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OPTIMAL CHOICE OF THE BEST AVAILABLE APPLICANT IN FULL-INFORMATION MODELS

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Abstract

The problem we consider here is a full-information best-choice problem in which n applicants appear sequentially, but each applicant refuses an offer independently of other applicants with known fixed probability $0 \leq q < 1$. The objective is to maximize the probability of choosing the best available applicant. Two models are distinguished according to when the availability can be ascertained; the availability is ascertained just after the arrival of the applicant (Model 1), whereas the availability can be ascertained only when an offer is made (Model 2). For Model 1, we can obtain the explicit expressions for the optimal stopping rule and the optimal probability for a given n. A remarkable feature of this model is that, asymptotically (i.e. $n \to \infty$), the optimal probability becomes insensitive to q and approaches 0.580 164. The planar Poisson process (PPP) model provides more insight into this phenomenon. For Model 2, the optimal stopping rule depends on the past history in a complicated way and seems to be intractable. We have not solved this model for a finite n but derive, via the PPP approach, a lower bound on the asymptotically optimal probability.

Keywords: Best-choice problem; secretary problem; optimal stopping; planar Poisson process; insensitivity; full-history dependence; Robbins' problem

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1. Introduction

A known number, n, of applicants appear sequentially in a random order, and one of them must be chosen. An applicant that has been rejected cannot be recalled later. In the classical best-choice problem, often referred to as the secretary problem, the objective is to maximize the probability of choosing the best applicant. At each time, we observe only the relative rank of the current applicant with respect to his/her predecessors. It is well known that the optimal rule lets approximately $e^{-1}n$ applicants go by and then selects the first relatively best applicant, if any. The optimal probability of choosing the overall best tends to $e^{-1} \approx 0.368$ as $n \to \infty$.

In contrast to the above *no-information* version of the problem, the *full-information* analogue is the problem in which the observations are the true values of n applicants $X_1, X_2, \ldots, X_n$, assumed to be independent and identically distributed random variables from a known continuous distribution, taken without loss of generality to be the uniform distribution on the interval [0, 1]. There exists a single increasing sequence of thresholds $\{b_k,\ 1 \leq k\}$, where b_k

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is defined as a solution $x \in (0, 1)$ to the equation

$$\sum_{i=1}^{k} \frac{1}{i}(x^{-i} - 1) = 1, \tag{1}$$

such that the optimal stopping rule is to select the jth applicant if this applicant is a relative maximum, i.e. $X_j = \max(X_1, \dots, X_j)$ and $X_j \geq b_{n-j}$. For example, $b_1 = \frac{1}{2}$ and $b_2 = (1 + \sqrt{6})/5$. Let $c \approx 0.804\,35$ be a root of the equation

$$\sum_{j=1}^{\infty} \frac{c^j}{j!\, j} = 1. \tag{2}$$

Then, as $n \to \infty$, $n(1 - b_n) \to c$, and the optimal probability of choosing the largest of $X_1, X_2, \dots, X_n$ tends to

$$\mathrm{e}^{-c} + (\mathrm{e}^c - c - 1) \int_1^{\infty} \frac{\mathrm{e}^{-cx}}{x} \, \mathrm{d}x, \tag{3}$$

which is numerically evaluated as 0.580 164. These best-choice problems were studied in Gilbert and Mosteller (1966) together with many other best-choice problems. See Samuels (1982), (1991), (2004) and Berezovsky and Gnedin (1984) for the limiting form (3). We refer to the above full-information problem as the GM problem.

Though we have so far implicitly assumed that the applicant is always *available*, that is, he/she accepts an offer of selection (employment) with certainty, Smith (1975) introduced the possibility of the applicant refusing an offer. He considered a best-choice problem in the no-information setting and Petruccelli (1982) later considered the corresponding full-information problems in a greater generality. However, the common objective of their problems is to choose the best overall, implying that the trial is automatically unsuccessful if the overall best is unavailable.

Another more appropriate objective may be to choose the best available applicant, i.e. the best among all the available applicants. Tamaki (1991) considered this problem in the no-information setting. The problem we consider in this paper is the full-information analogue of Tamaki (1991). We simply assume here that each applicant is available with a known fixed probability p $(0 < p \leq 1)$ and unavailable with the remaining probability $q = 1 - p$, independent of the value and also of the other applicants. The decision of when to make an offer is based on both the values of the applicants and their availabilities observed so far, and the objective is to maximize the probability of choosing the best available applicant. If all are unavailable, we lose. Two models are distinguished according to when the availability can be ascertained.

Model 1. The availability of the applicant can be ascertained just after his/her arrival. Thus, we make an offer only to an available applicant because nothing is lost in doing so.

Model 2. The availability of the applicant can be ascertained only when an offer is given. If the offer is accepted, the applicant proves to be available and the selection process terminates, but if rejected, the applicant proves to be unavailable and the next one must be observed.

In both models, we give an offer to the nth applicant if the final stage is reached. When $q = 0$, these two models are equivalent and reduced to the GM problem.

In Section 2 we consider Model 1. A remarkable feature of this model is that, asymptotically, the optimal success probability becomes insensitive to q and approaches 0.580 164 given by (3), though it in effect depends on q for a finite n. This feature can be directly derived via the limiting argument of the finite problem. However, as is shown in Section 3, the planar Poisson process model provides more insight into this phenomenon. This model is known to facilitate the derivation of the asymptotic values for some best-choice problems. Model 2 will be discussed in Section 4. When trying, by means of backward induction, to find the optimal rule for a given n, we immediately find it to be a formidable task. Our attempt to find the exact optimal probability has been unsuccessful. However, we can derive a lower bound on the asymptotically optimal probability by exploiting a reasonable rule suggested from Model 1.

2. Model 1

Let X_j and I_j respectively denote the value and the availability indicator of the jth applicant, $1 \le j \le n$. It is assumed that $(X_1, X_2, \dots, X_n)$ is a sequence of n independent random variables each uniformly distributed on [0, 1], and that $(I_1, I_2, \dots, I_n)$ is also a sequence of n independent random variables each taking a value of 1 or 0 with probability p and $q = 1 - p$ respectively according to whether the applicant is available or not, that is,

$$\mathrm{P}\{I_j = 1\} = p = 1 - \mathrm{P}\{I_j = 0\}.$$

The two sequences $(X_1, X_2, \dots, X_n)$ and $(I_1, I_2, \dots, I_n)$ are also assumed to be independent of each other.

Let J_i be the index set of the available applicants observed up to time i, i.e. $J_i = \{1 \le j \le i : I_j = 1\}$, $1 \le i \le n$, and define the largest value of the first i observations as $L_i = \max_{j \in J_i} X_j$ for $J_i \ne \varnothing$. (For $J_i = \varnothing$, define $L_i = 0$ for convenience.) Call X_i (or the ith applicant) a *candidate* if $I_i = 1$ and $X_i = L_i$. The objective is to maximize the probability of *success*, i.e. choosing the applicant having the largest value L_n for $J_n \ne \varnothing$. Obviously, an optimal rule at any decision point only chooses a candidate, depending on the past only through the largest value observed so far and on the future through the remaining number of observations. Thus, if we denote by $v_k(x)$ the maximal probability of success, provided that there are k applicants yet to observe and the largest value observed so far is x, the principle of optimality gives, for $1 \le k \le n - 1$,

$$v_k(x) = (q + px)v_{k-1}(x) + p \int_x^1 \max\{(q + pt)^{k-1}, v_{k-1}(t)\}\, \mathrm{d}t, \tag{4}$$

where $(q + pt)^{k-1}$ is the probability that the $(n - k + 1)$th applicant is the best available, that is, $X_{n-k+1} = L_n$, given that the applicant is a new candidate having value t (this is just the probability that all the subsequent X_js are smaller than t or unavailable). The boundary condition is $v_0(x) \equiv 0$.

The main results of Model 1 can be summarized as follows.

Theorem 1. (a) *Optimal stopping rule: let*

$$m = m(q) = \min\{k \ge 1 : b_k \ge q\},$$

where b_k is defined by (1); then, for a given q, there exists a nondecreasing sequence of the

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thresholds $\{b_k(q),\ 1 \le k\}$ *defined as* $b_k(q) = \max\{0, (b_k - q)/(1-q)\}$ *or*

$$b_k(q) = \begin{cases} \dfrac{b_k - q}{1-q} & \text{if } k \ge m, \\ 0 & \text{if } 1 \le k < m, \end{cases} \tag{5}$$

as a function of k*, such that the optimal rule is to choose the first candidate* X_j *that exceeds the threshold* $b_{n-j}(q)$.

(b) *Optimal probability: let* $\mathrm{P}_n^*(q)$ *denote the optimal success probability as a function of* n *and* q*; then*

$$\mathrm{P}_n^*(q) = \frac{1}{n}\Bigg[1 + \sum_{i=m}^{n-1}\sum_{r=1}^{i}\frac{1}{n-r}b_i^{n-r}\Bigg] + q^n \sum_{r=1}^{m}\frac{1}{r}\Bigg[\binom{m-r}{n-r}q^{-r} - \binom{m}{n}\Bigg] \tag{6}$$

for $n > m$ *and* $\mathrm{P}_n^*(q) = q^n \sum_{r=1}^{n}(q^{-r} - 1)/r$ *for* $n \le m$.

(c) *Asymptotics: let* $c \approx 0.804\,35$ *be a root of (2); then, as* $n \to \infty$,

$$\mathrm{P}_n^*(q) \to \mathrm{P}^*(q) = \mathrm{e}^{-c} + (\mathrm{e}^{c} - c - 1)\int_1^\infty \frac{\mathrm{e}^{-cx}}{x}\,\mathrm{d}x \approx 0.580\,164, \tag{7}$$

showing the insensitivity to $0 \le q < 1$ *of the asymptotic success probability.*

Proof. See Appendix A.

Remark 1. Besides the expression $\mathrm{P}_n^*(q) = q^n \sum_{r=1}^{n}(q^{-r} - 1)/r$, as given in Theorem 1(b) for $n \le m$, we have another expression:

$$\mathrm{P}_n^*(q) = \sum_{r=1}^{n}\frac{1}{r}\binom{n}{r}(1-q)^r q^{n-r}.$$

This follows because the total number, M, of available applicants is a binomial random variable with parameters (n, p), i.e. $\mathrm{P}\{M = r\} = \binom{n}{r}p^r(1-p)^{n-r},\ 0 \le r \le n$, and because the optimal rule chooses, due to $b_j(q) = 0$, the first available applicant, and if there are r available applicants, the first one has probability $1/r$ of being the best by exchangeability.

For $n = 2$ and 3, the optimal success probabilities are given as follows:

$$\mathrm{P}_2^*(q) = \begin{cases} \frac{3}{4} - \frac{1}{2}q^2 & \text{if } 0 \le q < b_1, \\ \frac{1}{2} + q - \frac{3}{2}q^2 & \text{if } b_1 \le q < 1, \end{cases}$$

$$\mathrm{P}_3^*(q) = \begin{cases} \dfrac{293 + 48\sqrt{6}}{600} - \dfrac{1}{3}q^3 & \text{if } 0 \le q < b_1, \\ \dfrac{67 + 12\sqrt{6}}{150} + \dfrac{1}{2}q^2 - q^3 & \text{if } b_1 \le q < b_2, \\ \dfrac{1}{3} + \dfrac{1}{2}q + q^2 - \dfrac{11}{6}q^3 & \text{if } b_2 \le q < 1, \end{cases}$$

where $b_1 = \frac{1}{2}$ and $b_2 = (1 + \sqrt{6})/5$, as mentioned before.

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The neat expression (6), inspired by Sakaguchi (1973), is crucial to derive (7) as the limit of the finite problems (see Appendix A). An interesting phenomenon is that no matter what the value of q, we have asymptotically the same success probability, 0.580 164, by allowing the stopping rule to depend on q. To give more perspective to this phenomenon, which prevails in greater generality, we reconsider this problem as a planar Poisson process model in the next section.

Remark 2. It is not known whether the insensitivity property also holds in the corresponding noninformative model. Though Tamaki (1991) considered this model, the paper is unfinished in the sense that it solves only the finite problems, so the asymptotic results remain unsolved. However, judging from the numerical results of his Table VI, which gives the optimal success probability for some selected values of $n(\leq 1000)$ and $p(= 1-q)$, we may conclude that such a property does not hold when the availability can be ascertained in advance (see Model 2 in Section 2 of Tamaki (1991), which corresponds to our Model 1.)

3. The planar Poisson process

The planar Poisson process (PPP) model is widely known to be an appropriate setting in which we can define the infinite version of the corresponding finite problems. Bruss and Rogers (1991) used a PPP to study no-information problems with a random or infinite number of applicants and then Gnedin (1996) showed that a PPP model with rate 1 on the semi-infinite strip $[0, 1]\times(-\infty, 0]$ serves as the desired setting for the GM problem. For further applications to the full-information problems related to the GM problem, see Samuels (2004), who preferred to use an *inverted* PPP, i.e. a PPP on the semi-infinite strip $[0, 1] \times [0, \infty)$, which is equivalent to Gnedin's PPP if one turns the problem upside down, making the 'best' become the 'smallest'. The process is scanned from left to right by shifting a vertical detector and the scanning can be stopped each time a point in the PPP, referred to as an *atom* henceforth, is detected.

It is obvious that, when we use an inverted PPP (as we do so throughout this paper), the appropriate setting for Model 1 is a PPP with rate p on the semi-infinite strip $[0, 1] \times [0, \infty)$ because only available applicants are under consideration for choice (i.e. unavailable applicants are neglected). A link to the finite problems can be established by embedding suitably the finite independent and identically distributed sequences in the PPP in a similar manner as given to the GM problem in Gnedin (1996, Section 3). However, for more generality, we consider here the PPP best-choice problem with parameters (a, λ) defined as the problem of maximizing the probability of stopping on the lowest atom in the PPP with rate λ on the semi-infinite strip $[0, a] \times [0, \infty)$, where $a, \lambda > 0$. This problem corresponds to the GM problem when $(a, \lambda) = (1, 1)$ and to Model 1 when $(a, \lambda) = (1, p)$. The typical properties of the PPP are

(a) the number of atoms in each bounded domain of the PPP with rate λ has a Poisson distribution with mean equal to $\lambda \times$ (the area of the domain),

(b) the numbers of atoms in disjoint domains are independent.

See Gnedin (2004, Section 2.1) for further properties of the PPP.

The goal in this section is to show that, though the optimal rule of the PPP best-choice problem with parameters (a, λ) in fact depends on the values of the parameters, its optimal success probability is insensitive to these values (Theorem 2, below). Let $u(t, y)$ denote the probability of success if we choose the point (t, y) in the PPP, i.e. we stop at time t with a relatively best atom having value y. Then, if we denote by $\mathrm{P}_{\mathrm{Poisson}}(k, \mu)$ the Poisson probability of k events, for a given mean μ, we have

$$u(t, y) = \mathrm{P}_{\mathrm{Poisson}}(0, \lambda y(a-t)), \tag{8}$$

because $u(t, y)$ is just the probability that there is no atom in the box domain $[t, a] \times [0, y]$ whose area is $y(a-t)$. On the other hand, let $v(t, y)$ denote the probability of success if we do not choose the point (t, y), but instead choose the point related to the next relatively best atom, if any; then

$$v(t, y) = \sum_{j=1}^{\infty} \frac{1}{j} \mathrm{P}_{\mathrm{Poisson}}(j, \lambda y(a-t)),$$

because, if there are j atoms in the box $[t, a] \times [0, y]$, the leftmost atom has probability $1/j$ of being best (lowest). Solving for the locus of point (t, y) at which $u(t, y) = v(t, y)$ yields $\lambda y(a-t) = c$ for $c \approx 0.804\,35$ given as a root of (2). Moreover, since $u(t, y) \geq v(t, y)$ implies that $u(t', y') \geq v(t', y')$ for $t' > t$ and $y' < y$, we are in the *monotone* case of optimal stopping and can conclude that the optimal rule stops with the first relatively best atom, if any, that lies below the threshold curve

$$y = \frac{c}{\lambda(a-t)}.$$

We have the following result.

Theorem 2. *Let* $\mathrm{P}_{a,\lambda}$ *denote the optimal success probability of the PPP best-choice problem with parameters* (a, λ)*. Then*

$$\mathrm{P}_{a,\lambda} = \mathrm{e}^{-c} + (\mathrm{e}^{c} - c - 1) \int_1^{\infty} \frac{\mathrm{e}^{-cx}}{x} \, \mathrm{d}x \approx 0.580\,164,$$

which is insensitive to (a, λ)*.*

Proof. Our proof is essentially the same as Samuels (2004, Section 10.2) established for the GM problem. Let T be the arrival time of the first (leftmost) atom that lies below the threshold curve $y = c/\lambda(a-t)$, and let S be the time when the value of the best (lowest) atom above the threshold is equal to the threshold. Then, from properties (a) and (b) of the PPP, T and S are independent and their distributions are given by

$$\mathrm{P}\{T > t\} = \mathrm{P}_{\mathrm{Poisson}}\bigg(0, \lambda \int_0^t g_{a,\lambda}(r) \, \mathrm{d}r\bigg), \tag{9}$$

$$\mathrm{P}\{S > s\} = \mathrm{P}_{\mathrm{Poisson}}\bigg(0, \lambda \int_0^s (g_{a,\lambda}(s) - g_{a,\lambda}(r)) \, \mathrm{d}r\bigg), \tag{10}$$

where $g_{a,\lambda}(r) = c/\lambda(a-r)$, $0 < r < a$. Exploiting the virtual stopping time, $\min(S, T)$, which makes the calculations simpler, we have

$$\begin{aligned}\mathrm{P}_{a,\lambda} = {} & \int_0^a \int_0^t u\bigg(s, \frac{c}{\lambda(a-s)}\bigg) f_S(s) f_T(t) \, \mathrm{d}s \, \mathrm{d}t \\ & + \int_0^a \int_0^s \bigg[\frac{\lambda(a-t)}{c} \int_0^{c/\lambda(a-t)} u(t, y) \, \mathrm{d}y\bigg] f_T(t) f_S(s) \, \mathrm{d}t \, \mathrm{d}s,\end{aligned} \tag{11}$$

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where $f_T(t)$ and $f_S(s)$ are the densities of T and S, respectively. Substituting $u(t, y) = \mathrm{e}^{-\lambda y(a-t)}$ from (8) into (11) yields

$$\begin{aligned} \mathrm{P}_{a,\lambda} &= \mathrm{e}^{-c}\,\mathrm{P}\{S \le T\} + \frac{1}{c}(1 - \mathrm{e}^{-c})\,\mathrm{P}\{S > T\} \\ &= \mathrm{e}^{-c} + \left(\frac{1}{c} - \frac{1}{c}\mathrm{e}^{-c} - \mathrm{e}^{-c}\right)\mathrm{P}\{S > T\}. \end{aligned} \tag{12}$$

By the way, we have, from (9) and (10),

$$\begin{aligned} \mathrm{P}\{S > T\} &= \int_0^a \mathrm{P}\{S > t\} f_T(t)\,\mathrm{d}t \\ &= \int_0^a \left(\frac{a}{a-t}\right)^c \mathrm{e}^{-ct/(a-t)} \frac{c(a-t)^{c-1}}{a^c}\,\mathrm{d}t \\ &= c\mathrm{e}^c \int_1^\infty \frac{\mathrm{e}^{-cx}}{x}\,\mathrm{d}x, \end{aligned}$$

which, combined with (12), gives the desired result.

Remark 3. The PPP best-choice problem with parameters $(p, 1)$ deserves special attention. Many authors have introduced various forms of uncertainty about the number, N, of applicants. Among those, Porosinski (1987) studied the full-information best-choice problem with such prior distributions on N as uniform, Poisson, and geometric. The GM problem corresponds to the case where $\mathrm{P}\{N = n\} = 1$. It is known that the asymptotic success probability heavily depends on the distribution of N, e.g. for the case where N is uniform on $\{1, 2, \dots, n\}$, the asymptotic success probability, as $n \to \infty$, tends to 0.435 17 (see Porosinski (1987) and Samuels (2004)). When N is a binomial random variable with parameters (n, p), not treated in detail by Porosinski, it is conjectured that the infinite version of this problem as $n \to \infty$ is the PPP best-choice problem with parameters $(p, 1)$, so has the same asymptotic success probability, 0.580 164, as Model 1. An intuitive reasoning is as follows (this statement is due to Gnedin (1996)). Consider an infinite sequence of independent random variables $X_1, X_2, \dots,$ each uniformly distributed on $[0, 1]$. For a fixed n, think of the two-dimensional random point set $\chi_n = \{(1/n, nX_1), (2/n, nX_2), \dots, (N/n, nX_N)\}$ as a sequence of N applicants arriving at fractional times $1/n, 2/n, \dots, N/n$ and consider the problem of stopping with the applicant having the smallest value of $(nX_1, nX_2, \dots, nX_N)$. The set χ_n contains the same order structure as $X_1, X_2, \dots, X_N$, implying that the stopping problem is equivalent to the original discrete-time problem. Since, as $n \to \infty$, $N/n \to p$ with probability 1 by the strong law of large numbers, χ_n has an asymptotic pattern, i.e. PPP with rate 1 on the semi-infinite strip $[0, p] \times [0, \infty)$. It is noted that, for a finite n, Model 1 has an advantage over the above binomial model in the sense that the optimal success probability of the former is at least as large as that of the latter. The reason is that Model 1 has more precise information than the binomial model at each decision epoch concerning the number of possible applicants that may yet appear; more specifically, in Model 1 we know not only the number of available applicants seen so far but also their arrival times, whereas in the binomial model we only know the number of applicants that have appeared. We have found that the advantage of Model 1 disappears asymptotically as $n \to \infty$ and that the binomial model is also insensitive to p. As a related work, see Das and Tsitsiklis (2008), whose main concern was, in our terminology, to make comparisons between Model 1 and the binomial model in several ways when the objective is to maximize the expected (true) value of the applicant chosen.

4. Model 2

In Model 2 we must give an offer to the applicant without knowing his/her availability. Let the two independent sequences $(X_1, X_2, \dots, X_n)$ and $(I_1, I_2, \dots, I_n)$ be defined as in Model 1. Imagine a situation where we have observed the first $(n-k)$ applicants with no one chosen previously. Let J_{n-k} be the index set of the applicants passed up previously (this implies that we have ascertained $I_j = 0$ for $j \in \{1, 2, \dots, n-k\} - J_{n-k}$ by giving an offer to be rejected). Obviously, the information necessary for the future decision is the observed values indexed by J_{n-k}, i.e. $\{X_j = x_j : j \in J_{n-k}\}$. However, since the arrival times of these values do not matter, they can be arranged, for ease of description, in an ascending order as a vector $\boldsymbol{y} = (y_1, y_2, \dots, y_m)$, referred to as the *history*, where y_t, $1 \le t \le m$, is the tth smallest x_j and m denotes the size of the set J_{n-k}. Denote by $v_k(\boldsymbol{y})$ the maximal probability of success starting from this situation, indicating that this quantity depends on the past through the history and on the future through the remaining number of observations. To derive the optimality equation, denote by $(k-1; t, \boldsymbol{y})$ a *state* where we have just observed the value of the next applicant to be t, i.e. $X_{n-(k-1)} = t$. Let $s_{k-1}(t, \boldsymbol{y})$ and $c_{k-1}(t, \boldsymbol{y})$ be the probabilities of success when we make an offer and when we make no offer, respectively, to the current applicant in state $(k-1; t, \boldsymbol{y})$ and proceed optimally thereafter. Define $a(t, \boldsymbol{y})$ as a history revised by adding a new value t $(0 < t \le 1)$ to the previous history $\boldsymbol{y}$, that is,

$$a(t, \boldsymbol{y}) = (y_1, \dots, y_i, t, y_{i+1}, \dots, y_m) \quad \text{if } y_i < t \le y_{i+1},$$

where $y_0 = 0$ and $y_{m+1} = 1$. Also, define $K(t, \boldsymbol{y}) = \#\{i \in J_{n-k} : y_i > t\}$ as the number of applicants indexed by J_{n-k} whose values are greater than t. Then we have the following optimality equation:

$$v_k(\boldsymbol{y}) = \int_0^1 \max\{s_{k-1}(t, \boldsymbol{y}), c_{k-1}(t, \boldsymbol{y})\} \, \mathrm{d}t, \tag{13}$$

where

$$s_{k-1}(t, \boldsymbol{y}) = pq^{K(t,\boldsymbol{y})}(q + pt)^{k-1} + q v_{k-1}(\boldsymbol{y}), \tag{14}$$

$$c_{k-1}(t, \boldsymbol{y}) = v_{k-1}(a(t, \boldsymbol{y})). \tag{15}$$

Equations (13)–(15), which are also valid for $J_{n-k} = \varnothing$ if $\boldsymbol{y}$ is interpreted as 0, can be solved recursively (in principle) to yield the optimal rule and the success probability $v_n(0)$ for a finite n starting with $v_0(\boldsymbol{y}) \equiv 0$.

From its definition, $v_{k-1}(\boldsymbol{y})$ is nonincreasing in each component of $\boldsymbol{y}$. Hence, from (14) and (15), for given q and $\boldsymbol{y}$, $s_{k-1}(t, \boldsymbol{y})$ is increasing in t whereas $c_{k-1}(t, \boldsymbol{y})$ is nonincreasing in t. This implies that there exists a threshold defined as

$$b_{k-1}(q, \boldsymbol{y}) = \max\{t : s_{k-1}(t, \boldsymbol{y}) \ge c_{k-1}(t, \boldsymbol{y})\}$$

such that, in state $(k-1; t, \boldsymbol{y})$, the optimal rule makes an offer to the $(n-k+1)$th applicant if $X_{n-k+1} = t \ge b_{k-1}(q, \boldsymbol{y})$. However, since the optimal rule is history dependent, it is a formidable task to derive the explicit expression for the threshold, even for the simplest threshold $b_1(q, \boldsymbol{y})$ for which the vector $\boldsymbol{y}$ has a single component y. After a considerable amount of arithmetic, we obtain, for $0 < q < 1$,

$$b_1(q, y) = \min\left(\max\left\{1 - \frac{1}{2p}, y\right\}, \max\left\{\frac{(1 - q - q^2) - p^2 y}{2pq}, 0\right\}\right),$$

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or, more specifically,

(i) for $0 < q < \frac{1}{2}$,

$$b_1(q, y) = \begin{cases} 1 - \dfrac{1}{2p} & \text{if } 0 \le y < 1 - 1/2p, \\ y & \text{if } 1 - 1/2p \le y < (1 - q - q^2)/p(1+q), \\ \dfrac{(1 - q - q^2) - p^2 y}{2pq} & \text{if } (1 - q - q^2)/p(1+q) \le y \le 1, \end{cases}$$

(ii) for $\frac{1}{2} \le q < (\sqrt{5} - 1)/2$,

$$b_1(q, y) = \begin{cases} y & \text{if } 0 \le y < (1 - q - q^2)/p(1+q), \\ \dfrac{(1 - q - q^2) - p^2 y}{2pq} & \text{if } (1 - q - q^2)/p(1+q) \le y < (1 - q - q^2)/p^2, \\ 0 & \text{if } (1 - q - q^2)/p^2 \le y \le 1, \end{cases}$$

(iii) for $(\sqrt{5} - 1)/2 \le q < 1$,

$$b_1(q, y) \equiv 0, \qquad 0 \le y \le 1.$$

It is observed that $b_1(q, y)$ is nonincreasing in q, but is not necessarily a monotone function of y.

We give up finding the optimal rule or the success probability for a finite n, but instead examine the asymptotic value achieved by a simple stopping rule. As a promising rule suggested from Model 1, we here consider and evaluate the rule that makes an offer to the applicant, say the kth applicant, as long as the applicant satisfies $X_k \ge \max\{b_{n-k}(q), y^*\}$, where y^* denotes the largest value passed up previously. Let $\tilde{\mathrm{P}}(q)$ represent the asymptotic success probability achieved by this tractable rule for a given q. Then, since this rule depends on the history only through the largest component, if any, $\tilde{\mathrm{P}}(q)$ gives a lower bound on the asymptotically optimal success probability for Model 2. This value is given as follows through the PPP argument.

Theorem 3. *We have*

$$\tilde{\mathrm{P}}(q) = \mathrm{e}^{-c} + (\mathrm{e}^{c} - c - 1)\left(\frac{p\mathrm{e}}{c}\right)^{qc/p} L\left(\frac{qc}{p}, \frac{c}{p}\right), \qquad 0 \le q < 1,$$

where $L(a, b) = \int_b^\infty x^{a-1}\mathrm{e}^{-x}\,\mathrm{d}x$ for $a, b > 0$ is an incomplete gamma function.

Proof. The proof is similar to that of Theorem 2. Remember that, for Model 1, the optimal rule is described as the threshold curve $y = c/p(1-t)$ defined on the semi-infinite strip $[0, 1] \times [0, \infty)$ with rate p. We use rate p because, as mentioned before, the availability of the atom is immediately ascertained upon its arrival, so only available atoms can be taken into consideration. We apply the same threshold curve to our problem. What differs from Model 1 is the rate of the PPP; the appropriate PPP for our problem must have rate 1 above the threshold and rate p below the threshold. This follows because the availability of the atom above the threshold is not ascertained, whereas the availability of the atom below the threshold is revealed successively (by making an offer) and our rule can stop on the first available atom that lies below the threshold. Now let T be the arrival time of the first available atom that lies below the threshold curve $y = c/p(1-t)$, and let S be the time when the value of the best atom above threshold is equal to the threshold. Then

$$\mathrm{P}\{T > t\} = \mathrm{P}_{\text{Poisson}}\bigg(0,\, p\int_0^t g_{1,p}(r)\,\mathrm{d}r\bigg), \tag{16}$$

$$\mathrm{P}\{S > s\} = \mathrm{P}_{\text{Poisson}}\bigg(0,\, \int_0^s (g_{1,p}(s) - g_{1,p}(r))\,\mathrm{d}r\bigg), \tag{17}$$

where $g_{1,p}(r) = c/p(1-r)$, $0 < r < 1$, as before. It is easy to see that expression (12) also holds for our problem, that is,

$$\tilde{\mathrm{P}}(q) = \mathrm{e}^{-c} + \bigg(\frac{1}{c} - \frac{1}{c}\mathrm{e}^{-c} - \mathrm{e}^{-c}\bigg)\mathrm{P}\{S > T\}. \tag{18}$$

We have, from (16) and (17),

$$\begin{aligned}\mathrm{P}\{S > T\} &= \int_0^1 \mathrm{P}\{S > t\} f_T(t)\,\mathrm{d}t \\ &= \int_0^1 (1-t)^{-c/p}\mathrm{e}^{-ct/p(1-t)} c(1-t)^{c-1}\,\mathrm{d}t \\ &= \mathrm{e}^c\bigg(\frac{p\mathrm{e}}{c}\bigg)^{qc/p} L\bigg(\frac{qc}{p}, \frac{c}{p}\bigg). \end{aligned} \tag{19}$$

Substituting (19) into (18) yields the desired result.

It is easy to see that, when a is a positive integer,

$$L(a,b) = (a-1)!\,\mathrm{e}^{-b}\sum_{i=0}^{a-1}\frac{b^i}{i!}. \tag{20}$$

Hence, for some special values of q, $\tilde{\mathrm{P}}(q)$ can be further simplified.

Corollary 1. *Define* $q_m = m/(m+c)$ *for a given positive integer* m*. Then*

$$\tilde{\mathrm{P}}(q_m) = \mathrm{e}^{-c} + (\mathrm{e}^c - c - 1)\mathrm{e}^{-c}\sum_{i=1}^{m}\frac{(m-1)!}{(m-i)!\,(m+c)^i}, \qquad m \ge 1. \tag{21}$$

Proof. Let $qc/p = m$ or, equivalently, $q = m/(m+c)$. Then $c/p = m + c$, so (21) is immediate from Theorem 3 and (20).

Table 1 presents the numerical values of $\tilde{\mathrm{P}}(q)$ for specified values of q. This shows that our rule works well, especially for small values of q, and gives a pretty good lower bound on the asymptotically optimal value whose trivial upper bound is 0.580 164. The exact asymptotic

Table 1: $\tilde{\mathrm{P}}(q)$ for some values of q.

q	$\tilde{\mathrm{P}}(q)$
0.1	0.577
0.3	0.569
0.5	0.558
0.7	0.542
0.9	0.512

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value is still unknown. Since Model 2 is history dependent in a complicated way, finding the exact value is closely related to the question of how the relevant history can be packed into tractable limiting form. In addition, there are two more questions of interest which are related to each other; one is whether the insensitivity to q of the optimal asymptotic value also holds in Model 2, and the other is whether Models 1 and 2 are asymptotically equivalent in a sense that the advantage of Model 1 fades away and these two models have the same asymptotic value. The difficulty of Model 2 is related with the one of Robbins' problem (see, e.g. Bruss (2005) for a review of this problem or Bruss and Swan (2009) for a version of Robbins' problem in a similar PPP setting).

Appendix A. Proof of Theorem 1

(a) Let $p_k(x) = (q + px)^k$, $k \ge 0$. To show that $p_k(x) \ge v_k(x)$ if and only if $x \ge b_k(q)$ for a nondecreasing threshold $b_k(q)$, it suffices to show that $v_k(x)/p_k(x)$ is decreasing in x and increasing in k with $v_k(x)/p_k(x) \to 0$ as $x \to 1$. This can be easily verified, because the repeated use of (4) yields

$$v_k(x) = p \sum_{i=1}^{k} p_{k-i}(x) \int_x^1 \max\{p_{i-1}(t), v_{i-1}(t)\}\,\mathrm{d}t, \tag{A.1}$$

or, equivalently,

$$\frac{v_k(x)}{p_k(x)} = p \sum_{i=1}^{k} \frac{1}{(q+px)^i} \int_x^1 \max\{p_{i-1}(t), v_{i-1}(t)\}\,\mathrm{d}t,$$

showing that $v_k(x)/p_k(x)$ has the desired properties as each term of the sum on the right-hand side is nonnegative and decreasing in x. To obtain the threshold (5), suppose that the $(n-k)$th applicant is a candidate and has value $X_{n-k} = x$. If we choose this candidate, the success probability is $p_k(x)$. If instead we continue and choose the next candidate, if any, we can expect, from (A.1), the corresponding probability to be

$$q_k(x) = p \sum_{i=1}^{k} p_{k-i}(x) \int_x^1 p_{i-1}(t)\,\mathrm{d}t = (q+px)^k \sum_{i=1}^{k} \frac{1}{i}[(q+px)^{-i} - 1].$$

Since $b_k(q)$ is nondecreasing in k, $b_k(q)$ must be the value of x which equates $p_k(x)$ and $q_k(x)$, i.e. satisfies the equation

$$\sum_{i=1}^{k} \frac{1}{i}[(q+px)^{-i} - 1] = 1 \tag{A.2}$$

if $p_k(0) < q_k(0)$ and $b_k(q)$ must be 0 otherwise. Comparing (A.2) with (1) (the definition of b_k) yields (5).

(b) A stopping rule $\boldsymbol{d} = \{d_j,\ 1 \le j \le n\}$ is called a *monotone thresholds rule* if it chooses the first candidate X_j if $X_j \ge d_j$ for a nonincreasing sequence d_j, i.e. $d_1 \ge d_2 \ge \cdots \ge d_n$ with $0 \le d_j \le 1$. We first give an explicit expression for $\mathrm{P}_n(\boldsymbol{d})$, the success probability achieved by monotone thresholds rule $\boldsymbol{d}$.

Lemma 1. *Let* $\mathrm{P}(j \mid \boldsymbol{d})$ *be the probability of success by stopping on the* j*th applicant using rule* $\boldsymbol{d}$. *Then* $\mathrm{P}_n(\boldsymbol{d}) = \sum_{j=1}^n \mathrm{P}(j \mid \boldsymbol{d})$, *where*

$$\mathrm{P}(1 \mid \boldsymbol{d}) = \frac{1}{n}[1 - (pd_1 + q)^n],$$

$$\mathrm{P}(r+1 \mid \boldsymbol{d}) = \frac{1}{r(n-r)} \sum_{i=1}^{r} (pd_i + q)^r - \frac{1}{n(n-r)} \sum_{i=1}^{r} (pd_i + q)^n - \frac{1}{n}(pd_{r+1} + q)^n, \qquad 1 \le r < n.$$

Proof. Of many possible proofs, the shortest one is to recognize that the argument of Gilbert and Mosteller (1966) can apply to our problem by simply replacing d_i by $pd_i + q$, $1 \le i \le n$, in their Theorem 4 (3c-1).

Since the optimal rule is a monotone thresholds rule, to show (6), it suffices to show that $\mathrm{P}_n(\boldsymbol{d})$ can be written as (6) if we set $d_j = b_{n-j}(q)$, $1 \le j \le n$ for $n > m$ (recall that the index of $b_{n-j}(q)$ represents the number of observations remaining). If we write $pd_i + q = v_{n-i}$ for convenience, Lemma 1 yields

$$\begin{aligned}
\mathrm{P}_n(\boldsymbol{d}) - \frac{1}{n}(1 - v_{n-1}^n) &= \sum_{r=1}^{n-1} \sum_{i=1}^{r} \frac{1}{n-r}\left(\frac{1}{r} v_{n-i}^r - \frac{1}{n} v_{n-i}^n\right) - \frac{1}{n} \sum_{i=0}^{n-2} v_i^n \\
&= \sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{k}\left(\frac{1}{n-k} v_j^{n-k} - \frac{1}{n} v_j^n\right) - \frac{1}{n} \sum_{i=0}^{n-2} v_i^n \\
&= \frac{1}{n} \sum_{j=1}^{n-1} \sum_{k=1}^{j} \left[\left(\frac{1}{n-k} + \frac{1}{k}\right) v_j^{n-k} - \frac{1}{k} v_j^n\right] - \frac{1}{n}\left(\sum_{j=1}^{n-1} v_j^n + v_0^n - v_{n-1}^n\right) \\
&= \frac{1}{n} \sum_{j=1}^{n-1} A_j - \frac{1}{n}(v_0^n - v_{n-1}^n), \qquad \text{(A.3)}
\end{aligned}$$

where

$$A_j = \sum_{k=1}^{j} \left[\left(\frac{1}{n-k} + \frac{1}{k}\right) v_j^{n-k} - \frac{1}{k} v_j^n\right] - v_j^n, \qquad 1 \le j < n.$$

From now on, we consider the optimal success probability by setting $d_j = b_{n-j}(q)$, or, from (5),

$$v_j = \begin{cases} b_j & \text{if } m \le j < n, \\ q & \text{if } 0 \le j < m. \end{cases}$$

We then have, for $j \ge m$,

$$\begin{aligned}
A_j &= \sum_{k=1}^{j} \left[\left(\frac{1}{n-k} + \frac{1}{k}\right) b_j^{n-k} - \frac{1}{k} b_j^n\right] - b_j^n \\
&= \sum_{k=1}^{j} \frac{1}{n-k} b_j^{n-k} + b_j^n \left[\sum_{k=1}^{j} \frac{1}{k}(b_j^{-k} - 1) - 1\right] \\
&= \sum_{k=1}^{j} \frac{1}{n-k} b_j^{n-k}, \qquad \text{(A.4)}
\end{aligned}$$

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where the last equality follows from the definition of b_j, (1). On the other hand, for $j < m$,

$$A_j = \sum_{k=1}^{j}\left[\left(\frac{1}{n-k}+\frac{1}{k}\right)q^{n-k} - \frac{1}{k}q^n\right] - q^n$$
$$= nq^n\left[\sum_{k=1}^{j}\frac{1}{k(n-k)}q^{-k} - \frac{1}{n}\left(1+\sum_{k=1}^{j}\frac{1}{k}\right)\right],$$

and, hence,

$$\sum_{j=1}^{m-1} A_j = nq^n\left[\sum_{j=1}^{m-1}\sum_{k=1}^{j}\frac{1}{k(n-k)}q^{-k} - \frac{1}{n}\sum_{j=1}^{m-1}\left(1+\sum_{k=1}^{j}\frac{1}{k}\right)\right]$$
$$= nq^n\left[\sum_{k=1}^{m-1}\frac{m-k}{k(n-k)}q^{-k} - \frac{m}{n}\sum_{k=1}^{m-1}\frac{1}{k}\right]$$
$$= nq^n\sum_{k=1}^{m-1}\frac{1}{k}\left[\left(\frac{m-k}{n-k}\right)q^{-k} - \left(\frac{m}{n}\right)\right]$$
$$= nq^n\sum_{k=1}^{m}\frac{1}{k}\left[\left(\frac{m-k}{n-k}\right)q^{-k} - \left(\frac{m}{n}\right)\right] + q^n. \tag{A.5}$$

Substituting (A.4) and (A.5) into (A.3), combined with $v_0^n = q^n$, yields (6). For $n \le m$, the result is immediate from (A.3) by letting $v_j = q$.

(c) Let $c_k = k(1-b_k)$. Then $c_k \to c$ as $k \to \infty$ (see (2)) and (6) is now written as

$$\mathrm{P}_n^*(q) = \frac{1}{n}\left[1+\sum_{i=m}^{n-1}\sum_{r=1}^{i}\frac{1}{n-r}\left(1-\frac{c_i}{i}\right)^{n-r}\right] + q^n\sum_{r=1}^{m}\frac{1}{r}\left[\left(\frac{m-r}{n-r}\right)q^{-r} - \left(\frac{m}{n}\right)\right].$$

Since m is finite for a given $q < 1$, and so the second term vanishes as $n \to \infty$, we have

$$\mathrm{P}_n^*(q) \to \mathrm{P}^*(q) = \int_0^1\int_0^u \frac{1}{1-v}(\mathrm{e}^{-c})^{(1-v)/u}\,\mathrm{d}v\,\mathrm{d}u.$$

To evaluate the double integral, make the change of variables $s = 1/u$ and $t = (1-v)/u$. Since this transformation has Jacobian $J(s,t) = \partial(u,v)/\partial(s,t) = 1/s^3$, it follows that

$$\mathrm{P}^*(q) = \int_0^1\left(\int_1^{t+1}\frac{\mathrm{d}s}{s^2}\right)\frac{1}{t}\mathrm{e}^{-ct}\,\mathrm{d}t + \int_1^\infty\left(\int_t^{t+1}\frac{\mathrm{d}s}{s^2}\right)\frac{1}{t}\mathrm{e}^{-ct}\,\mathrm{d}t$$
$$= \int_0^1\frac{\mathrm{e}^{-ct}}{1+t}\,\mathrm{d}t + \int_1^\infty\left(\frac{1}{t^2}-\frac{1}{t}+\frac{1}{1+t}\right)\mathrm{e}^{-ct}\,\mathrm{d}t$$
$$= \int_0^\infty\frac{\mathrm{e}^{-ct}}{1+t}\,\mathrm{d}t + \int_1^\infty\frac{\mathrm{e}^{-ct}}{t^2}\,\mathrm{d}t - \int_1^\infty\frac{\mathrm{e}^{-ct}}{t}\,\mathrm{d}t$$
$$= \mathrm{e}^{c}\int_1^\infty\frac{\mathrm{e}^{-ct}}{t}\,\mathrm{d}t + \left(\mathrm{e}^{-c} - c\int_1^\infty\frac{\mathrm{e}^{-ct}}{t}\,\mathrm{d}t\right) - \int_1^\infty\frac{\mathrm{e}^{-ct}}{t}\,\mathrm{d}t$$
$$= \mathrm{e}^{-c} + (\mathrm{e}^{c}-c-1)\int_1^\infty\frac{\mathrm{e}^{-ct}}{t}\,\mathrm{d}t,$$

which proves (7).

Dedication

The author dedicates this paper with great respect to Professor Minoru Sakaguchi, who made enormous contributions to the field of optimal stopping and game theory. His work will remain a source of inspiration to many researchers. He died on 16 June 2009 at the age of 83.

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