

$$\text{BMO}[X] = \left\{ f \in L^1(X) : \|f\|_{\text{BMO}[X]} < \infty \right\},$$

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where for every $f=\sum_{I\in\mathcal{D}} f_I h_I \in L^1(X)$, $f_I\in X$, $I\in\mathcal{D}$, the norm is given by

$$\|f\|^2_{\mathrm{BMO}[X]} = \sup_{I\in\mathcal{D}} \frac{1}{|I|}\int_I \Big\|\sum_{J\subset I} f_J h_J(x)\Big\|_X^2\, dx.$$

Taking Davis' inequality into account, we observe that for UMD spaces X Bourgain [3, theorem 12] proved that the dual of $H^1[X]$ is $\mathrm{BMO}[X^*]$. Finally, we define $\mathrm{VMO}[X]$ as the norm closure of $\operatorname{span}\{x_I h_I : x_I\in X,\ I\in\mathcal{D}\}$ in $\mathrm{BMO}[X]$.

Moreover, let $\mathcal{R}=\{I\times J : I, J\in\mathcal{D}\}$ be the collection of dyadic rectangles contained in the unit square, and set

$$h_{I,J}(x,y)=h_I(x)h_J(y), \qquad I\times J\in\mathcal{R},\ x,y\in[0,1).$$

For $1\le p,q<\infty$, the *mixed-norm Hardy space* $H^p(H^q)$ is the completion of

$$\operatorname{span}\{h_{I,J} \ :\ I\times J\in\mathcal{R}\}$$

under the square function norm

$$\|f\|_{H^p(H^q)} = \left(\int_0^1\left(\int_0^1\left(\sum_{I,J} a^2_{I,J} h^2_{I,J}(x,y)\right)^{q/2} dy\right)^{p/q} dx\right)^{1/p}, \qquad (2{\cdot}3)$$

where $f=\sum_{I,J} a_{I,J}h_{I,J}$. The system $(h_{I,J})_{I\times J\in\mathcal{R}}$ is a 1-unconditional basis of $H^p(H^q)$, called the *bi-parameter Haar system*. Note that in view of the Khinchin–Kahane inequality, the norms in the spaces $H^p(H^q)$ and $H^p[H^q]$ are equivalent for $1\le p,q<\infty$; to be precise, the identity operator $J\colon H^p(H^q)\to H^p[H^q]$ satisfies

$$\|J\|\cdot\|J^{-1}\|\le C(p,q).$$

We refer to [9, theorem 4, p.20].

First note that H^p, $1<p<\infty$ is a UMD space; the UMD constant depends only on p. We will now recall that the dual of $\mathrm{VMO}(H^p)$, $1<p<\infty$, is $H^1(H^{p'})$, where $p'=p/(p-1)$.

THEOREM 2·5. *Let $1<p<\infty$, define $p'=p/(p-1)$ and $J\colon H^1(H^{p'})\to (VMO(H^p))^*$ by $f\mapsto (g\mapsto\langle f,g\rangle)$. Then J is an isomorphism with $\|J\|\cdot\|J^{-1}\|\le C(p)$; hence, $\big(VMO(H^p)\big)^*=H^1(H^{p'})$.*

Proof. Define $J\colon H^1(H^{p'})\to(\mathrm{VMO}(H^p))^*$ by $f\mapsto(g\mapsto\langle f,g\rangle)$. First, we observe that by Bourgain's vector-valued version of Fefferman's inequality [3], we know that $\|J\|\le C(p)$. Secondly, let $f\in H^1(H^{p'})$ be given as the finite linear combination $f=\sum_{I\in\mathcal{D}} f_I h_I$, $f_I\in H^{p'}$. Define the family of functions $f_t=\sum_{I\in\mathcal{D}} r_I(t) f_I h_I$, where the (r_I) are independent Rademacher functions. Since

$$\int_0^1 \|f_t\|_{L^1(H^{p'})} dt \ge c(p)\|f\|_{H^1(H^{p'})},$$

we find a $t_0\in[0,1]$ such that

$$\|f_{t_0}\|_{L^1(H^{p'})}\ge c(p)\|f\|_{H^1(H^{p'})}.$$

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Next, we choose $g \in L^\infty(H^p)$ with $\|g\|_{L^\infty(H^p)} = 1$ such that $\langle f_{t_0}, g\rangle = \|f_{t_0}\|_{L^1(H^{p'})}$. Since, f is a finite linear combination of h_I's, so is g, and we write $g = \sum_{I\in\mathcal{D}} g_I h_I$. Now, we define $h = \sum_{I\in\mathcal{D}} r_I(t_0) g_I h_I$ and note

$$\langle f, h\rangle = \langle f_{t_0}, g\rangle = \|f_{t_0}\|_{L^1(H^{p'})} \geq c(p)\|f\|_{H^1(H^{p'})}.$$

Taking into account that H^p, $1 < p < \infty$ is a UMD space, we observe

$$\|h\|_{\mathrm{BMO}(H^p)} \leq C(p)\|g\|_{\mathrm{BMO}(H^p)} \leq 4C(p)\|g\|_{L^\infty(H^p)} = 4C(p).$$

We summarise the above calculation

$$\|Jf\|_{(\mathrm{VMO}(H^p))^*} \geq c(p)\|f\|_{H^1(H^{p'})}, \qquad f \in H^1(H^{p'}).$$

Finally, let $L\colon \mathrm{VMO}(H^p) \to \mathbb{R}$ denote any bounded linear functional. Define the conditional expectation $\mathbb{E}_n$ by

$$\mathbb{E}_n\left(\sum_{I,J\in\mathcal{D}} a_{I,J} h_{I,J}\right) = \sum_{I,J\in\mathcal{D}^{n-1}} a_{I,J} h_{I,J}$$

and note that $\mathbb{E}_n$ is a contraction on $\mathrm{BMO}(H^p)$. Next, we now define $h = \sum_{I,J\in\mathcal{D}} L(h_{I,J})h_{I,J}/|I\times J|$ and calculate

$$\begin{aligned}
\|h\|_{H^1(H^{p'})} &= \sup_n \|\mathbb{E}_n(h)\|_{H^1(H^{p'})} \leq C(p)\sup_n \sup\big\{\langle \mathbb{E}_n(h), g\rangle : \|g\|_{\mathrm{BMO}(H^p)} \leq 1\big\}\\
&= C(p)\sup_n \sup\left\{\sum_{I,J\in\mathcal{D}^n} g_{I,J} L(h_{I,J}) : \|g\|_{\mathrm{BMO}(H^p)} \leq 1\right\}\\
&= C(p)\sup_n \sup\left\{L\big(\mathbb{E}_n(g)\big) : \|g\|_{\mathrm{BMO}(H^p)} \leq 1\right\} \leq C(p)\|L\|.
\end{aligned}$$

It follows that $L(f) = (Jh)(f)$, $f \in \mathrm{VMO}(H^p)$.

Remark 2·6. Later, in Lemma 2·10, we will show that $H^1(H^{p'})$ does not contain c_0. This observation allows us to give another proof of Theorem 2·5, which we will discuss below. Since $H^1(H^{p'})$ has a 1-unconditional basis and it does not contain c_0, we obtain by James characterisation [**7**, lemma 1] (see also [**15**, theorem 1·c·10]) that the biparameter Haar basis of $H^1(H^{p'})$ is boundedly complete. By 1-unconditionality $H^1(H^{p'})$ is isometrically isomorphic to the dual of the $(H^1(H^{p'}))^*$-norm-closed linear span of $\{h_{I,J} : I, J \in \mathcal{D}\}$ in $(H^1(H^{p'}))^*$ [**15**, theorem 1·b·4]. Hence, $H^1(H^{p'})$ is isomorphic to the dual of $\mathrm{VMO}(H^p) = [h_{I,J} : I, J \in \mathcal{D}] \subset \mathrm{BMO}(H^p)$ and the isomorphism constant between them depends just on the isomorphism constant between $H^1(H^p)^*$ and $\mathrm{BMO}(H^p)$.

PROPOSITION 2·7. *Let X denote a Banach space with a normalized shrinking basis (e_j) and assume that $(e_j^*/\|e_j^*\|)$ is C-strategically reproducible in X^*. Then (e_j) is C-strategically reproducible in X.*

Proof. For the sake of simplicity, we assume that (e_j) is bimonotone (and thus, $\|e_j^*\| = \|e_j\| = 1$); the statement still holds without that assumption and can be proved by slightly modifying the argument given below.

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We are now describing a winning strategy for Player II, assuming he has a winning strategy in X^*. Assume that in Turn n Step 1 Player I picks $W_n \in \mathrm{cof}(X)$ and $G_n \in \mathrm{cof}_{w^*}(X^*)$. Using his winning strategy in X^* for the spaces $\widetilde{G}_n = \overline{W_n}^{w*} \in \mathrm{cof}_{w^*}(X^{**})$ and $\widetilde{W}_n = G_n \in \mathrm{cof}(X^*)$, Player II completes Step 2 of Turn n. Obviously, (i), (ii) and (iii) are satisfied, while (iv) follows from the fact that $\mathrm{dist}(x_n, \overline{W_n}^{w^*}) = \mathrm{dist}(x_n, W_n)$, which is a consequence of the Hahn–Banach theorem.

Remark 2·8. Using Proposition 2·7, we are able to deduce the following two assertions. By [13, theorem 5·2] the Haar basis (h_I) is strategically reproducible in H^1, and hence is also strategically reproducible in VMO. Moreover, the biparameter Haar system $(h_{I,J})$ in $\mathrm{VMO}(H^p)$ is C_p-strategically reproducible for a constant $C_p > 0$, which satisfies $\sup_{p_0 \le p \le p_1} C_p \le C_{p_0,p_1} < \infty$ whenever $1 < p_0 \le p_1 < \infty$ [13, theorem 5·3].

Definition 2·9. Let X be a Banach space with a basis (e_n). We say that X is *asymptotically curved* (with respect to (e_j)) if for every bounded block basis (x_n)

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{j=1}^{n} x_n \right\| = 0.$$

As already defined in the introduction we call the sequence of Banach spaces (X_k) *uniformly asymptotically curved* with respect to the array $(e_{(k,j)})$, if for every bounded array $(x_{(k,j)})_{k,j}$, for which $(x_{(k,j)})_j$ is for every $k \in \mathbb{N}$ a block basis of $(e_{(k,j)})_j$, we have

$$\lim_n \sup_k \left\| \frac{1}{n} \sum_{j=1}^{n} x_{(k,j)} \right\|_{X_k} = 0.$$

The following special case of [23, proposition 3] is well known. It can also be easily shown directly.

LEMMA 2·10. *Let X denote a Banach space with a Schauder basis (e_j), and let $1 \le r \le \infty$, $1 \le s \le \infty$ be such that $1/r + 1/s = 1$. Assume that each block sequence (x_j^*) of the coordinate functionals (e_j^*) of (e_j) satisfies the lower r-estimate*

$$\left\| \sum_{j=1}^{n} x_j^* \right\|_{X^*} \ge c \left(\sum_{j=1}^{n} \|x_j^*\|_{X^*}^r \right)^{1/r}, \qquad n \in \mathbb{N},$$

for some constant $c > 0$ independent of n. Then each block sequence (x_j) of (e_j) satisfies the upper s-estimate

$$\left\| \sum_{j=1}^{n} x_j \right\|_{X} \le \frac{1}{c} \left(\sum_{j=1}^{n} \|x_j\|_{X}^s \right)^{1/s}, \qquad n \in \mathbb{N}.$$

The following Lemma is proved easily.

LEMMA 2·11. *Let $1 < s < \infty$. Assume that the array $(e_{k,j})_{k,j}$ is such that $(e_{k,j})_j$ satisfies an upper s-estimate for each k, where the constant C is independent of k. Then the array $(e_{k,j})_{k,j}$ is uniformly asymptotically curved.*

PROPOSITION 2·12. *Let $1 \le p, q < \infty$. Then every block sequence of the biparameter Haar system in $H^p(H^q)$ satisfies the lower* $\max(2, p, q)$*-estimate with constant* 1 *and the upper* $\min(2, p, q)$*-estimate also with constant* 1.

Proof. Before we begin with the actual proof, we define the biparameter square function $\mathbb{S}$ by

$$\mathbb{S}\left(\sum_{I,J\in\mathcal{D}} a_{I,J}h_{I,J}\right) = \left(\sum_{I,J\in\mathcal{D}} a_{I,J}^2 h_{I,J}^2\right)^{1/2}.$$

Let (f_i) denote a block sequence of the biparameter Haar system. Note that

$$\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} = \left(\int\left(\int\left(\sum_{j=1}^n (\mathbb{S}f_j)^2\right)^{q/2} dy\right)^{p/q} dx\right)^{1/p}. \tag{2·4}$$

First, we will show that $H^p(H^q)$ satisfies the upper $\min(2, p, q)$-estimate with constant 1.

Case $p \ge 2$, $q \ge 2$. Since $q/2 \ge 1$, we reinterpret (2·4) and use Minkowski's inequality to obtain

$$\begin{aligned}
\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} &= \left(\int\left\|\sum_{j=1}^n (\mathbb{S}f_j)^2\right\|_{L^{q/2}(y)}^{p/2} dx\right)^{1/p} \le \left(\int\left(\sum_{j=1}^n \|(\mathbb{S}f_j)^2\|_{L^{q/2}(y)}\right)^{p/2} dx\right)^{1/p}\\
&= \left(\int\left(\sum_{j=1}^n\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right)^{p/2} dx\right)^{1/p} = \left\|\sum_{j=1}^n\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right\|_{L^{p/2}(x)}^{1/2}\\
&\le \left(\sum_{j=1}^n\left\|\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right\|_{L^{p/2}(x)}\right)^{1/2} = \left(\sum_{j=1}^n \|f_j\|_{H^p(H^q)}^2\right)^{1/2}.
\end{aligned}$$

Case $p \le 2$, $q \ge 2$. The first step is the same as in the previous case, *i.e.*, we have

$$\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} \le \left(\int\left(\sum_{j=1}^n\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right)^{p/2} dx\right)^{1/p}.$$

Since $p/2 \le 1$, we obtain

$$\begin{aligned}
\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} &\le \left(\int\left(\sum_{j=1}^n\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right)^{p/2} dx\right)^{1/p}\\
&\le \left(\sum_{j=1}^n\int\left(\int(\mathbb{S}f_j)^q dy\right)^{p/q} dx\right)^{1/p} = \left(\sum_{j=1}^n \|f_j\|_{H^p(H^q)}^p\right)^{1/p}.
\end{aligned}$$

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Case $p \geq 2$, $q \leq 2$. Since $q/2 \leq 1$, (2·4) yields

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \leq \left(\int \left(\sum_{j=1}^{n} \int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right)^{1/p} = \left\| \sum_{j=1}^{n} \int (\mathbb{S} f_j)^q \, dy \right\|_{L^{p/q}(x)}^{1/q}$$
$$\leq \left(\sum_{j=1}^{n} \left(\int \left(\int (\mathbb{S} f_j)^q \, dy \right)^{p/q} dx \right)^{q/p} \right)^{1/q} = \left(\sum_{j=1}^{n} \| f_j \|_{H^p(H^q)}^q \right)^{1/q}.$$

Case $p \leq 2$, $q \leq 2$. The first step is similar to the previous case, *i.e.*,

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \leq \left(\int \left(\sum_{j=1}^{n} \int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right)^{1/p}. \tag{2·5}$$

If $p \geq q$, we use Minkowski's inequality in $L^{p/q}$ and obtain

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \leq \left\| \sum_{j=1}^{n} \left\| \int (\mathbb{S} f_j)^q dy \right\|_{L^{p/q}(x)} \right\|^{1/q} = \left(\sum_{j=1}^{n} \| f_j \|_{H^p(H^q)}^q \right)^{1/q}.$$

If $p \leq q$, (2·5) yields

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \leq \left(\sum_{j=1}^{n} \int \left(\int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right)^{1/p} = \left(\sum_{j=1}^{n} \| f_j \|_{H^p(H^q)}^p \right)^{1/p}.$$

Secondly, we will prove that $H^p(H^q)$ satisfies the lower $\max(2, p, q)$-estimate with constant 1.

Case $p \geq q \geq 2$. Since $q/2 \geq 1$, we obtain from (2·4)

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \geq \left(\int \left(\sum_{j=1}^{n} \int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right)^{1/p}. \tag{2·6}$$

Using $p/q \geq 1$ yields

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \geq \left(\sum_{j=1}^{n} \int \left(\int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right)^{1/p} = \left(\sum_{j=1}^{n} \| f_j \|_{H^p(H^q)}^p \right)^{1/p}.$$

Case $q \geq p \geq 2$. Using (2·6) and Minkowski's inequality yields

$$\left\| \sum_{j=1}^{n} f_j \right\|_{H^p(H^q)} \geq \left(\int \left(\sum_{j=1}^{n} \int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right)^{1/p} = \left(\int \left\| \left(\int (\mathbb{S} f_j)^q dy \right)^{p/q} \right\|_{\ell^{q/p}(j)} dx \right)^{1/p}$$
$$\geq \left(\left\| \int \left(\int (\mathbb{S} f_j)^q dy \right)^{p/q} dx \right\|_{\ell^{q/p}(j)} \right)^{1/p} = \left(\sum_{j=1}^{n} \| f_j \|_{H^p(H^q)}^q \right)^{1/q}.$$

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Case $p \geq 2 \geq q$. By (2·4) and Minkowski's inequality, and since $2/q \geq 1$, $p/2 \geq 1$ we obtain

$$\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} = \left(\int\left(\int \|(\mathbb{S}f_j)^q\|_{\ell^{2/q}(j)}\,dy\right)^{p/q} dx\right)^{1/p} \geq \left(\int \left\|\int (\mathbb{S}f_j)^q dy\right\|_{\ell^{2/q}(j)}^{p/q} dx\right)^{1/p}$$

$$= \left(\int\left(\sum_{j=1}^n\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right)^{p/2} dx\right)^{1/p}$$

$$\geq \left(\sum_{j=1}^n\int\left(\int(\mathbb{S}f_j)^q dy\right)^{p/q} dx\right)^{1/p} = \left(\sum_{j=1}^n \|f_j\|_{H^p(H^q)}^p\right)^{1/p}.$$

Case $p, q \leq 2$. The first step is the same as in the previous case, *i.e.*,

$$\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} \geq \left(\int\left(\sum_{j=1}^n\left(\int(\mathbb{S}f_j)^q dy\right)^{2/q}\right)^{p/2} dx\right)^{1/p}$$

$$= \left(\int\left\|\left(\int(\mathbb{S}f_j)^q dy\right)^{p/q}\right\|_{\ell^{2/p}(j)} dx\right)^{1/p}.$$

Here, $2/p \geq 1$, hence, by Minkowski's inequality, we obtain

$$\left\|\sum_{j=1}^n f_j\right\|_{H^p(H^q)} \geq \left(\left\|\int\left(\int(\mathbb{S}f_j)^q dy\right)^{p/q} dx\right\|_{\ell^{2/p}(j)}\right)^{1/p} = \left(\sum_{j=1}^n \|f_j\|_{H^p(H^q)}^2\right)^{1/2}.$$

Remark 2·13. Let $1 < r, s < \infty$, then the identity operator provides an isomorphism between $H^r(H^s)$ and $L^r(L^s)$ (see Capon [5]); hence, by Proposition 2·12, each block sequence with respect to the biparameter Haar system in $L^r(L^s)$ satisfies an upper $\min(r, s)$-estimate with constant $C = C_{r,s}$. Moreover, $\sup_{p_0 \leq r,s \leq p_1} C_{r,s} \leq C_{p_0,p_1} < \infty$ whenever $1 < p_0 \leq p_1 < \infty$.

3. *Simultaneous strategical reproducibility and statement of the main results*

In order to state our main results, we will now state precisely the necessary definitions. Recall that we defined Z as the space

$$Z = \ell^\infty(X_k : k \in \mathbb{N}) = \left\{(x_k) : x_k \in X_k,\ k\in\mathbb{N},\ \|(x_k)\| = \sup_{k\in\mathbb{N}} \|x_k\|_{X_k} < \infty\right\}. \qquad (3·1)$$

We also put

$$Y = c_0(X_k : k \in \mathbb{N}) = \left\{(x_k) : x_k \in X_k,\ k\in\mathbb{N},\ \lim_{k\to\infty} \|x_k\|_{X_k} = 0\right\}.$$

If for some space X we have $X_k = X$, for all $k \in \mathbb{N}$, we will write $\ell^\infty(X)$ and $c_0(X)$ instead of $\ell^\infty(X_k : k \in \mathbb{N})$ and $c_0(X_k : k \in \mathbb{N})$.

For $\overline{x} = (x_k)$ in Z (or Y) we call the set $\mathrm{supp}(\overline{x}) = \{k \in \mathbb{N} : x_k \neq 0\}$ *the support of* $\overline{x}$ *in* Z *(or* Y*)*.

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For $N \subset \mathbb{N}$, and $\overline{x} = (x_k) \in Z$ we let $P_N(x) \in Z$ be the projection of x on the coordinates in N,*i.e.*,

$$P_N(\overline{x}) = (y_k), \text{ with } y_k = \begin{cases} x_k & \text{if } k \in N, \text{ and} \\ 0 & \text{if } k \notin N, \end{cases} \tag{3·2}$$

and we put $Z_N := P_N(Z)$ which is isometrically isomorphic to $\ell^\infty(X_k : k \in N)$ and will be identified with that space. For $\overline{x} = (x_k) \in Z$ and $k \in \mathbb{N}$ we put $P_k(\overline{x}) = x_k$. In particular we identify X_k with its image under the canonical embedding into Z. We also identify X_k^* in the canonical way as a subspace of Z^* ($x^* \in X_k^*$ is acting on the k-component of $\bar{z} = (z_k) \in Z$). Note that X_k^* is thus a w^*-closed subspace of Z^*. For $k, j \in \mathbb{N}$ we also consider $e_{(k,j)}$ to be an element of Z, and $e^*_{(k,j)}$ to be an element of Z^* in the obvious way.

Convention 3·1. We fix from now on a bijective map $\nu(\cdot,\cdot) : \mathbb{N}^2 \to \mathbb{N}$, $(k, j) \mapsto \nu(k, j)$ with the property that for any $i, j, k \in \mathbb{N}$ we have $\nu(k, i) < \nu(k, j)$ if and only if $i < j$. We denote the inverse map by $(\kappa, \iota) : \mathbb{N} \to \mathbb{N}^2$, $n \mapsto (\kappa(n), \iota(n))$. We order the array $(e_{(k,j)} : k, j \in \mathbb{N})$ into a sequence (e_n), by putting $e_n = e_{(\kappa(n),\iota(n))}$ and $e_n^* = e^*_{(\kappa(n),\iota(n))}$. More generally, whenever $(x_{(k,j)})_j$ is a sequence in X_k, $k \in \mathbb{N}$, then we order the array $(x_{(k,j)})$ into the sequence (x_n) defined by $x_n = x_{(\kappa(n),\iota(n))}$; we do the same for $(x^*_{(k,j)})$.

Let $\mathcal{P}$ denote the product topology on Z, *i.e.*, the coarsest topology such that all the P_k, $k \in \mathbb{N}$, are continuous. Let $\overline{z}^{(j)} \in Z$, $j \in \mathbb{N}$, and $\overline{z} \in Z$. Then $(\overline{z}^{(j)})$ converges to $\overline{z}$ with respect to $\mathcal{P}$, if and only if

$$\lim_{j\to\infty} P_k \overline{z}^{(j)} = P_k \overline{z}, \quad \text{for all } k \in \mathbb{N}.$$

Whenever a sequence converges in Z, we implicitly refer to convergence with respect to the product topology $\mathcal{P}$. Whenever a sequence converges in some X_k, we refer to the norm topology in X_k.

Remark 3·2. For each $k \in \mathbb{N}$, assume $(e_{(k,j)})_j$ has basis constant $\lambda \geq 1$. Let $\sum_{n=1}^\infty a_n e_n \in Y$, where the series converges in the relative topology $\mathcal{P}|_Y$. Then the series $\sum_{n=1}^\infty a_n e_n$ converges in the norm topology of Y.

We now consider the following "simultaneous version" of the game described in [**13**].

Definition 3·3. Let $C \geq 1$. We say that the array $(e_{(k,j)})$ is C*-simultaneously strategically reproducible in* Z if for every $k \in \mathbb{N}$ $(e_{(k,j)})_j$ is C-strategically reproducible in X_k.

Remark 3·4. Note that we can also describe simultaneous strategic reproducibility in terms of the following two-player game: The array $(e_{(k,j)})$ is C- simultaneously strategically reproducible if and only if for every $\eta > 0$, Player II has a winning strategy for the game $\mathrm{Rep}_{(Z,(e_{(k,j)}))}(C, \eta)$ between Player I and Player II:

Assume the space Z, P_N, $(e_n : n \in \mathbb{N})$ and $(e_n^* : n \in \mathbb{N})$, are defined as in (3·1), (3·2) and in Convention 3·1.

Pregame. Before the first turn Player I is allowed to choose a partition of $\mathbb{N} = N_1 \cup N_2$. For $k \in \mathbb{N}$, and $r = 1, 2$ let $N_r^{(k)} = \{\nu(k, j) : j \in \mathbb{N}\} \cap N_r$.

Turn n, Step 1. Player I chooses $\eta_n > 0$, $G_n \in \mathrm{cof}_{w^*}(X^*_{\kappa(n)})$, and $W_n \in \mathrm{cof}(X_{\kappa(n)})$.

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Turn n, Step 2. Player II chooses $i_n \in \{1, 2\}$, a finite subset E_n of $N_{i_n}^{(\kappa(n))}$ and sequences of non-negative real numbers $(\lambda_i^{(n)})_{i\in E_n}$, $(\mu_i^{(n)})_{i\in E_n}$ satisfying

$$1 - \eta < \sum_{i\in E_n} \lambda_i^{(n)} \mu_i^{(n)} < 1 + \eta.$$

Turn n, Step 3. Player I chooses $(\varepsilon_i^{(n)})_{i\in E_n}$ in $\{-1, 1\}^{E_n}$.

We say that Player II has a winning strategy in the game $\mathrm{Rep}_{Z,(e_{(k,j)})}(C, \eta)$ if he can force the following properties on the result:

For all $k, j \in \mathbb{N}$ we set $n = \nu(k, j)$ and put

$$x_n = x_{(k,j)} = \sum_{i\in E_n} \varepsilon_i^{(n)} \lambda_i^{(n)} e_{(k,i)} \qquad \text{and} \qquad x_n^* = x_{(k,j)}^* = \sum_{i\in E_n} \varepsilon_i^{(n)} \mu_i^{(n)} e_{(k,i)}^*$$

and demand:

(i) the sequences $(x_{(k,j)})_j$ and $(e_{(k,j)})_j$ are impartially $(C + \eta)$-equivalent for each $k \in \mathbb{N}$;
(ii) the sequences $(x_{(k,j)}^*)_j$ and $(e_{(k,j)}^*)_j$ are impartially $(C + \eta)$-equivalent for each $k \in \mathbb{N}$;
(iii) for all $n \in \mathbb{N}$ we have $\mathrm{dist}(x_n^*, G_n) < \eta_n$;
(iv) for all $n \in \mathbb{N}$ we have $\mathrm{dist}(x_n, W_n) < \eta_n$.

Completely analogous to Definition 2·1, we define the corresponding notions in Z, below.

Definition 3·5. Let $T : Z \to Z$ be an operator.

(i) We call T a *diagonal operator on* Z, if $e_m^*(Te_n) = 0$, whenever $m \neq n$.
(ii) We say that T has *a large diagonal* if $\inf_n \left|e_n^*(Te_n)\right| > 0$.

We are now in the position to state our two main results.

THEOREM 3·6. *Assume that there are $C, \lambda \geq 1$, and a map $K : (0, \infty) \to (0, \infty)$ so that:*

(i) *the basis constant of $(e_{(k,j)})_j$, is at most λ in X_k, for each $k \in \mathbb{N}$;*
(ii) *$(e_{(k,j)})_j$ has the K-diagonal factorisation property in X_k, for each $k \in \mathbb{N}$;*
(iii) *the array $(e_{(k,j)})_{k,j}$ is C-simultaneously strategically reproducible in Z.*

Let $T : Z \to Z$ be bounded and linear, with

$$\delta = \inf_{k,j\in\mathbb{N}} \left|e_{(k,j)}^*(Te_{(k,j)})\right| > 0.$$

Then for each sequence of infinite subsets (Ω_l) of $\mathbb{N}$, there is an infinite $\Gamma \subset \mathbb{N}$ so that $\Gamma \cap \Omega_l$ is infinite for all $l \in \mathbb{N}$ and the identity on Z_Γ $\lambda C K(\delta)$-factors through T.

Remark 3·7. Note, that we did not simply state in Theorem 3·6 that there is an infinite Γ so that the identity on Z_Γ factors through T. However, we show more: additionally we require that the intersection of Γ with any prespecified sequence of infinite sets $\Omega_l \subset N$, $l \in \mathbb{N}$, also has to stay infinite. This means that if an infinite number of elements of the spaces X_n belong to a certain category, then also the spaces in an infinite subset of Γ will belong to that category. In Proposition 7·2 we will provide an application of that additional condition on Γ.

COROLLARY 3·8. *Assume that X is a Banach space with a normalised basis which is C-strategically reproducible and has the K-diagonal factorisation property for some $K:(0,\infty)\to(0,\infty)$. Define $e_{(k,j)}$ to be the jth basis element of the kth component in $\ell^\infty(X)$, for $k, j\in\mathbb{N}$.*

Then the array $(e_{(k,j)})_{k,j}$ is simultaneously C-strategically reproducible in $Z=\ell^\infty(X)$, and the identity on Z factors through every operator $T: Z\to Z$ with large diagonal.

The following result describes a situation where it is not necessary, like in Theorem 3·6, to pass to an infinite subset Γ of $\mathbb{N}$.

THEOREM 3·9. *Assume the array $(e_{(k,j)})$ is uniformly asymptotically curved (see (1·2)), and furthermore, that there are $C,\lambda\geq 1$, and a map $K:(0,\infty)\to(0,\infty)$ so that:*

(i) *the basis constant of $(e_{(k,j)})_j$, is at most λ in X_k, for each $k\in\mathbb{N}$;*
(ii) *$(e_{(k,j)})_j$ has the K-diagonal factorisation property in X_k, for each $k\in\mathbb{N}$;*
(iii) *the array $(e_{(k,j)})_{k,j}$ is simultaneously C-strategically reproducible in Z.*

Let $T: Z\to Z$ be bounded and linear, with

$$\delta=\inf_{k,j\in\mathbb{N}}\left|e^*_{(k,j)}(Te_{(k,j)})\right|>0.$$

Then the identity on Z $\lambda CK(\delta)$-factors through T.

4. *Factorisation through diagonal operators*

The main purpose of this section is to prove the pivotal Proposition 4·5.

LEMMA 4·1. *Let $S: Z\to Z$ be a bounded operator, and let (Ω_k) denote a sequence of infinite subsets of $\mathbb{N}$. For any $\rho>0$ there is an infinite $\Gamma\subset\mathbb{N}$ so that*

$$\Gamma\cap\Omega_k \text{ is infinite for all } k\in\mathbb{N}$$

and

$$\left\|P_\Gamma\circ S|_{Z_\Gamma}\right\|\leq 2\left\|S|_Y\right\|+\rho.$$

Proof. Let $l\in\mathbb{N}$. We first observe that for a fixed $x^*\in X_l^*$, and for any infinite set $\Lambda\subset\mathbb{N}$, such that $\Lambda\cap\Omega_k$ is infinite for all $k\in\mathbb{N}$, there is an infinite $\Lambda'\subset\Lambda$ so that $\Lambda'\cap\Omega_k$ is still infinite, for all $k\in\mathbb{N}$, and $\|x^*\circ P_l\circ S\circ P_{\Lambda'}\|\leq\rho$. Indeed, if that were not true we could choose for any $n\in\mathbb{N}$ a partition of Λ into n infinite subsets $\Lambda_1,\Lambda_2,\ldots\Lambda_n$ such that $\Lambda_j\cap\Omega_k$ is infinite, for all $1\leq j\leq n$, $k\in\mathbb{N}$, and find $\overline{x}_1,\overline{x}_2,\ldots\overline{x}_n$ in Z with $\|\overline{x}_j\|\leq 1$, $\mathrm{supp}(\overline{x}_j)\subset\Lambda_j$ and $x^*\big(P_l\circ S(\overline{x}_j)\big)\geq\rho$. But then we would have for $\overline{x}=\sum_{j=1}^n\overline{x}_j$ that $\|\overline{x}\|\leq 1$ and $x^*\big(P_l\circ S(\overline{x})\big)\geq n\rho$, which is impossible assuming that n is chosen large enough.

We now choose a sequence $(y_j^*: j\in\mathbb{N})$ in X_l^* with $\|y_j^*\|=1$ which norms the elements of X_l. Using our previous observation we find infinite sets $\Lambda_1\supset\Lambda_2\supset\cdots$, so that for all $j,k\in\mathbb{N}$ theset $\Lambda_j\cap\Omega_k$ is infinite and

$$\left\|y_j^*\circ P_l\circ S\circ P_{\Lambda_j}\right\|\leq\rho.$$

Then choose a diagonal set $\Gamma'=\{\lambda_j: j\in\mathbb{N}\}$ of the sets Γ_j, $j\in\mathbb{N}$, which has furthermore the property that $\Gamma'\cap\Omega_k$ is infinite for all $k\in\mathbb{N}$.

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Now, we can choose for a given $\varepsilon > 0$ and an $\overline{x} = (x_k) \in Z$ a number $j \in \mathbb{N}$ for which

$$y_j^*\big(P_l \circ S \circ P_{\Gamma'}(\overline{x})\big) \geq (1-\varepsilon)\big\|P_l \circ S \circ P_{\Gamma'}(\overline{x})\big\|$$

and thus,

$$\begin{aligned}(1-\varepsilon)\|P_l \circ S \circ P_{\Gamma'}(\overline{x})\| &\\ &\leq \big|y_j^* \circ P_l \circ S \circ P_{\Gamma' \setminus \Gamma_j}(\overline{x})\big| + \big|y_j^* \circ P_l \circ S \circ P_{\Gamma_j} \circ P_{\Gamma'}(\overline{x})\big| \\ &\leq \|P_l \circ S|_Y\| + \|y_j^* \circ P_l \circ S \circ P_{\Gamma_j}\| \leq \|S|_Y\| + \rho.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\big\|P_l \circ S|_{Z_{\Gamma'}}\| \leq \|S|_Y\big\| + \rho.$$

Let $(k_j) \subset \mathbb{N}$ be a sequence in which every $k \in \mathbb{N}$ appears infinitely often. Starting by letting $\Gamma_0 = \mathbb{N}$ and $\gamma_1 \in \Omega_{k_1}$, we can apply our observation and recursively choose infinite sets $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$, and $\gamma_1 < \gamma_2 < \cdots$ so that

$$\Gamma_j \cap \Omega_k \text{ is infinite for all } j, k \in \mathbb{N}, \gamma_j \in \Gamma_{j-1} \cap \Omega_{k_j} \text{ and } \big\|P_{\gamma_j} \circ S|_{Z_{\Gamma_j}}\| \leq \|S|_Y\big\| + \rho.$$

Finally, letting $\Gamma = \{\gamma_j : j \in \mathbb{N}\}$, we deduce that

$$\begin{aligned}\|P_\Gamma S|_{Z_\Gamma}\| &= \sup_{j \in \mathbb{N}} \|P_{\gamma_j} S|_{Z_\Gamma}\| \\ &\leq \sup_{j \in \mathbb{N}} \Big(\big\|P_{\gamma_j} S|_{Z_{\{\gamma_1,\gamma_2,\ldots,\gamma_j\}}}\big\| + \big\|P_{\gamma_j} S|_{Z_{\Gamma_j}}\big\|\Big) \leq 2\big\|S|_Y\big\| + \rho,\end{aligned}$$

which proves our claim.

LEMMA 4·2. *We assume that the array $(e_{(k,j)})_{k,j}$ is uniformly asymptotically curved. Let $z^* \in Z^*$ and $\eta > 0$. Then there exists $(m_k) \subset \mathbb{N}$ so that for every $w = (w_k) \in Z$, $\|w\| \leq 1$ with $w_k \in [e_{(k,j)} : j \geq m_k]$, $k \in \mathbb{N}$, it follows that $|z^*(w)| \leq \eta$. In other words, letting $W_k = [e_{(k,j)} : j \geq m_k]$ it follows that*

$$\|z^*|_{\ell^\infty(W_k : k \in \mathbb{N})}\| \leq \eta.$$

Proof. Assume that our claim is not true for some $z^* \in Z^*$ and $\eta > 0$. Then we can choose inductively for every $n \in \mathbb{N}$, sequences $(m_k^{(n)})_{k \in \mathbb{N}} \subset \mathbb{N}$, and $\overline{z}_n = (w_k^{(n)})_k \in B_Z$, so that

$$m_k^{(n-1)} < m_k^{(n)} \text{ for all } k \in \mathbb{N} \text{ (with } m_k^{(0)} = 0), \tag{4·1}$$

$$w_k^{(n)} \in [e_{(k,j)} : m_k^{(n-1)} < j \leq m_k^{(n)}], \text{ for all } k \in \mathbb{N}, \text{ and} \tag{4·2}$$

$$z^*((w_k^{(n)})_k) > \eta/2. \tag{4·3}$$

Then for $n \in \mathbb{N}$ define

$$\overline{u}_n = \left(\frac{1}{n}\sum_{m=1}^{n} w_k^{(m)} : k \in \mathbb{N}\right) \in Z,$$

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It follows from our assumption on the spaces X_k that $\lim_n \|\overline{u}_n\|_Z = 0$, but on the other hand we have

$$z^*\big(\overline{u}_n\big) = \frac{1}{n}\sum_{m=1}^{n} z^*\big((w_k^{(n)})_k\big) > \eta/2$$

which for large enough n leads to a contradiction.

Let (e_j) denote a basic sequence in a Banach space X. We say that a sequence (x_j) in X is a *perturbation of a block basic sequence of* (e_j) if there exists a block basis sequence $(\widetilde{x}_j)$ of (e_j) such that $\sum_{j=1}^{\infty} \|x_j - \widetilde{x}_j\|_X < \infty$.

Notation 4·3. Let $\lambda \geq 1$ and assume the basis constant of $(e_{(k,j)})_j$, is not larger than λ, for all $k \in \mathbb{N}$. Assume that for each $k \in \mathbb{N}$, $(x_{(k,j)})_j$ is a sequence in X_k and that $(x^*_{(k,j)})_j$ is a perturbation of a block basic sequence of $(e^*_{(k,j)})_j$ in X_k^*. Moreover, assume that:

(i) $(x_{(k,j)})_j$ and $(e_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;
(ii) $(x^*_{(k,j)})_j$ and $(e^*_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;
(iii) $1 - \eta < x^*_{(k,i)}(x_{(k,i)}) < 1 + \eta$, for all $k, i \in \mathbb{N}$.

Then for each $k,\ j \in \mathbb{N}$, we define

$$A_k : X_k \longrightarrow X_k, \qquad A_k e_{(k,j)} = x_{(k,j)},$$

$$B_k : X_k \longrightarrow X_k, \qquad B_k x = \sum_{j=1}^{\infty} x^*_{(k,j)}(x) e_{(k,j)}.$$

and their respective vector operator version

$$A : Z \longrightarrow Z, \qquad A\big((z_k)_k\big) = \big(A_k z_k\big)_k,$$

$$B : Z \longrightarrow Z, \qquad B\big((z_k)_k\big) = \big(B_k z_k\big)_k.$$

Remark 4·4. In view of our hypothesis, the operators A_k, B_k, $k \in \mathbb{N}$ and consequently A, B in Notation 4·3 are well defined and satisfy $\|A\| \leq \sqrt{C}$ and $\|B\| \leq \lambda\sqrt{C}$. Indeed, for $k \in \mathbb{N}$ we have

$$\sup_{x \in X_k, \|x\| \leq 1} \Big\| \sum_j x_j^*(x) e_{k,j} \Big\| = \sup_{\substack{x \in X_k \|x\| \leq 1 \\ x^* \in X_k^* \|x^*\| \leq 1}} \sum_j x_j^*(x) x^*(e_{k,j})$$

$$= \sup_{\substack{x \in X_k, \|x\| \leq 1 \\ x^* \in X_k^* \|x^*\| \leq 1}} \Big(\sum_j x^*(e_{k,j}) x_j^* \Big)(x)$$

$$= \sup_{x^* \in X_k^*, \|x^*\| \leq 1} \Big\| \sum_j x^*(e_{k,j}) x_j^* \Big\| \leq \lambda \sup_{x^* \in \operatorname{span}(e^*_{k,j}), \|x^*\| \leq 1} \Big\| \sum_j x^*(e_{k,j}) x_j^* \Big\|$$

$$\leq \lambda\sqrt{C} \sup_{x^* \in \operatorname{span}(e^*_{k,j}), \|x^*\| \leq 1} \Big\| \sum_j x^*(e_j) e^*_{k,j} \Big\| = \lambda\sqrt{C}.$$

Moreover, for each $k, i \in \mathbb{N}$ we have

$$BTA(e_{(k,i)}) = \left(\sum_{j=1}^{\infty} x^*_{(l,j)}(Tx_{(k,i)})e_{(l,j)} : l \in \mathbb{N} \right) \in Z \tag{4·4}$$

it follows (infinite sums are meant to converge with respect to the topology $\mathcal{P}$, introduced in Convention 3·1) for an $x = \sum_{n=1}^{\infty} a_n e_n \in Z$ that

$$\begin{aligned} BTA\left(\sum_{n=1}^{\infty} a_n e_n\right) &= \sum_{m=1}^{\infty} x^*_m \left(TA \sum_{n=1}^{\infty} a_n e_n\right) e_m \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} a_n x^*_m (Tx_n) e_m \\ &\quad + \sum_{m=1}^{\infty} a_m x^*_m (Tx_m) e_m + \sum_{m=1}^{\infty} x^*_m \big(T \sum_{n=m+1}^{\infty} a_n x_n\big) e_m. \end{aligned} \tag{4·5}$$

(Note that T might not be continuous with respect to $\mathcal{P}$, we only used the linearity of T). If $x = \sum_{n=1}^{\infty} a_n e_n \in Y$ (which implies that this series is norm convergent) then

$$BTA\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n x^*_m \big(Tx_n\big) e_m, \tag{4·6}$$

where the convergence of the double sum is meant with respect to the topology $\mathcal{P}$.

We now formulate and prove a rather technical proposition which presents the heart of the proof of Theorem 3·6 and Theorem 3·9.

PROPOSITION 4·5. *Assume that for some $\lambda \geq 1$ the basis constant of $(e_{(k,j)})_j$, is not larger than λ. Let $T : Z \to Z$ be a bounded linear operator, for each $k \in \mathbb{N}$ let $(x_{(k,j)})_j$ be a sequence in X_k and let $(x^*_{(k,j)})_j$ be a perturbation of a block basis of $(e^*_{(k,j)})_j$ in X^*_k.*

Let $0 < \eta \leq 1$, $C \geq 1$ and $(\eta_n) \subset (0, 1]$ so that $\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \eta_n < \eta/2$. Consider the following conditions:

(i) *$(x_{(k,j)})_j$ and $(e_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;*
(ii) *$(x^*_{(k,j)})_j$ and $(e^*_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;*
(iii) *$\sum_{n=1}^{m-1} |x^*_m(Tx_n)| < \eta_m$, for all $m \in \mathbb{N}$;*
(iv) *$\sum_{n=m+1}^{\infty} |x^*_m(Tx_n)| < \eta_m$, for all $m \in \mathbb{N}$.*

In order to formulate the last condition, we assume that for each $n \in \mathbb{N}$, we are given a sequence $(W^{(n)}_k)_k$, where $W^{(n)}_k$ is a cofinite-dimensional subspace of X_k, with $W^{(n+1)}_k \subset W^{(n)}_k$ for $k, n \in \mathbb{N}$.

(v) *For each $n \in \mathbb{N}$, assume that*

$$\|T^* x^*_n|_{\ell^\infty(W^{(n+1)}_k : k \in \mathbb{N})}\| < \eta_n \qquad \text{and} \qquad dist\big(x_n, W^{(n)}_{\kappa(n)}\big) < \eta_n,$$

for all $n \in \mathbb{N}$.

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Let $D\colon Z \to Z$ denote the diagonal operator given by

$$De_{(k,j)} = x^*_{(k,j)}(Tx_{(k,j)})e_{(k,j)}, \qquad \text{for all } k, j \in \mathbb{N}.$$

Then the following assertions (a) and (b) hold true:

(a) *If (i)–(iv) is satisfied, then $BTA - D\colon Y \to Y$ is well defined and*

$$\|BTA - D : Y \longrightarrow Y\| \le 2\lambda\eta.$$

D is a bounded operator from Z to Z and thus also $D - BTA$. Moreover, for each sequence of infinite subsets (Ω_l) of $\mathbb{N}$, there exists an infinite set $\Gamma \subset \mathbb{N}$ such that

$$\Gamma \cap \Omega_l \text{ is infinite, for all } l \in \mathbb{N}, \text{ and } \|D_\Gamma - P_\Gamma BTA : Z_\Gamma \longrightarrow Z_\Gamma\| < 5\lambda\eta,$$

where $D_\Gamma = P_\Gamma \circ D$.

If we additionally assume that $K \ge 1$ is such that the identity K-factors through $P_\Gamma D|_{Z_\Gamma}$ and $\eta < 1/(5\lambda K)$, then the identity on $Z_\Gamma (\lambda KC/1 - 5\lambda K\eta)$-factors through T.

(b) *If alternatively, (i)–(iii) and (v) are satisfied, then*

$$\|BTA - D\colon Z \to Z\| < 2\lambda\sqrt{C}(3 + \|T\|)\eta.$$

If we additionally assume that $K \ge 1$ is such that the identity on Z K-factors through D and $\eta < 1/(2\lambda\sqrt{C}(3+\|T\|)K)$ then the identity on Z $\left(\lambda KC/1 - 2\lambda\sqrt{C}(3 + \|T\|)K\eta\right)$-factors through T.

Proof. Naturally, the proof splits into two parts.

Proof of (a). Let $y = \sum_{n=1}^{\infty} a_n e_n \in Y$ be finitely supported and observe that by (4·6)

$$(BTA - D)y = \sum_{m=1}^{\infty} \sum_{n:n\neq m} a_n x^*_m(Tx_n)e_m. \tag{4·7}$$

Hence, for each $k \in \mathbb{N}$ we obtain

$$P_k(BTA - D)y = \sum_{m:\kappa(m)=k} \sum_{n:n\neq m} a_n x^*_m(Tx_n)e_m,$$

and thus

$$\begin{aligned} \|P_k(BTA - D)y\|_{X_k} &\le \sum_{m:\kappa(m)=k} \sum_{n\neq m}^{\infty} |a_n||x^*_m(Tx_n)| \\ &\le 2\lambda\|y\|_Y \sum_{m:\kappa(m)=k} \sum_{n\neq m}^{\infty} |x^*_m(Tx_n)|. \end{aligned} \tag{4·8}$$

By (iii) and (iv) we have $\sum_{m,n:m\neq n} |x^*_m(Tx_n)| < \infty$, and hence

$$\lim_{k\to\infty} \sum_{m:\kappa(m)=k} \sum_{n\neq m}^{\infty} |x^*_m(Tx_n)| = 0.$$

Together with estimate (4·8) we obtain $(BTA - D)y \in Y$.

Using (iii) and (iv), (4·7) yields

$$\|(BTA - D)y\|_Y \le \sum_{m=1}^{\infty}\sum_{n=1}^{m-1} |a_n||x_m^*(Tx_n)| + \sum_{m=1}^{\infty}\sum_{n=m+1}^{N} |a_n||x_m^*(Tx_n)| \le 2\lambda\eta\|y\|_Y.$$

We conclude

$$\|BTA - D\colon Y \longrightarrow Y\| \le 2\lambda\eta.$$

Next, we observe that since D is a diagonal operator, which means that D is of the form

$$D\colon Z \longrightarrow Z, \quad (x_k) \longrightarrow (D_k x_k),$$

where the operators $D_k\colon X_k \to X_k$, $k \in \mathbb{N}$, are uniformly bounded. Thus, D is also a bounded operator on Z with norm $\sup_k \|D_k\colon X_k \to X_k\|$ and the operator $P_\Gamma BTA|_{Z_\Gamma} - D$ is a well defined and a bounded operator on all of Z. Our conclusion follows therefore from Lemma 4·1 for some infinite set $\Gamma \subset \mathbb{N}$.

For the additional part, assume that $\hat{B}\colon Z_\Gamma \to Z_\Gamma$ and $\hat{A}\colon Z_\Gamma \to Z_\Gamma$ are such that $\|\hat{B}\|\|\hat{A}\| \le K$ and $I = \hat{B}D_\Gamma\hat{A}$. It follows that $\|I - \hat{B}P_\Gamma BTA|_{Z_\Gamma}\hat{A}\| = \|\hat{B}(D_\Gamma - P_\Gamma BTA|_{Z_\Gamma})\hat{A}\| < 5\lambda K\eta < 1$. Hence, the map $Q = \hat{B}P_\Gamma BTA|_{Z_\Gamma}\hat{A}$ is invertible with $\|Q^{-1}\| \le 1/(1 - 5\lambda K\eta)$. In conclusion, if we set $\tilde{B} = Q^{-1}\hat{B}B$, $\tilde{A} = A\hat{A}$ then $\tilde{B}T\tilde{A} = I$ and $\|\tilde{B}\|\|\tilde{A}\| \le \lambda KC/(1 - 5\lambda K\eta)$.

Proof of (b). Note that since A is a bounded operator (see Remark 4·4), it follows that

$$A(z) = \left(\sum_{j=1}^{\infty} a_{\nu(k,j)}x_{(k,j)} : k \in \mathbb{N}\right) = \sum_{n=1}^{\infty} a_n x_n,$$

whenever $z = \sum_{n=1}^{\infty} a_n e_n \in Z$, where the series converges in the product topology $\mathcal{P}$. So, in particular the sum $\sum_{n=1} a_n x_n$ is well defined in Z if the sum $\sum_{n=1} a_n e_n$ is well defined. Let us assume that $z = \sum_{n=1}^{\infty} a_n e_n \in S_Z$ and thus $\|A(z)\| \le \sqrt{C}$.

By (4·5) and the definition of D we obtain

$$(BTA - D)\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{m-1} a_n x_m^*(Tx_n)e_m + \sum_{m=1}^{\infty} x_m^*\big(T\sum_{n=m+1}^{\infty} a_n x_n\big)e_m.$$

The norm of the first sum is dominated by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{m-1} |a_n||x_m^*(Tx_n)| < 2\lambda\sum_{m=1}^{\infty}\eta_m \le 2\lambda\eta. \tag{4·9}$$

To estimate the norm of the second sum, first choose $y_n \in W_{\kappa(n)}^{(n)}$ according to (v) such that $\|x_n - y_n\| < \eta_n$. Let $m \in \mathbb{N}$ be fixed. We claim that $y = \sum_{n=m+1}^{\infty} a_n y_n$ is a well defined element in $\ell^\infty\big(W_k^{(m+1)} : k \in \mathbb{N}\big)$. Indeed, it is well defined since $\sum_{n=m+1}^{\infty} a_n x_n \in Z$ and $\sum_{n=1}^{\infty} \eta_n < \eta < \infty$. Moreover, by the properties of our enumeration (see Convention 3·1) and since $W_k^{(n+1)} \subset W_k^{(n)}$, $k, n \in \mathbb{N}$, we have that

$$y = \sum_{n=m+1}^{\infty} a_n y_n = \sum_{k=1}^{\infty}\sum_{\substack{n>m\\ \kappa(n)=k}} a_n y_n \in \ell^\infty\big(W_k^{(n_k)} : k \in \mathbb{N}\big),$$

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where $n_k = \min\{n > m : \kappa(n) = k\} \geq m+1$; thus we proved $y \in \ell^\infty\big(W_k^{(m+1)} : k\in\mathbb{N}\big)$. By a standard perturbation argument, we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\left|x_m^*\left(T\sum_{n=m+1}^{\infty}a_n x_n\right)\right| &\leq \sum_{m=1}^{\infty}\left|T^*x_m^*\left(\sum_{n=m+1}^{\infty}a_n y_n\right)\right| + \sum_{m=1}^{\infty}\left|T^*x_m^*\left(\sum_{n=m+1}^{\infty}a_n(x_n-y_n)\right)\right|\\ &\leq \sum_{m=1}^{\infty}\eta_m\left\|\sum_{n=m+1}^{\infty}a_n y_n\right\| + \sum_{m=1}^{\infty}\|T^*x_m^*\|\sum_{n=m+1}^{\infty}|a_n|\eta_n \quad \text{(by (vi))}\\ &\leq \sum_{m=1}^{\infty}\eta_m(2\lambda\sqrt{C}+2\lambda\eta) + \sum_{m=1}^{\infty}\|T\|\sqrt{C}2\lambda\sum_{n=m+1}^{\infty}\eta_n\\ &\leq \eta 2\lambda\sqrt{C}(2+\|T\|)\end{aligned}$$

which, together with (4·9), establishes the first part of (b).

For the additional part, assume that $\hat{B}\colon Z\to Z$ and $\hat{A}\colon Z\to Z$ are such that $\|\hat{B}\|\|\hat{A}\| \leq K$ and $I = \hat{B}D\hat{A}$. It follows that $\|I - \hat{B}BTA\hat{A}\| = \|\hat{B}(D-BTA)\hat{A}\| < 2\lambda\sqrt{C}(3+\|T\|)K\eta < 1$. Hence, the map $Q = \hat{B}BTA\hat{A}$ is invertible with $\|Q^{-1}\| \leq 1/(1-2\lambda\sqrt{C}(3+\|T\|)K\eta)$. Setting $\tilde{B} = Q^{-1}\hat{B}B$, $\tilde{A} = A\hat{A}$ then $\tilde{B}T\tilde{A} = I$ and

$$\|\tilde{B}\|\|\tilde{A}\| \leq \lambda KC/(1-2\lambda\sqrt{C}(3+\|T\|)K\eta)$$

concludes the proof.

We finally prepare the work of the next section by isolating the following technical Lemma 4·6.

LEMMA 4·6. *Assume that $\lambda \geq 1$ and $K\colon (0,\infty)\to(0,\infty)$ is continuous, so that for each k the basis constant of $(e_{(k,j)})_j$, is not larger than λ and which has the $K(\delta)$-factorisation property in X_k.*

Then for every bounded diagonal operator $D\colon Z\to Z$ (i.e. $D(e_{k,j})$ is a multiple of $e_{k,j}$, for $k, j\in\mathbb{N}$), for which

$$\delta = \inf_{k,j\in\mathbb{N}}\left|e^*_{(k,j)}(De_{(k,j)})\right| > 0,$$

the identity almost $K(\delta)$-factors through D.

Proof. Let $D\colon Z\to Z$ be a bounded diagonal operator with $\delta = \inf_{k,j\in\mathbb{N}}\left|e^*_{(k,j)}\big(Te_{(k,j)}\big)\right| > 0$. For $k\in\mathbb{N}$ let $D_k = D|_{X_k}$. Then D_k is a diagonal operator from X_k to X_k. Next, let $\eta > 0$. Then there are for each $k\in\mathbb{N}$ bounded operators $A_k\colon X_k\to X_k$ and $B_k\colon X_k\to X_k$, so that $\|A_k\|\cdot\|B_k\| \leq K(\delta)+\eta$ and $I_k = B_k\circ D_k\circ A_k$, where I_k is the identity on X_k. We can assume that $\|A_k\| = 1$ and that $\|B_k\| \leq K(\delta)+\eta$. Putting

$$A\colon Z\longrightarrow Z,\quad (z_k)\longmapsto\big(A_k z_k\big),\qquad B\colon Z\longrightarrow Z,\quad (z_k)\longmapsto\big(B_k z_k\big),$$

it follows that $\|A\| = 1$ and $\|B\| \leq K(\delta)+\eta$, and $I_Z = B\circ D\circ A$.

5. *Proof of Theorem 3·6 and Theorem 3·9*

The proof of both theorems Theorem 3·6 and Theorem 3·9 is organised as depicted in the flowchart Figure 1. According to the flowchart Figure 1, their respective proofs deviate at

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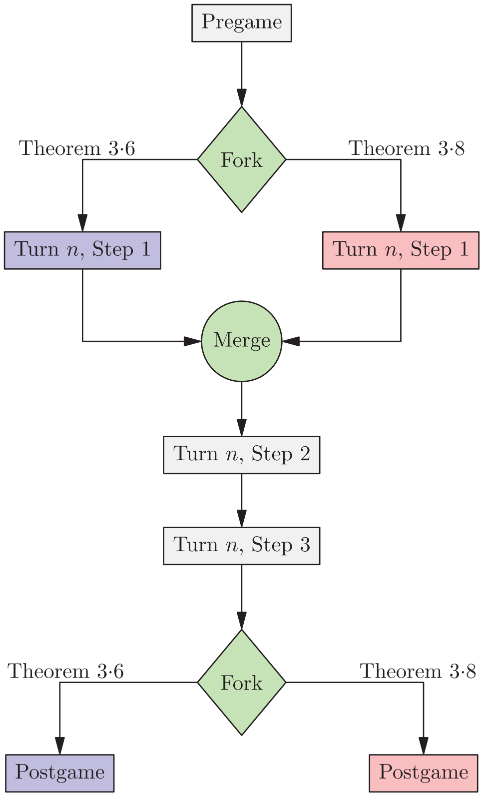

Fig. 1. Flowchart of the proof of Theorem 3·6 and Theorem 3·9.

two distinct points. In the proof below we indicate that by arranging the text side by side in separate columns.

Let $T: Z \to Z$ be a bounded linear operator with $\inf_{k,j\in\mathbb{N}} \left|e^*_{(k,j)}(Te_{(k,j)})\right| =: \delta > 0$. By Remark 3·4, Player II has a winning strategy in the game $\mathrm{Rep}_{(Z,(e_{(k,j)}))}(C, \eta)$. Let us fix $\eta > 0$ to be determined later.

We will now describe a strategy for Player I in a game $\mathrm{Rep}_{(X,(e_n))}(C, \eta)$, and assume Player II answers by following his winning strategy.

Pregame

At the beginning, Player I chooses $N_1 = \{n \in \mathbb{N} : e^*_n(Te_n) \geq \delta\}$ and $N_2 = \{n \in \mathbb{N} : e^*_n(Te_n) \leq -\delta\}$. For $k \in \mathbb{N}$, and $i = 1, 2$ let $N_i^{(k)} = \{\nu(k, j) : j \in \mathbb{N}\} \cap N_i$.

Turn n, Step 1

During the nth turn, Player I proceeds as follows. In the first step of the nth turn he chooses

$$\eta_n < \frac{\eta}{2^{n+2}n(1 + \|T\|)\sqrt{C+\eta}}. \tag{5·1}$$

Let $l_n = \max(\bigcup_{m=1}^{n-1} E_m)$ if $n > 1$ (where the finite sets E_1, E_2,...E_{n-1} were chosen in previous turns) and put $l_1 = 1$. Define

$$A_n = \left\{x_m, Tx_m : m < n\right\} \cup \{e_{(\kappa(n),i)}, Te_{(\kappa(n),i)} : i \leq l_n\}$$

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and put

$$G_n = A_n^{\perp} \cap X^*_{\kappa(n)} \tag{5·2}$$

as subspace of $X^*_{\kappa(n)}$ and Z^*.

Choice of W_n in Theorem 3·6.

Now, let

$$B_n = \{x^*_m, T^*x^*_m : m < n\} \cup \{e^*_{(\kappa(n),i)}, T^*e^*_{(\kappa(n),i)} : i \le l_n\}$$

and define

$$W_n = (B_n)_\perp \cap X_{\kappa(n)} \tag{5·3}$$

as a subspace of $X_{\kappa(n)}$ and Z.

Choice of W_n in Theorem 3·9.

In order to choose the required cofinite-dimensional subspace W_n of $X_{\kappa(n)}$ Player I also chooses at the n-th step a sequence $(V_l^{(n)})_l$, where $V_l^{(n)}$ is a finite codimensional subspace of X_l of the form

$$V_l^{(n)} = [e_{(l,j)} : j \ge m(n,l)], \tag{5·4}$$

with $V_l^{(n)} \subset V_l^{(m)}$ for $l \in \mathbb{N}$, $0 \le m < n$ and we put $V_l^{(0)} = X_l$ for $l \in \mathbb{N}$. By using Lemma 4·2 finitely many times Player I can find for each l a subspace $V_l^{(n)}$ of $V_l^{(n-1)}$ so that for all $j < n$

$$\|T^*x^*_j|_{\ell^\infty(V_l^{(n)}:l\in\mathbb{N})}\| \le \eta_n \tag{5·5}$$

and he chooses

$$W_n = V^{(n)}_{\kappa(n)} = [e_{(\kappa(n),j)} : j \ge m_n], \tag{5·6}$$

where $m_n = m(n, \kappa(n))$.

Turn n, Step 2

Player II, following a winning strategy, chooses $i_n \in \{1,2\}$, picks a finite set $E_n \subset N_{i_n}^{(\kappa(n))}$ and sequences of non-negative scalars $(\lambda_i^{(n)})_{i\in E_n}$, $(\mu_i^{(n)})_{i\in E_n}$ with

$$1-\eta < \sum_{i\in E_n} \lambda_i^{(n)}\mu_i^{(n)} < 1+\eta.$$

Turn n, Step 3

Then Player I picks signs $(\varepsilon_i^{(n)})_{i\in E_n} \in \{-1,+1\}^{E_n}$ so that whenever

$$x_n = \sum_{i\in E_n} \lambda_i^{(n)}\varepsilon_i^{(n)} e_{(\kappa(n),i)} \qquad \text{and} \qquad x^*_n = \sum_{i\in E_n} \mu_i^{(n)}\varepsilon_i^{(n)} e^*_{(\kappa(n),i)},$$

we have

$$|x^*_n(Tx_n)| > (1-\eta)\delta. \tag{5·7}$$

That it is possible to choose such signs $(\varepsilon_i^{(n)})_{i\in E_n} \in \{-1,+1\}^{E_n}$. Indeed, $r = (r_j)_{j\in E_n}$ be a Rademacher sequence, meaning that r_j, $j \in E_n$, are independent random variables on some

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probability space $(\Omega, \Sigma, \mathbb{P})$, with $\mathbb{P}(r_j{=}1) = \mathbb{P}(r_j{=}{-1}) = 1/2$.

$$\mathbb{E}\left(\left(\sum_{i\in E_n} r_i \mu_i^{(n)} e^*_{(\kappa(n),i)}\right)\left(T\Big(\sum_{j\in E_n} r_j \lambda_j^{(n)} e_{(\kappa(n),j)}\Big)\right)\right) = \mathbb{E}\left(\sum_{i,j\in E_n} r_i r_j \mu_i^{(n)} \lambda_j^{(n)} e_j^*(T e_{(\kappa(n),i)})\right)$$
$$= \sum_{i\in E_n} \mu_i^{(n)} \lambda_i^{(n)} e^*_{(\kappa(n),i)}(T e_{(\kappa(n),i)}) \begin{cases} > \delta(1-\eta) & \text{if } i_n = 1, \\ < -\delta(1-\eta) & \text{if } i_n = 2. \end{cases}$$

The latter inequality follows from the large diagonal of the operator.

Postgame

After the game is completed the conditions (i) to (iv) of Remark 3·4 are satisfied.

Now, let $m, n \in \mathbb{N}$, $m < n$. By the winning strategy of Player II (see (iii) of Remark 3·4) and (5·2), we obtain

$$|x_n^*(T x_m)| \le \|T x_m\| \cdot \operatorname{dist}(x_n^*, G_n) < \|T\| \sqrt{C+\eta} \cdot \eta_n. \tag{5·8}$$

We now estimate $|x_n^*(T x_m)|$ if $m > n$.

Postgame for Theorem 3·6.

By the winning strategy of Player II (see Remark 3·4 (iv)) together with (5·3) yields for all $n < m$ that

$$\begin{aligned} |x_n^*(T x_m)| &= |T^* x_n^*(x_m)| \\ &\le \|T^* x_n^*\| \cdot \operatorname{dist}(x_m, W_m) \\ &< \|T^*\| \sqrt{C+\eta} \cdot \eta_m. \end{aligned}$$

This inequality, (5·1) and (5·8) imply that the conditions of Proposition 4·5 are satisfied for $\eta'_m = m\eta_m$, $m \in \mathbb{N}$.

By Proposition 4·5 (a), there is an infinite set $\Gamma \subset \mathbb{N}$ so that the diagonal operator $D_\Gamma : Z_\Gamma \to Z_\Gamma$ given by

$$D_\Gamma e_{(k,i)} = x^*_{(k,i)}(T x_{(k,i)}) e_{(k,i)}$$

is bounded. By Lemma 4·6, the identity on Z_Γ $(K\delta(1-\eta)+\xi)$ factors through D_Γ for any $\xi > 0$. Hence, if η is sufficiently small then by the additional assertion in Proposition 4·5 (a), the identity $\left(\frac{(\lambda K(\delta(1-\eta))+\xi)(C+\eta)}{1-5\lambda(K\delta(1-\eta))+\xi)\eta}\right)$-factors through T.

Postgame for Theorem 3·9.

Note that by (5·8), condition (iii) of Proposition 4·5 is satisfied if we replace η_n by $\eta/2^{n+2}$. Secondly, for $n_0 < n$ if $w_k \in V_k^{(n)} \subset V_k^{(n_0+1)}$, for $k \in \mathbb{N}$ with $\|(w_k)\|_Z \le 1$ and $z = (w_k)$, then by (5·5)

$$\left|\left(T^*_{n_0} x^*_{n_0}\right)(z)\right| \le \eta_{n_0} < \eta/2^{n_0+2}.$$

Thus, condition (v) of Proposition 4·5 is satisfied, as well. By Proposition 4·5 (b), the diagonal operator $D : Z \to Z$ given by

$$D e_{(k,i)} = x^*_{(k,i)}(T x_{(k,i)}) e_{(k,i)}$$

is bounded. By Lemma 4·6, the identity on Z $(K(\delta(1-\eta))+\xi)$-factors through D for any $\xi > 0$. Hence, if η is sufficiently small, then by the additional assertion of Proposition 4·5 (b), the identity $\left(\frac{(\lambda K(\delta(1-\eta))+\xi)(C+\eta)}{1-2\lambda\sqrt{C+\eta}(3+\|T\|)K(\delta(1-\eta))+\xi)\eta}\right)$-factors through T.

Recall that the function $K : (0, \infty) \to \mathbb{R}$ is continuous (see [13, remark 3·11]). As we could have picked η and ξ arbitrarily close to zero we deduce that the identity on Z_Γ (for

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Theorem 3·6), or the identity on Z, (for Theorem 3·9), respectivly, almost $\lambda K(\delta)C$-factors through T.

6. *Heterogenous ℓ^∞-sums of classical Banach spaces*

Given $1 < p_0 < p_1 < \infty$, we define the sets of Banach spaces

$$\mathcal{W} = \{L^p, H^p, \mathrm{VMO}, \mathrm{VMO}(H^r), H^p(H^q), L^r(L^s) : 1 \le p, q < \infty,\ p_0 \le r, s \le p_1\} \qquad (6{\cdot}1)$$

$$\mathcal{X} = \{L^r, H^p, \mathrm{VMO}, \mathrm{VMO}(H^r), H^p(H^q), L^r(L^s) : p_0 \le p, q < \infty,\ p_0 \le r, s \le p_1\}. \qquad (6{\cdot}2)$$

In this section, we use the following notation: Assume that we are given a sequence of Banach spaces (V_k) in either $\mathcal{W}$ or $\mathcal{X}$. Then for each $k, j \in \mathbb{N}$, $e_{k,j}$ denotes the jth Haar function whenever $V_k \in \{L^r, H^p, \mathrm{VMO}\}$, and $e_{k,j}$ denotes the jth biparameter Haar function if $V_k \in \{\mathrm{VMO}(H^r), H^p(H^q), L^r(L^s)\}$. For details on the enumeration of the biparameter Haar system we refer to [**10**]; see also [**13**].

COROLLARY 6·1. *Let $(W_k)_{k=1}^\infty$ denote a sequence of Banach spaces in $\mathcal{W}$ and let $T : \ell^\infty(W_k : k\in\mathbb{N}) \to \ell^\infty(W_k : k\in\mathbb{N})$ be bounded and linear, with*

$$\delta = \inf_{k,j\in\mathbb{N}} \left| e^*_{k,j}(Te_{k,j}) \right| > 0.$$

Then for each sequence of infinite subsets (Ω_l) of $\mathbb{N}$, there is an infinite $\Gamma \subset \mathbb{N}$ so that $\Gamma \cap \Omega_l$ is infinite for all $l \in \mathbb{N}$ and the identity on $\ell^\infty(W_k : k \in \Gamma)$ C/δ-factors through T, where C depends only on p_0, p_1.

Proof. First, we note that by [**13**, proof of theorem 5·1, theorem 5·2, theorem 5·3, theorem 6·1, remark 6·6] the one- or two-parameter Haar system in each of the spaces L^p, H^p, $H^p(H^q)$, $1 \le p, q < \infty$ is C-strategically reproducible for some universal constant, and $L^r(L^s)$, $p_0 \le r, s \le p_1$ is C_{p_0,p_1}-strategically reproducible. By Proposition 2·7, the Haar system is strategically reproducible in VMO, and the biparameter Haar system is C_{p_0,p_1}-strategically reproducible in $\mathrm{VMO}(H^p)$. In each of the spaces in $\mathcal{W}$, the Haar system has the C/δ-diagonal factorisation property (for the L^1 case we refer to [**13**, proposition 6·2], for the other cases, this follows by unconditionality) for some universal constant C. The assertion follows from Theorem 3·6.

Before we can proceed to our next application, we need the following observation.

LEMMA 6·2. *Let $(X_k)_{k=1}^\infty$ denote a sequence of Banach spaces in $\mathcal{X}$. Then the array $(e_{k,j})$ is uniformly asymptotically curved.*

Proof. For each $k \in \mathbb{N}$, let $(f_{(k,j)})_j$ be a normalized block basis of (h_I). Let $k \in \mathbb{N}$ be fixed for now. Since the $(f_{(k,j)})_j$ have disjoint Haar spectra, observe that for each $n \in \mathbb{N}$

$$\left\| \sum_{j=1}^n f_{(k,j)} \right\|_{X_k} \le C \int_0^1 \left\| \sum_{j=1}^n r_j(t) f_{(k,j)} \right\|_{X_k} dt \le C \left(\sum_{j=1}^n \| f_{(k,j)} \|_{X_k}^r \right)^{1/r},$$

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where $r_1, \ldots, r_K$ are independent Rademacher functions and $r = p_0$ if $X_n \in \{L^p, H^p\}$ and $r = 2$ if $X_n = \mathrm{VMO}$ [**18**, proposition 5·1·1, p. 268]. The constant C depends on p_0 and p_1 if $X_n = L^p$, and $C = 1$ if $X_n \in \{H^p, \mathrm{VMO}\}$. In either case, we obtain

$$\sup_k \left\| \sum_{j=1}^{n} f_{(k,j)} \right\|_{X_k} \leq C n^{1/r}.$$

Moving on to the biparameter spaces, let r' be such that $1/r + 1/r' = 1$. By Proposition 2·12, $H^1(H^{r'})$ satisfies a lower r'-estimate with constant 1; hence by Lemma 2·10 and Lemma 2·11, the array formed by the biparameter Haar system in the predual of $H^1(H^{r'})$, which is $\mathrm{VMO}(H^r)$, is uniformly asymptotically curved. For the spaces $H^p(H^q)$ and $L^r(L^s)$ we refer to Proposition 2·12, Remark 2·13 and Lemma 2·11.

Lemma 6·2 and Theorem 3·9 combined yield the following factorisation result.

COROLLARY 6·3. *Let* $(X_k)_{k=1}^\infty$ *denote a sequence of Banach spaces in* $\mathcal{X}$*, and let* $T : \ell^\infty(X_k : k\in\mathbb{N}) \to \ell^\infty(X_k : k\in\mathbb{N})$ *be bounded and linear, with*

$$\delta = \inf_{k,j\in\mathbb{N}} \left| e^*_{k,j}(T e_{k,j}) \right| > 0.$$

Then the identity on $\ell^\infty(X_k : k\in\mathbb{N})$ C/δ*-factors through* T*, where* C *depends only on* p_0, p_1*.*

Proof. Let $(e_{k,j})$ be defined as in Lemma 6·2. First, note that $(e_{k,j})_j$ has unconditional basis constant C_{p_0,p_1} for each $k \in \mathbb{N}$; hence, $(e_{k,j})_j$ has the $C_{p_0,q_1}/\delta$-diagonal factorisation property for all $k \in \mathbb{N}$. Secondly, for the simultaneous strategical reproducibility of the array $(e_{k,j})$, we refer to the argument presented in the proof of Corollary 6·1. Finally, applying Lemma 6·2 and Theorem 3·9 concludes the proof.

7. *Final remarks and open questions*

Our first question asks whether Theorem 3·9 can be true if we drop the assumption that the array $(e_{(k,j)})$ is uniformly asymptotically curved.

Question 1. Assume that

(i) the basis constant of $(e_{(k,j)})_j$, is at most λ in X_k, for each $k \in \mathbb{N}$;
(ii) $(e_{(k,j)})_j$ has the K-diagonal factorisation property in X_k, for each $k \in \mathbb{N}$;
(iii) the array $(e_{(k,j)})_{k,j}$ is simultaneously C-strategically reproducible in Z.

Is it true that the array $(e_{(k,j)})$ has the factorisation property in $Z = \ell^\infty(X_k)$?

The following example exhibits a sequence of spaces the array of which is *not* uniformly asymptotically curved, yet the array has the factorisation property in Z.

Example 7·1. Let (p_k) be a dense sequence in $[1, \infty)$. Then:

(i) the basis constant of $(h_{k,I})_I$ is 1 in L^{p_k};
(ii) $(h_{k,I})_I$ has the $\frac{1}{\delta}$-diagonal factorisation property [**13**, remark 6·6];
(iii) the array $(h_{k,I})$ is simultaneously 1-strategically reproducible [**13**, Remark 6·6];
(iv) the array $(h_{k,I})$ is *not* uniformly asymptotically curved.

In contrast, the array $(h_{k,I})$ has in $Z = \ell^\infty(L^{p_k} : k\in\mathbb{N})$ the factorisation property.

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In order to show this is indeed true, we need the following proposition.

PROPOSITION 7·2. *Let* (p_k) *and* (q_k) *denote two dense sequences in* $[1, \infty)$. *Then* $\ell^\infty(L^{p_k} : k\in\mathbb{N})$ *is isomorphic to* $\ell^\infty(L^{q_k} : k\in\mathbb{N})$.

Proof. In the first step of this proof we fix $p \in [1, \infty)$ and a dense sequence $(q_k)_k$ in $[1, \infty)$ and show that L^p is isomorphic to a 2-complemented subspace of $\ell^\infty(L^{q_k} : k\in\mathbb{N})$. By the density of $(q_k)_k$ we may pick a subsequence $(q_{k_n})_n$ so that for each $n \in \mathbb{N}$ the identity map $I_n : L_n^{q_{k_n}} \to L_n^p$ is a 2-isomorphism. We then define $T : L^p \to \ell^\infty(L^{q_{k_n}} : n\in\mathbb{N})$ given as follows: if $f \in L^p$ can be written as $f = \sum_j \sum_{|I|=2^{-j}} a_I h_I$ then $Tf = (\sum_{j\le n} \sum_{|I|=2^{-j}} a_I h_I)_n$. Then clearly, T is a 2-isomorphic embedding of the image. We will now show that the image of T is 4-complemented in $\ell^\infty(L^{q_{k_n}} : n\in\mathbb{N})$. To see this, we define $Q\colon \ell^\infty(L^{q_{k_n}} : n \in \mathbb{N}) \to L^p$ by

$$Q\left(\sum_{j=0}^{\infty} \sum_{|I|=2^{-j}} a_{n,I} h_I\right)_n = \sum_{j=0}^{\infty} \sum_{|I|=2^{-j}} (\lim_{n\in\mathcal{U}} a_{n,I}) h_I,$$

where $\mathcal{U}$ is some fixed non-principal ultrafilter on $\mathbb{N}$. For each $m \in \mathbb{N}$, define $R_m\colon L^p \to L^p$ by

$$R_m\left(\sum_{j=0}^{\infty} \sum_{|I|=2^{-j}} a_I h_I\right) = \sum_{j=0}^{m} \sum_{|I|=2^{-j}} a_I h_I.$$

We now verify that Q is indeed well defined. Observe that for $(f_n)_n \in \ell^\infty(L^{q_{k_n}} : n\in\mathbb{N})$, where $f_n = \sum_{j=0}^{\infty} \sum_{|I|=2^{-j}} a_{n,I} h_I \in L^{q_{k_n}}$, $n \in \mathbb{N}$, we have

$$\begin{aligned}\|R_m Q((f_n)_n)\|_{L^p} &= \left\|\sum_{j=0}^{m} \sum_{|I|=2^{-j}} (\lim_{n\in\mathcal{U}} a_{n,I}) h_I\right\|_{L^p} = \lim_{n\in\mathcal{U}} \left\|\sum_{j=0}^{m} \sum_{|I|=2^{-j}} a_{n,I} h_I\right\|_{L^p}\\ &= \lim_{n\in\mathcal{U}} \|R_m f_n\|_{L^p} \le \lim_{n\in\mathcal{U}} \|R_n f_n\|_{L^p}\\ &\le 2 \sup_n \|R_n f_n\|_{L^{q_{k_n}}} \le 2\|(f_n)\|_{\ell^\infty(L^{q_{k_n}} : n\in\mathbb{N})}.\end{aligned}$$

Evidently, $QTf = f$ for all $f \in L^p$, hence, the image of T is 4-complemented. Moreover, T can be extended to a 2-isomorphism $\tilde{T} : L^p \to \ell^\infty(L^{q_k} : k\in\mathbb{N})$ with 4-complemented image. This type of argument goes back to Johnson [8].

In the second step we show that if $(p_k)_k$, $(q_k)_k$ are as in the assumption then $\ell^\infty(L^{p_k} : k\in\mathbb{N})$ is 2-isomorphic to a 4-complemented subspace of $\ell^\infty(L^{q_k} : k\in\mathbb{N})$. Indeed, we may decompose $\mathbb{N}$ into infinite disjoint sets $(M_n)_n$ so that for all $n \in \mathbb{N}$ the sequence $(p_k)_{k\in M_n}$ is dense in $[1, \infty)$. By the first step, for each $n \in \mathbb{N}$ we can find a 2-embedding of L^{p_n} into $\ell^\infty(L^{q_k} : k\in M_n)$ with 4-complemented image. The second step then easily follows.

For the final step we fix a dense sequence $(q_k)_k$ in $[1, \infty)$ with the property that each term q_k is repeated infinitely many times. This implies that the space $X = \ell^\infty(L^{q_k} : k\in\mathbb{N})$ is isometrically isomorphic to $\ell^\infty(X)$, *i.e.*, it satisfies the *accordion property* (see [27, II·B·24]). To conclude, we show that for an arbitrary dense sequence $(q_k)_k$ the space $V = \ell^\infty(L^{p_k} : k\in\mathbb{N})$ is isomorphic to X. Indeed, by the second step we have that X is complemented in V and V is complemented in X. By the accordion property of X we deduce that X is isomorphic to V.

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Verification of Example 7·1. Given an operator $T: Z \to Z$ with large diagonal, choose infinite sets $\Omega_k \subset \mathbb{N}$ so that $p_k = \lim_{j\in\Omega_k} p_j$, $k \in \mathbb{N}$. By Theorem 3·6 there exists an infinite set $\Gamma \subset \mathbb{N}$ for which $\Gamma \cap \Omega_k$ is infinite for each $k \in \mathbb{N}$ so that I_{Z_Γ} factors through T. Since $\{p_\gamma : \gamma \in \Gamma\}$ is again dense in $[1, \infty)$, we obtain by Proposition 7·2 that Z_Γ is isomorphic to Z. Hence, I_Z factors through T.

An interesting special case of Question 1 is the following

Question 2. Assume that (p_k), $1 \le p_k < \infty$ either converges to 1 or diverges to ∞. Does the array $(h_{k,I})$ have the factorisation property in $Z = \ell^\infty(L^{p_k} : k\in\mathbb{N})$?

Acknowledgements. We would like to thank the anonymous referee for their diligence and helpful suggestions.

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