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MORE TRANSIENT RESULTS FOR GENERALISED STATE-DEPENDENT ERLANGIAN QUEUES

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Setting examination questions is often a nuisance, but occasionally it can be a productive exercise. Recently it enabled some gaps in Conolly (1975) to be filled. In that paper single-server queueing systems driven by (λ_n, μ_n) arrival and service mechanisms were discussed. These basically exponential mechanisms fluctuate with system state n. Three models were considered.

Model A, having $\lambda_n = \lambda$, $\mu_n = \mu n$ $(\lambda, \mu > 0)$, embodies the notion of server cooperation and it possesses compound Poisson state probabilities

$$p_n(t) = e^{-\rho\Lambda(t)}\{\rho\Lambda(t)\}^n/n!,$$

where $\rho = \lambda/\mu$, $\Lambda(t) = 1 - e^{-\mu t}$, for all non-negative state sizes n, given that $p_0(0) = 1$. This system mimics $M/M/\infty$.

Model B has $\lambda_n = \lambda/(n+1)$, $\mu_n = \mu$ $(\lambda, \mu > 0)$. This embodies customer reluctance or discouragement. Although the asymptotic form of $p_n(t)$ as $t \to \infty$ is identical with that of Model A, the time-dependent behaviour is less elegant (Natvig (1974), Van Doorn (1981)).

Model C incorporates the mechanisms of A and B. Thus $\lambda_n = \lambda/(n+1)$, $\mu_n = \mu n (\lambda, \mu > 0)$. Service is cooperative and customers reluctant. The asymptotic form of the state probabilities is easily obtained, namely, as $t \to \infty$, $p_n(t) \to \bar{p}_n$, where

$$\bar{p}_n = \frac{\rho^n}{n!\, n!\, I_0(2\rho^{\frac{1}{2}})}.$$

Here and in the following

$$I_n(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{n+2j}}{j!\,(j+n)!} \quad (n \text{ integer})$$

is the modified Bessel function of the first kind with argument x and order n. It satisfies the differential equation

$$x^2y'' + xy' - (x^2 + n^2) = 0.$$

The examination exercises referred to enable it to be stated that $p_n(t)$, with initial

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value $p_0(0)=1$, is generated by $P(x,t)=\sum x^n p_n(t)$, where

(1) $$P(x,t)=\frac{I_0(2(\rho x)^{\frac{1}{2}})}{I_0[2\{\rho(1+(x-1)e^{-\mu t})\}^{\frac{1}{2}}]}.$$

This result is obtained by showing in the usual way that $P(x,t)$ satisfies

$$x\frac{\partial P(x,t)}{\partial t}+\mu x(x-1)\frac{\partial P(x,t)}{\partial x}=\lambda(x-1)G(x,t)$$

where

$$G(x,t)=\int_0^x P(y,t)\,dy,$$

and using Lagrange's method to obtain the first obvious integral $u=(x-1)e^{-\mu t}$ of the characteristics. The other pair of equations can be written as

$$\frac{\partial^2 G(x,t)}{\partial x^2}-\frac{\rho G(x,t)}{x}=0,$$

which, by transformation of the Bessel function differential equation, can be seen to be satisfied by

$$G(x,t)=x^{\frac{1}{2}}I_1(2(\rho x)^{\frac{1}{2}}).$$

This gives for the second integral

$$\frac{P(x,t)}{v}=\frac{d}{dx}[x^{\frac{1}{2}}I_1(2(\rho x)^{\frac{1}{2}})],$$

where v is the integration constant. Since $I_1(x)=I_0'(x)$, and by use of the fundamental differential equation, we can express the general solution $v=F(u)$ in the form

$$P(x,t)=\rho^{\frac{1}{2}}I_0(2(\rho x)^{\frac{1}{2}})F[(x-1)e^{-\mu t}],$$

where F is a function whose form is revealed by the initial condition $P(x,0)=1$. Thus

$$F(x)=[\rho^{\frac{1}{2}}I_0\{2(\rho(x+1))^{\frac{1}{2}}\}]^{-1},$$

and (1) follows.

In particular,

$$p_0(t)=\frac{1}{I_0(2(\rho\Lambda(t))^{\frac{1}{2}})},$$

(2) $$p_1(t)=\frac{\rho}{I_0(2(\rho\Lambda(t))^{\frac{1}{2}})}\left[\Lambda(t)+e^{-\mu t}\frac{I_2(2(\rho\Lambda(t))^{\frac{1}{2}}}{I_0(2(\rho\Lambda(t))^{\frac{1}{2}})}\right],$$

and, for the record, the mean $m(t)$ is

(3) $$m(t)=\frac{\rho^{\frac{1}{2}}I_1(2\rho^{\frac{1}{2}})}{I_0(2\rho^{\frac{1}{2}})}\Lambda(t).$$

Conolly (1975) mentions an even more efficient mechanism than Model C. This is characterised by $\lambda_n=\lambda/(2n+1)$, $\mu_n=2\mu n$ ($\lambda,\mu>0$) and, under the same initial condition $p_0(0)=1$, this can be dealt with similarly. A specimen result is

(4) $$p_0(t)=\frac{1}{\cosh(\rho(1-e^{-2\mu t}))^{\frac{1}{2}}}.$$

No doubt future generations of students will produce others.

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