J. Austral. Math. Soc. (*Series A*) **61** (1996), 1–13

HODGE CYCLES ON KUGA FIBER VARIETIES

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(Received 22 April 1993)

Communicated by H. R. Rubinstein

Abstract

We determine the dimension of the space of Hodge cycles for the generic fibers of the Kuga fiber varieties associated to certain quaternion algebras.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 14C30, 14K10.

1. Introduction

One of the well-known conjectures in algebraic geometry is the Hodge conjecture which states that every Hodge cycle on a complex projective variety is an algebraic cycle. In this paper, we consider Hodge cycles on generic fibers of certain Kuga fiber varieties.

Let V be a vector space of dimension $2n$ over $\mathbb{Q}$, and let L be a lattice in V. Let β be a nondegenerate alternating bilinear form on V such that $\beta(L, L) \subset \mathbb{Z}$. Let $Sp(V, \beta)$ be the symplectic group of the pair (V, β), and let $\mathscr{H}$ denote the Siegel half space determined by β (see Section 4). Then each element $J \in \mathscr{H}$ defines a complex structure on $V(\mathbb{R})$ and there is a unique complex analytic structure on $\mathscr{H} \times V(\mathbb{R})$ such that the natural projection $\mathscr{H} \times V(\mathbb{R}) \to \mathscr{H}$ is a complex vector bundle over $\mathscr{H}$. For each J, if we denote the complex vector space $(V(\mathbb{R}), J)$ by V_J, then the complex torus V_J/L is an abelian variety with polarization β. Let $A_{\mathscr{H}}$ denote the quotient space $L \setminus \mathscr{H} \times V(\mathbb{R})$, where L acts on $\mathscr{H} \times V(\mathbb{R})$ by $l \cdot (J, v) = (J, v + l)$ for $J \in \mathscr{H}$, $v \in V(\mathbb{R})$ and $l \in L$. The projection map $\mathscr{H} \times V(\mathbb{R}) \to \mathscr{H}$ induces the fiber bundle $\pi_{\mathscr{H}} : A_{\mathscr{H}} \to \mathscr{H}$ whose fibers are abelian varieties isomorphic to V_J/L polarized by β. Let $Sp(L, \beta)$ be the subgroup of $Sp(V, \beta)$ of elements g with $gL = L$, and take a subgroup Γ_0 of $Sp(L, \beta)$ of finite index that contains no

https://doi.org/10.1017/S1446788700000033 Published online by Cambridge University Press

elements of finite order. Then the quotient $X_0 = \Gamma_0 \backslash \mathscr{H}$ is an arithmetic variety that can be considered as a Zariski open subset of a complex projective variety. Now the fiber bundle $\pi_{\mathscr{H}} : A_{\mathscr{H}} \to \mathscr{H}$ induces the standard family of abelian varieties $\pi_0 : Y_0 \to X_0$ over X_0 (see for example [4, 8], [9, Chapter 4]).

Let $\tilde{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$, and let $\tilde{K}$ be a maximal compact subgroup of the semisimple Lie group $\tilde{G}(\mathbb{R})$. We assume that the symmetric space $\tilde{D} = \tilde{G}(\mathbb{R})/\tilde{K}$ has a $\tilde{G}(\mathbb{R})$-invariant complex structure. Let $\tilde{\Gamma} \subset \tilde{G}(\mathbb{Q})$ be a torsion-free cocompact arithmetic subgroup of $\tilde{G}$, and let $\tilde{X} = \tilde{\Gamma}\backslash\tilde{D}$ be the corresponding arithmetic variety. Let $\tilde{\rho} : \tilde{G} \to Sp(V, \beta)$ be a homomorphism, and let $\tilde{\tau} : \tilde{D} \to \mathscr{H}$ be a holomorphic map such that $\tilde{\rho}(\tilde{\Gamma}) \subset \Gamma_0$ and $\tilde{\tau}(\tilde{g}\tilde{y}) = \tilde{\rho}(\tilde{g})\tilde{\tau}(\tilde{y})$ for all $\tilde{g} \in \tilde{G}(\mathbb{R})$ and $\tilde{y} \in \tilde{D}$. Then the pair $(\tilde{\rho}, \tilde{\tau})$ determines a fiber variety $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ called a Kuga fiber variety over the arithmetic variety $\tilde{X}$ whose fibers are abelian varieties. Such a Kuga fiber variety can be constructed as follows. The semidirect product $\tilde{\Gamma} \ltimes_{\tilde{\rho}} L$ with respect to the representation $\tilde{\rho} : \tilde{\Gamma} \to \operatorname{Aut}(L)$ operates on the product manifold $\tilde{D} \times V(\mathbb{R})$ properly discontinuously by $(\gamma, l) \cdot (y, v) = (\gamma y, \gamma v + l)$ for $(\gamma, l) \in \tilde{\Gamma} \ltimes_{\rho} L$ and $(y, v) \in \tilde{D} \times V(\mathbb{R})$. We set $\tilde{Y} = \tilde{\Gamma} \ltimes_{\rho} L \backslash \tilde{D} \times V(\mathbb{R})$, and denote by $\tilde{\pi}$ the natural projection of $\tilde{Y}$ onto $\tilde{X} = \tilde{\Gamma}\backslash\tilde{D}$. Then $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is a fiber bundle over $\tilde{X}$, which is in fact the pullback of the standard fiber bundle $\pi_0 : Y_0 \to X_0$ over $X_0 = \Gamma_0\backslash\mathscr{H}$ via the map $\tilde{X} \to X_0$ induced by $\tilde{\tau} : \tilde{D} \to \mathscr{H}$.

Let K be a totally real number field with $[K : \mathbb{Q}] = m$, and let $S = \{\varphi_1, \dots, \varphi_m\}$ be the set of embeddings of K into $\mathbb{R}$. Let B be a quaternion algebra over K, and let G be the algebraic group $\operatorname{Res}_{K/\mathbb{Q}}(SL_1(B))$ over $\mathbb{Q}$, where Res is the Weil restriction map. Then $G(\mathbb{C})$ can be identified with $SL_2(\mathbb{C})^m$. We denote by ρ_j be the projection of $G(\mathbb{C})$ onto the jth factor of $SL_2(\mathbb{C})^m$. We fix a subset $R = \{\varphi_a, \varphi_b, \varphi_c, \varphi_d\}$ of S, and define the representation $\rho : G(\mathbb{C}) \to SL_{16}(\mathbb{C}) \subset Sp(8, \mathbb{C})$ to be the tensor product $\rho_a \otimes \rho_b \otimes \rho_c \otimes \rho_d$. Let Γ be a torsion-free arithmetic subgroup of G, and let $\tau : D \to \mathscr{H}$ a holomorphic map such that $\tau(gy) = \rho(g)\tau(y)$ for $y \in D$ and $g \in G(\mathbb{R})$, where D is the quotient of $G(\mathbb{R})$ by a maximal compact subgroup. Let $\pi : Y \to X$ be the Kuga fiber variety over X determined by the pair (ρ, τ). Given a point $x \in X$, we denote by

$$HH^{2k}(Y_x, \mathbb{Q}) = H^{(k,k)}(Y_x) \cap H^{2k}(Y_x, \mathbb{Q})$$

the space of Hodge cycles in the fiber Y_x over $x \in X$. Such Hodge cycles have been studied in a number of papers (see for example [1, 3, 5, 6, 7, 10]). The purpose of this paper is to determine the dimension of the space $HH^{2k}(Y_x, \mathbb{Q})$ for $0 \le k \le 8$ for a generic point x in X.

2. Representations determined by quaternion algebras

In this section we state a theorem which determines exterior powers of the representation of a complex Lie group associated to a quaternion algebra. Let K be a totally real number field with $[K : \mathbb{Q}] = m$, and let B be a quaternion algebra over K. Let $S = \{\varphi_1, \dots, \varphi_m\}$ be the set of all embeddings of K into $\mathbb{R}$, and let K_j be the completion of K by the embedding $\varphi_j : K \hookrightarrow \mathbb{R}$ for each $j \in \{1, \dots, m\}$. Then the algebra $B \otimes_K K_j$ is isomorphic to either the algebra $M_2(\mathbb{R})$ of 2×2 real matrices or the Hamiltonian quaternion $\mathbb{H}$. We denote by S_0 the set of mappings φ_j with $B \otimes_K K_j \cong M_2(\mathbb{R})$ and for later purposes assume that $S_0 = \{1, \dots, n\}$ with $1 \le n \le m-3$.

Let $\mathbb{B} = \operatorname{Res}_{K/\mathbb{Q}}(B)$, where Res is Weil's restriction operator. Then we have

$$\mathbb{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{j=1}^{m} (B \otimes_K K_j) \cong M_2(\mathbb{R})^n \times \mathbb{H}^{m-n}.$$

As a ring, $\mathbb{B}$ is isomorphic to B. We fix a ring isomorphism $\iota : B \to \mathbb{B}$. We identify $\mathbb{B} \otimes_{\mathbb{Q}} \mathbb{R}$ with $M_2(\mathbb{R})^n \times \mathbb{H}^{m-n}$ and denote by Pr_j its projection map onto the jth factor for $1 \le j \le m$. Then $\mathrm{Pr}_j \circ \iota$ is an isomorphism of B onto $M_2(\mathbb{R})$ for $1 \le j \le n$ and onto $\mathbb{H}$ for $n+1 \le j \le m$.

Let G be the algebraic group $\operatorname{Res}_{K/\mathbb{Q}}(SL_1(B))$ over $\mathbb{Q}$. Then we have

$$G(\mathbb{Q}) \cong B_1^\times = \{x \in B^\times \mid \nu(x) = 1\},$$

where ν is the reduced norm of the quaternion algebra B. We identify $G(\mathbb{Q})$ with the subgroup $\iota(B_1^\times)$ of $\mathbb{B}^\times$. Then the Lie group $G(\mathbb{R})$ can be identified with the subgroup $SL_2(\mathbb{R})^n \times (\mathbb{H}_1^\times)^{m-n}$ of $(\mathbb{B} \otimes \mathbb{R})^\times = (M_2(\mathbb{R})^n \times \mathbb{H}^{m-n})^\times$ and $G(\mathbb{C})$ can be identified with $SL_2(\mathbb{C})^m$. If we also identify $\mathbb{H}_1^\times$ with SU_2, we have

$$G(\mathbb{Q}) \subset G(\mathbb{R}) = SL_2(\mathbb{R})^n \times SU_2^{m-n} \subset G(\mathbb{C}) = SL_2(\mathbb{C})^m.$$

Let $R = \{\varphi_a, \varphi_b, \varphi_c, \varphi_d\}$ be a subset of S with $|R| = 4$. We associate to R a representation ρ_R of $G(\mathbb{C}) = SL_2(\mathbb{C})^m$ to $SL_{16}(\mathbb{C})$ by $\rho_R = \rho_a \otimes \rho_b \otimes \rho_c \otimes \rho_d$, where ρ_j is the projection onto the jth factor of $SL_2(\mathbb{C})^m$ for $1 \le j \le m$. We shall denote the representation ρ_R with $R = \{\varphi_a, \varphi_b, \varphi_c, \varphi_d\}$ simply by $abcd$. We shall also denote by a_k, for example, the kth symmetric power $S^k(\rho_a)$ of ρ_a, and denote the tensor product operation $\otimes$ by $\cdot$ and the direct sum operation $\oplus$ by $+$ respectively.

THEOREM 1. *Given the set of embeddings $S = \{\varphi_1, \dots, \varphi_m\}$ of K into $\mathbb{R}$ and a subset $R = \{\varphi_a, \varphi_b, \varphi_c, \varphi_d\}$ of S, let ρ_R be the representation $\rho_R = abcd = \rho_a \otimes \rho_b \otimes \rho_c \otimes \rho_d$ of $G(\mathbb{C}) = SL_2(\mathbb{C})^m$ in the complex vector space $\mathbb{C}^{16}$ for $1 \le a, b, c, d \le m$ as described above. If ρ_U denotes the representation of the compact real form $(SU_2)^m$ of $G(\mathbb{C}) = SL_2(\mathbb{C})^m$ induced by ρ_R, then up to equivalence of representations the exterior powers $\wedge^k(\rho_U)$ for $0 \le k \le 16$ are as follows:*

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$$\begin{aligned}
\wedge^0(\rho_U) &= \wedge^{16}(\rho_U) = 1,\\
\wedge^1(\rho_U) &= \wedge^{15}(\rho_U) = abcd,\\
\wedge^2(\rho_U) &= \wedge^{14}(\rho_U) = a_2b_2c_2 + a_2b_2d_2 + a_2c_2d_2 + b_2c_2d_2 + a_2 + b_2 + c_2 + d_2,\\
\wedge^3(\rho_U) &= \wedge^{13}(\rho_U) = a_3b_3cd + a_3bc_3d + a_3bcd_3 + a_3bcd + ab_3c_3d + ab_3cd_3\\
&\quad + ab_3cd + abc_3d_3 + abc_3d + abcd_3 + 3abcd,\\
\wedge^4(\rho_U) &= \wedge^{12}(\rho_U) = 3a_2b_2c_2d_2 + a_2b_2c_2d_4 + 2a_2b_2c_2 + a_2b_2c_4d_2 + a_2b_2c_4\\
&\quad + 2a_2b_2d_2 + a_2b_2d_4 + 2a_2b_2 + a_2b_4c_2d_2 + a_2b_4c_2 + a_2b_4d_2\\
&\quad + 2a_2c_2d_2 + a_2c_2d_4 + 2a_2c_2 + a_2c_4d_2 + 2a_2d_2 + a_4b_2c_2d_2\\
&\quad + a_4b_2c_2 + a_4b_2d_2 + a_4b_4 + a_4c_2d_2 + a_4c_4 + a_4d_4 + a_4\\
&\quad + 2b_2c_2d_2 + b_2c_2d_4 + 2b_2c_2 + b_2c_4d_2 + 2b_2d_2 + b_4c_2d_2 + b_4c_4\\
&\quad + b_4d_4 + b_4 + 2c_2d_2 + c_4d_4 + c_4 + d_4 + 3,\\
\wedge^5(\rho_U) &= \wedge^{11}(\rho_U) = 5abcd + 4abcd_3 + abcd_5 + 4abc_3d + 3abc_3d_3 + abc_3d_5\\
&\quad + abc_5d + abc_5d_3 + 4ab_3cd + 3ab_3cd_3 + ab_3cd_5 + 3ab_3c_3d\\
&\quad + 2ab_3c_3d_3 + ab_3c_5d + ab_5cd + ab_5cd_3 + ab_5c_3d + 4a_3bcd\\
&\quad + 3a_3bcd_3 + a_3bcd_5 + 3a_3bc_3d + 2a_3bc_3d_3 + a_3bc_5d\\
&\quad + 3a_3b_3cd + 2a_3b_3cd_3 + 2a_3b_3c_3d + a_3b_3c_3d_3 + a_3b_5cd\\
&\quad + a_5bcd + a_5bcd_3 + a_5bc_3d + a_5b_3cd,\\
\wedge^6(\rho_U) &= \wedge^{10}(\rho_U) = 6a_2b_2c_2d_2 + 4a_2b_2c_2d_4 + 6a_2b_2c_2 + 4a_2b_2c_4d_2 + a_2b_2c_4d_4\\
&\quad + 2a_2b_2c_4 + a_2b_2c_6 + 6a_2b_2d_2 + 2a_2b_2d_4 + a_2b_2d_6 + a_2b_2\\
&\quad + 4a_2b_4c_2d_2 + a_2b_4c_2d_4 + 2a_2b_4c_2 + a_2b_4c_4d_2 + a_2b_4c_4\\
&\quad + 2a_2b_4d_2 + a_2b_4d_4 + 2a_2b_4 + a_2b_6c_2 + a_2b_6d_2 + 6a_2c_2d_2\\
&\quad + 2a_2c_2d_4 + a_2c_2d_6 + a_2c_2 + 2a_2c_4d_2 + a_2c_4d_4 + 2a_2c_4\\
&\quad + a_2c_6d_2 + a_2d_2 + 2a_2d_4 + 3a_2 + 4a_4b_2c_2d_2 + a_4b_2c_2d_4\\
&\quad + 2a_4b_2c_2 + a_4b_2c_4d_2 + a_4b_2c_4 + 2a_4b_2d_2 + a_4b_2d_4 + 2a_4b_2\\
&\quad + a_4b_4c_2d_2 + a_4b_4c_2 + a_4b_4d_2 + a_4b_4 + 2a_4c_2d_2 + a_4c_2d_4\\
&\quad + 2a_4c_2 + a_4c_4d_2 + a_4c_4 + 2a_4d_2 + a_4d_4 + a_6b_2c_2 + a_6b_2d_2\\
&\quad + a_6c_2d_2 + a_6 + 6b_2c_2d_2 + 2b_2c_2d_4 + b_2c_2d_6 + b_2c_2 + 2b_2c_4d_2\\
&\quad + b_2c_4d_4 + 2b_2c_4 + b_2c_6d_2 + b_2d_2 + 2b_2d_4 + 3b_2 + 2b_4c_2d_2\\
&\quad + b_4c_2d_4 + 2b_4c_2 + b_4c_4d_2 + b_4c_4 + 2b_4d_2 + b_4d_4 + b_6c_2d_2\\
&\quad + b_6 + c_2d_2 + 2c_2d_4 + 3c_2 + 2c_4d_2 + c_4d_4 + c_6 + 3d_2 + d_6,
\end{aligned}$$

$$\begin{aligned}
\wedge^7(\rho_U) = \wedge^9(\rho_U) = {} & 7abcd + 6abcd_3 + 3abcd_5 + abcd_7 + 6abc_3d + 6abc_3d_3 \\
& + 2abc_3d_5 + 3abc_5d + 2abc_5d_3 + abc_7d + 6ab_3cd + 6ab_3cd_3 \\
& + 2ab_3cd_5 + 6ab_3c_3d + 4ab_3c_3d_3 + ab_3c_3d_5 + 2ab_3c_5d \\
& + ab_3c_5d_3 + 3ab_5cd + 2ab_5cd_3 + 2ab_5c_3d + ab_5c_3d_3 + ab_7cd \\
& + 6a_3bcd + 6a_3bcd_3 + 2a_3bcd_5 + 6a_3bc_3d + 4a_3bc_3d_3 \\
& + a_3bc_3d_5 + 2a_3bc_5d + a_3bc_5d_3 + 6a_3b_3cd + 4a_3b_3cd_3 \\
& + a_3b_3cd_5 + 4a_3b_3c_3d + 3a_3b_3c_3d_3 + a_3b_3c_5d + 2a_3b_5cd \\
& + a_3b_5cd_3 + a_3b_5c_3d + 3a_5bcd + 2a_5bcd_3 + 2a_5bc_3d \\
& + a_5bc_3d_3 + 2a_5b_3cd + a_5b_3cd_3 + a_5b_3c_3d + a_7bcd,
\end{aligned}$$

$$\begin{aligned}
\wedge^8(\rho_U) = {} & 10a_2b_2c_2d_2 + 5a_2b_2c_2d_4 + a_2b_2c_2d_6 + 5a_2b_2c_2 + 5a_2b_2c_4d_2 \\
& + 2a_2b_2c_4d_4 + 4a_2b_2c_4 + a_2b_2c_6d_2 + a_2b_2c_6 + 5a_2b_2d_2 + 4a_2b_2d_4 \\
& + a_2b_2d_6 + 4a_2b_2 + 5a_2b_4c_2d_2 + 2a_2b_4c_2d_4 + 4a_2b_4c_2 + 2a_2b_4c_4d_2 \\
& + a_2b_4c_4 + 4a_2b_4d_2 + a_2b_4d_4 + a_2b_4 + a_2b_6c_2d_2 + a_2b_6c_2 \\
& + a_2b_6d_2 + a_2b_6 + 5a_2c_2d_2 + 4a_2c_2d_4 + a_2c_2d_6 + 4a_2c_2 + 4a_2c_4d_2 \\
& + a_2c_4d_4 + a_2c_4 + a_2c_6d_2 + a_2c_6 + 4a_2d_2 + a_2d_4 + a_2d_6 + 5a_4b_2c_2d_2 \\
& + 2a_4b_2c_2d_4 + 4a_4b_2c_2 + 2a_4b_2c_4d_2 + a_4b_2c_4 + 4a_4b_2d_2 + a_4b_2d_4 \\
& + a_4b_2 + 2a_4b_4c_2d_2 + a_4b_4c_2 + a_4b_4c_4 + a_4b_4d_2 + a_4b_4d_4 + 2a_4b_4 \\
& + 4a_4c_2d_2 + a_4c_2d_4 + a_4c_2 + a_4c_4d_2 + a_4c_4d_4 + 2a_4c_4 + a_4d_2 \\
& + 2a_4d_4 + 3a_4 + a_6b_2c_2d_2 + a_6b_2c_2 + a_6b_2d_2 + a_6b_2 + a_6c_2d_2 + a_6c_2 \\
& + a_6d_2 + a_8 + 5b_2c_2d_2 + 4b_2c_2d_4 + b_2c_2d_6 + 4b_2c_2 + 4b_2c_4d_2 \\
& + b_2c_4d_4 + b_2c_4 + b_2c_6d_2 + b_2c_6 + 4b_2d_2 + b_2d_4 + b_2d_6 + 4b_4c_2d_2 \\
& + b_4c_2d_4 + b_4c_2 + b_4c_4d_2 + b_4c_4d_4 + 2b_4c_4 + b_4d_2 + 2b_4d_4 + 3b_4 \\
& + b_6c_2d_2 + b_6c_2 + b_6d_2 + b_8 + 4c_2d_2 + c_2d_4 + c_2d_6 + c_4d_2 + 2c_4d_4 \\
& + 3c_4 + c_6d_2 + c_8 + 3d_4 + d_8 + 4,
\end{aligned}$$

where a_j for instance denotes the jth symmetric power $S^j(a) = S^j(\rho_a)$ of ρ_a, the products denote tensor products, and sums denote the direct sums as before.

3. Proof of Theorem 1

In this section we give a proof of Theorem 1 stated in the previous section. To each representation μ of a complex semisimple Lie group $\mathscr{G}$ in $\mathbb{C}^N$ and an element $g \in \mathscr{G}$ we

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associate a polynomial $P_{\mu,g}(t)$ of degree N in t given by $P_{\mu,g}(t) = \det(1_N + \mu(g)t)$, where 1_N is the $N \times N$ identity matrix. Then we have

$$P_{\mu,g}(t) = \sum_{k=1}^{N} \operatorname{tr}((\wedge^k \mu)g)t^k.$$

We also denote by

$$P_\mu(t) = \det(1_N + \mu t) = \sum_{k=1}^{N} \operatorname{tr}(\wedge^k \mu)t^k$$

the map that associates $P_{\mu,g}(t)$ to each $g \in \mathscr{G}$.

LEMMA 2. *Let ρ_U be the representation of $G_U = (SU_2)^m$ in $\mathbb{C}^{16}$ as described in Theorem 1. We fix an element $g = (g_1, \dots, g_m)$ in $G_U = (SU_2)^m$ and assume that the eigenvalues of the 2×2 matrix g_d are λ and λ^{-1}. Then we have*

$$P_{\rho_U,g}(t) = P_{(abc)_U,g}(\lambda t) P_{(abc)_U,g}(\lambda^{-1}t),$$

where $(abc)_U$ is the representation of $G_U = (SU_2)^m$ induced by $abc = \rho_a \otimes \rho_b \otimes \rho_c$.

PROOF. Since λ, λ^{-1} are the eigenvalues of g_d, we have

$$g_d = v^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} v$$

for some $v \in SU_2$ and $\lambda \in \mathbb{C}$. We have

$$P_{\rho_U,g}(t) = \det\left(\begin{pmatrix} 1_N & 0 \\ 0 & 1_N \end{pmatrix} + v_N \begin{pmatrix} abc(g)\lambda t & 0 \\ 0 & abc(g)\lambda^{-1}t \end{pmatrix} v_N^{-1} \right),$$

where

$$v_N = \begin{pmatrix} \alpha 1_N & \beta 1_N \\ \gamma 1_N & \delta 1_N \end{pmatrix} \quad \text{if} \quad v = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Thus we have

$$\begin{aligned} P_{\rho_U,g}(t) &= \begin{pmatrix} 1_N + abc(g)\lambda t & 0 \\ 0 & 1_N + abc(g)\lambda^{-1}t \end{pmatrix} \\ &= \det(1_N + abc(g)\lambda t) \det(1_N + abc(g)\lambda^{-1}t) \\ &= P_{(abc)_U,g}(\lambda t) P_{(abc)_U,g}(\lambda^{-1}t). \end{aligned}$$

Hence the lemma follows.

From now on we shall denote the representation $(a_j b_k c_l d_m)_U$ of G_U induced by the representation $a_j b_k c_l d_m$ of $G(\mathbb{C})$ simply by $a_j b_k c_l d_m$, where j, k, l, m are nonnegative integers.

LEMMA 3. *Using the notational convention in the previous paragraph, the exterior powers of the representations a, ab and abc for $1 \le a, b, c \le m$ of $G_U = (SU_2)^m$ are as follows:*

(i) $\wedge^0(a) = \wedge^2(a) = 1, \quad \wedge^1(a) = a.$

(ii) $\wedge^0(ab) = \wedge^4(ab) = 1, \quad \wedge^1(ab) = \wedge^3(ab) = ab, \quad \wedge^2(ab) = a_2 + b_2.$

(iii)
$$\begin{aligned}
&\wedge^0(abc) = \wedge^8(abc) = 1,\\
&\wedge^1(abc) = \wedge^7(abc) = abc,\\
&\wedge^2(abc) = \wedge^6(abc) = a_2b_2 + b_2c_2 + c_2a_2 + 1,\\
&\wedge^3(abc) = \wedge^5(abc) = a_3bc + ab_3c + abc_3 + abc,\\
&\wedge^4(abc) = a_4 + b_4 + c_4 + a_2b_2c_2 + a_2b_2 + b_2c_2 + c_2a_2 + 1.
\end{aligned}$$

PROOF. See [6, Lemma 2.2.1].

Now we go back to the proof of Theorem 1. By Lemma 2, we have

$$P_{\rho_R, g}(t) = \sum_{k=0}^{16} \operatorname{tr}(\wedge^k(\rho_R)(g))t^k = \Big(\sum_{k=0}^{8} \operatorname{tr}(C_k(g))(\lambda t)^k\Big)\Big(\sum_{k=0}^{8} \operatorname{tr}(C_k(g))(\lambda^{-1}t)^k\Big),$$

where $C_k = \wedge^k(abc)$ for each k given by Lemma 3(iii). Since we are interested in the representations up to equivalence, from now on we shall identify representations with their traces. Thus we have

$$\sum_{k=0}^{16} \wedge^k(\rho_R)(g)t^k = \Big(\sum_{k=0}^{8} C_k(g)(\lambda t)^k\Big)\Big(\sum_{k=0}^{8} C_k(g)(\lambda^{-1}t)^k\Big).$$

By comparing the coefficients of t^k in the above relation, we obtain

$$\begin{aligned}
\wedge^0(\rho_U) &= \wedge^{16}(\rho_U) = 1\\
\wedge^1(\rho_U) &= \wedge^{15}(\rho_U) = abc(\lambda + \lambda^{-1})\\
\wedge^2(\rho_U) &= \wedge^{14}(\rho_U) = (a_2b_2 + b_2c_2 + c_2a_2 + 1)(\lambda^2 + \lambda^{-2}) + a^2b^2c^2\\
\wedge^3(\rho_U) &= \wedge^{13}(\rho_U) = (a_3bc + ab_3c + abc_3 + abc)(\lambda^3 + \lambda^{-3})\\
&\qquad + abc(a_2b_2 + b_2c_2 + c_2a_2 + 1)(\lambda + \lambda^{-1})\\
\wedge^4(\rho_U) &= \wedge^{12}(\rho_U) = (a_4 + b_4 + c_4 + a_2b_2c_2 + a_2b_2 + b_2c_2 + c_2a_2 + 1)(\lambda^4 + \lambda^{-4})\\
&\qquad + abc(a_3bc + ab_3c + abc_3 + abc)(\lambda^2 + \lambda^{-2})\\
&\qquad + (a_2b_2 + b_2c_2 + c_2a_2 + 1)^2
\end{aligned}$$

$$
\begin{aligned}
\wedge^5(\rho_U) = \wedge^{11}(\rho_U) = {} & (a_3bc + ab_3c + abc_3 + abc)(\lambda^5 + \lambda^{-5}) \\
& + abc(a_4 + b_4 + c_4 + a_2b_2c_2 + a_2b_2 + b_2c_2 + c_2a_2 + 1) \\
& \quad \times (\lambda^3 + \lambda^{-3}) \\
& + (a_2b_2 + b_2c_2 + c_2a_2 + 1)(a_3bc + ab_3c + abc_3 + abc) \\
& \quad \times (\lambda + \lambda^{-1}) \\
\wedge^6(\rho_U) = \wedge^{10}(\rho_U) = {} & (a_2b_2 + b_2c_2 + c_2a_2 + 1)(\lambda^6 + \lambda^{-6}) \\
& + abc(a_3bc + ab_3c + abc_3 + abc)(\lambda^4 + \lambda^{-4}) \\
& + (a_2b_2 + b_2c_2 + c_2a_2 + 1) \\
& \quad \times (a_4 + b_4 + c_4 + a_2b_2c_2 + a_2b_2 + b_2c_2 + c_2a_2 + 1) \\
& \quad \times (\lambda^2 + \lambda^{-2}) \\
& + (a_3bc + ab_3c + abc_3 + abc)^2 \\
\wedge^7(\rho_U) = \wedge^9(\rho_U) = {} & abc(\lambda^7 + \lambda^{-7}) + abc(a_2b_2 + b_2c_2 + c_2a_2 + 1)(\lambda^5 + \lambda^{-5}) \\
& + (a_2b_2 + b_2c_2 + c_2a_2 + 1)(a_3bc + ab_3c + abc_3 + abc) \\
& \quad \times (\lambda^3 + \lambda^{-3}) \\
& + (a_3bc + ab_3c + abc_3 + abc) \\
& \quad \times (a_4 + b_4 + c_4 + a_2b_2c_2 + a_2b_2 + b_2c_2 + c_2a_2 + 1) \\
& \quad \times (\lambda + \lambda^{-1}) \\
\wedge^8(\rho_U) = {} & (\lambda^8 + \lambda^{-8}) + (abc)^2(\lambda^6 + \lambda^{-6}) \\
& + (a_2b_2 + b_2c_2 + c_2a_2 + 1)^2(\lambda^4 + \lambda^{-4}) \\
& + (a_3bc + ab_3c + abc_3 + abc)^2(\lambda^2 + \lambda^{-2}) \\
& + (a_4 + b_4 + c_4 + a_2b_2c_2 + a_2b_2 + b_2c_2 + c_2a_2 + 1)^2.
\end{aligned}
$$

By the Clebsch-Gordon formula we have

$$
\begin{aligned}
a_k \otimes a_l &= S^k(\rho_a) \otimes S^l(\rho_a) \\
&= S^{k+l}(\rho_a) \oplus S^{k+l-2} \oplus \cdots \oplus S^{|k-l|}(\rho_a) \\
&= a_{k+l} \oplus a_{k+l-2} \oplus \cdots \oplus a_{|k-l|},
\end{aligned}
$$

where K, l are non-negative integers. Thus, using our notational convention, we have

$$a_k a_l = a_{k+l} + a_{k+l-2} + \cdots + a_{|k-l|}.$$

Similar formulas are obtained for b, c and d.

LEMMA 4. *If* λ *and* λ^{-1} *are the eigenvalues of* g_d *as before, then we have*

$$\lambda^n + \lambda^{-n} = d_n - d_{n-2}$$

for all $n \geq 2$.

PROOF. We use induction on n. Since $d^2 = d_2 + 1$ by the Clebsch-Gordon formula, it follows that

$$\lambda^2 + \lambda^{-2} = (\lambda + \lambda^{-1})^2 - 2 = d^2 - 2 = d_2 - 1;$$

hence the statement is true for $n = 2$. Assuming that it is true for all $k \leq n$, we have

$$\begin{aligned}
\lambda^{n+1} + \lambda^{-(n+1)} &= (\lambda^n + \lambda^{-n})(\lambda + \lambda^{-1}) - (\lambda^{n-1} + \lambda^{-(n-1)}) \\
&= (d_n - d_{n-2})d - (d_{n-1} - d_{n-3}) \\
&= d_n d - d_{n-2} d - d_{n-1} + d_{n-3} \\
&= d_{n+1} + d_{n-1} - (d_{n-1} + d_{n-3}) - d_{n-1} + d_{n-3} \\
&= d_{n+1} - d_{n-1}.
\end{aligned}$$

So the statement is true for $n + 1$ and the lemma follows.

To complete the proof of Theorem 1 we first use Lemma 4 to replace the expressions of the form $(\lambda^k + \lambda^{-k})$ in the relations for $\wedge^0(R), \dots, \wedge^{16}(R)$ above by $d_k - d_{k-2}$, and then use the Clebsch-Gordon formula with the aid of a computer to obtain the formulas given in Theorem 1.

4. Kuga fiber varieties

In this section, we review the construction of Kuga fiber varieties over arithmetic varieties. Let V be a vector space of dimension $2n$ over $\mathbb{Q}$, and let L be a lattice in V. Let β be a nondegenerate alternating bilinear form on V such that $\beta(L, L) \subset \mathbb{Z}$. Let

$$Sp(V, \beta) = \{g \in GL(V) \mid \beta(gx, gy) = \beta(x, y) \text{ for all } x, y \in V\}$$

be the symplectic group of the pair (V, β), and let $\mathscr{H}$ denote the Siegel half space

$$\mathscr{H} = \{J \in GL(V(\mathbb{R})) \mid J^2 = -1, \quad \beta(x, Jy) \text{ is a positive definite symmetric bilinear form in } x, y \in V(\mathbb{R})\}.$$

Then each element $J \in \mathscr{H}$ defines a complex structure on $V(\mathbb{R})$ and there is a unique complex analytic structure on $\mathscr{H} \times V(\mathbb{R})$ such that the projection $P : \mathscr{H} \times V(\mathbb{R}) \to$

$\mathcal{H}$ is a complex vector bundle over $\mathcal{H}$. For each J if we denote the complex vector space $(V(\mathbb{R}), J)$ by V_J, then the complex torus $A_J = V_J/L$ is an abelian variety with the polarization β. We set

$$A_{\mathcal{H}} = L \setminus \mathcal{H} \times V(\mathbb{R}),$$

where the action of L on $\mathcal{H} \times V(\mathbb{R})$ is given by

$$l \cdot (J, v) = (J, v + l) \quad \text{for} \quad J \in \mathcal{H}, v \in V(\mathbb{R}) \quad \text{and} \quad l \in L.$$

Then the vector bundle $P : \mathcal{H} \times V(\mathbb{R}) \to \mathcal{H}$ induces the fiber bundle $\pi_{\mathcal{H}} : A_{\mathcal{H}} \to \mathcal{H}$ whose fibers are abelian varieties polarized by β. We set

$$Sp(L, \beta) = \{g \in Sp(V, \beta) \mid gL = L\},$$

and take a subgroup Γ_0 of $Sp(L, \beta)$ of finite index that contains no elements of finite order. Then the quotient $X_0 = \Gamma_0 \setminus \mathcal{H}$ is an arithmetic variety that can be considered as a Zariski open subset of a complex projective variety. Now the fiber bundle $\pi_{\mathcal{H}} : A_{\mathcal{H}} \to \mathcal{H}$ induces the standard family of abelian varieties $\pi_0 : Y_0 \to X_0$ over X_0.

Let $\tilde{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$, and let $\tilde{K}$ be a maximal compact subgroup of the semisimple Lie group $\tilde{G}(\mathbb{R})$. We assume that the symmetric space $\tilde{D} = \tilde{G}(\mathbb{R})/\tilde{K}$ has a $\tilde{G}(\mathbb{R})$-invariant complex structure. Let $\tilde{\Gamma} \subset \tilde{G}(\mathbb{Q})$ be a torsion-free cocompact arithmetic subgroup $\tilde{G}$, and let $\tilde{X} = \tilde{\Gamma} \backslash \tilde{D}$ be the corresponding arithmetic variety. Let $\tilde{\rho} : \tilde{G} \to Sp(V, \beta)$ be a symplectic representation and $\tilde{\tau} : \tilde{D} \to \mathcal{H}$ a holomorphic map such that $\tilde{\rho}(\tilde{\Gamma}) \subset \Gamma_0$ and

$$\tilde{\tau}(gy) = \tilde{\rho}(g)\tilde{\tau}(y) \quad \text{for all} \quad g \in \tilde{G}(\mathbb{R}) \quad \text{and} \quad y \in \tilde{D}.$$

Then the pair $(\tilde{\rho}, \tilde{\tau})$ determines a fiber variety $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ over the arithmetic variety $\tilde{X}$ whose fibers are abelian varieties called a Kuga fiber variety. It is constructed as follows. The semidirect product $\tilde{\Gamma} \ltimes_{\tilde{\rho}} L$ with respect to the representation $\tilde{\rho} : \tilde{\Gamma} \to \operatorname{Aut}(L)$ operates on the product manifold $\tilde{D} \times V(\mathbb{R})$ properly discontinuously by

$$(\gamma, l) \cdot (y, v) = (\gamma y, \gamma v + l)$$

for $(\gamma, l) \in \tilde{\Gamma} \ltimes_{\rho} L$ and $(y, v) \in \tilde{D} \times V(\mathbb{R})$. We set $\tilde{Y} = \tilde{\Gamma} \ltimes_{\tilde{\rho}} L \backslash \tilde{D} \times V(\mathbb{R})$, and denote by $\tilde{\pi}$ the natural projection of $\tilde{Y}$ onto $\tilde{X} = \tilde{\Gamma} \backslash \tilde{D}$. Then $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is a fiber bundle over $\tilde{X}$, which is in fact the pullback of the standard fiber bundle $\pi_0 : Y_0 \to X_0$ via the map $\tilde{X} \to X_0$ induced by $\tilde{\tau} : \tilde{D} \to \mathcal{H}$. It is known that $\tilde{Y}$ has a structure of a complex projective variety and that the fiber $\tilde{Y}_x$ over each $x \in \tilde{X}$ is an abelian variety polarized by β. Such a fiber variety $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is called a Kuga fiber variety (see [4, 8], [9, Chapter 4]).

5. Hodge cycles

In this section we consider Hodge cycles on generic fibers of Kuga fiber varieties associated to quaternion algebras and prove the main theorem of the paper. Let G be the algebraic group defined over $\mathbb{Q}$ considered in Section 1. Thus G is the algebraic group $\mathrm{Res}_{K/\mathbb{Q}}(SL_1(B))$ where B is a quaternion algebra over a totally real number field with $[k:\mathbb{Q}] = m$. Let $\rho : G \to Sp(V, \beta)$ with $V = \mathbb{C}^8$ be a symplectic representation of G associated to a subset $R = \{\varphi_a, \varphi_b, \varphi_c, \varphi_d\}$ of S as in Section 1, and let Γ be a torsion free arithmetic subgroup of G with $\rho(\Gamma) \subset \Gamma_S$. Let $\tau : D \to \mathscr{H}$ be a holomorphic map such that ρ and τ are equivariant, and let $\phi : X \to X_0$ be the morphism of varieties induced by τ. By pulling back the fiber bundle $\pi_0 : Y_0 \to X_0$ via the morphism $\phi : X \to X_0$, we obtain the Kuga fiber variety $\pi : Y \to X$ over the arithmetic variety X.

We fix a generic point x in X, and identify Γ with the fundamental group $\pi_1(X, x)$ of X at x. We also identify the fiber Y_x of Y over $x \in X$ with V/L, which induces the following further identifications:

$$H_1(Y_x, \mathbb{Q}) = L \otimes \mathbb{Q} = V, \quad H_k(Y_x, \mathbb{Q}) = \wedge^k(V), \qquad H^k(Y, \mathbb{Q}) = \wedge^k(V)^*;$$

here $*$ denotes the dual of the vector space. The action of $\pi_1(X, x)$ on $H^k(Y_x, \mathbb{Q})$ corresponds to the action $\wedge^k(\rho^*)$ of Γ on $\wedge^k(V)^*$; hence we have

$$H^k(Y_x, \mathbb{Q})^{\pi_1(X,x)} = (\wedge^k(V)^*)^{\Gamma}.$$

DEFINITION. Let $\pi : Y \to X$ be a Kuga fiber variety associated to $\rho : G \to Sp(V, \beta)$ and $\tau : D \to \mathscr{H}$, and let $\mathfrak{g}(\mathbb{R})$ be the Lie algebra of $G(\mathbb{R})$. The Kuga fiber variety (Y, π) is of *inner type* if there is a map $r : D \to \mathfrak{g}(\mathbb{R})$ such that

$$\cos(\pi t/2)I + \sin(\pi t/2)\tau(x)I = \rho(\exp(r(x)t))$$

for $x \in D$ and $t \in \mathbb{R}$, where I is the identity map on V.

REMARK 5.1. A Kuga fiber variety that does not allow deformations is said to be *rigid*. Any rigid Kuga fiber variety is of inner type. For example, if $R = \{\varphi_a, \varphi_b, \varphi_c, \varphi_d\} \subset S$ is $\mathrm{Gal}(K/\mathbb{Q})$-invariant and if $|R \cap S_0| = 1$, then the Kuga fiber variety associated to R is rigid and therefore of inner type (see [5]).

We shall denote by $HH^{2k}(Y_x, \mathbb{Q})$ the space of Hodge cycles of codimension k in Y_x, that is,

$$HH^{2k}(Y_x, \mathbb{Q}) = H^{(k,k)}(Y_x) \cap H^{2k}(Y_x, \mathbb{Q}).$$

The Hodge conjecture states that the space $HH^{2k}(Y_x, \mathbb{Q})$ coincides with the space of algebraic cycles of codimension k for $0 \le k \le \dim_{\mathbb{C}} Y_x$.

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PROPOSITION 5.1. *Let Y_x be a generic fiber over $x \in X$ of a Kuga fiber variety $\pi : Y \to X$ of inner type. Then*

$$HH^{2k}(Y_x, \mathbb{Q}) = H^{2k}(Y_x)^{\pi_1(X,x)}$$

for all even integers k with $0 \le k \le \dim_{\mathbb{C}} Y_x$.

PROOF. See [10].

Now we state the main theorem of the paper about the Hodge cycles on Kuga fiber varieties associated to quaternion algebras.

THEOREM 5.2. *Let Y_x be a generic fiber over $x \in X$ of a Kuga fiber variety $\pi : Y \to X$ of inner type associated to the quaternion algebra B in Section 2 and the pair (ρ, τ). Then we have*

$$\begin{aligned}
\dim HH^0(Y_x, \mathbb{Q}) &= \dim HH^{16}(Y_x, \mathbb{Q}) = 1,\\
\dim HH^2(Y_x, \mathbb{Q}) &= \dim HH^{14}(Y_x, \mathbb{Q}) = 0,\\
\dim HH^4(Y_x, \mathbb{Q}) &= \dim HH^{12}(Y_x, \mathbb{Q}) = 3,\\
\dim HH^6(Y_x, \mathbb{Q}) &= \dim HH^{10}(Y_x, \mathbb{Q}) = 0,\\
\dim HH^8(Y_x, \mathbb{Q}) &= 4.
\end{aligned}$$

PROOF. Since Γ is Zariski-dense in G, the action $\wedge^k(\rho^*)$ of Γ in $\wedge^k(V)^*$ can be extended to the action $\wedge^k(\rho^*)$ of $G(\mathbb{C})$ on $\wedge^k(V(\mathbb{C}))^*$. Thus we have

$$\left(\wedge^k(V(\mathbb{C}))^*\right)^{\Gamma} = \left(\wedge^k(V(\mathbb{C}))^*\right)^{G(\mathbb{C})},$$

On the other hand, by the unitary trick, we have

$$\left(\wedge^k(V(\mathbb{C}))^*\right)^{G(\mathbb{C})} = \left(\wedge^k(V(\mathbb{C}))^*\right)^{G_U},$$

where $G_U = (SU_2)^m$ is the compact real form of $G(\mathbb{C}) = SL_2(\mathbb{C})^m$ as in Theorem 1. Hence it follows that

$$\dim HH^{2k}(Y_x, \mathbb{Q}) = \dim_{\mathbb{C}}\left(H^{2k}(Y_x)^{\pi_1(X,x)}\right) = \dim_{\mathbb{C}}\left(\wedge^k(V(\mathbb{C}))^*\right)^{G_U}.$$

Since the symplectic representation ρ is equivalent to its dual ρ^*, we have

$$\dim_{\mathbb{C}}\left(\wedge^k(V(\mathbb{C}))^*\right)^{G_U} = \int_{G_U} \operatorname{tr}(\wedge^{2k}(\rho_U))(g)\, dg,$$

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where dg is the Haar measure of G_U normalized by $\int_{G_U} dg = 1$. On the other hand, the integral

$$\int_{G_U} \operatorname{tr}(\wedge^{2k}(\rho_U))(g)\, dg$$

is equal to the multiplicity $M_{2k} = \left(\wedge^{2k}(\rho_U) : 1\right)$ of the trivial representation 1 in the representation $\wedge^{2k}(\rho_U)$. By Theorem 1 we have

$$M_0 = M_{16} = 1, \quad M_2 = M_{14} = 0, \quad M_4 = M_{12} = 3, \quad M_6 = M_{10} = 0, \quad M_8 = 4;$$

hence the theorem follows.

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