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Some Elementary Theorems regarding Surds.

By Professor Chrystal.

1. If p and q be both commensurable, and if $p^{\frac{1}{n}} = q^{\frac{1}{r}}$, then, if n be prime to r, both the roots must be commensurable.

For, since n is prime to r, we can find two integers λ and μ such that $\lambda n + \mu r = 1$.

Hence $p = p^{\lambda n + \mu r} = p^{\lambda n} p^{\mu r}$.

Now, by data, $\left(p^{\frac{1}{n}}\right)^{nr} = \left(q^{\frac{1}{r}}\right)^{nr}$, that is, $p^r = q^n$. Hence

$$p = p^{\lambda n} q^{\mu n} = \left(p^{\lambda} q^{\mu}\right)^n.$$

Hence $p^{\frac{1}{n}}$, and therefore also $q^{\frac{1}{r}}$, is commensurable.

2. If $p^{\frac{1}{n}} = q^{\frac{1}{r}}$, where p and q are both commensurable, and $p^{\frac{1}{n}}$ and $q^{\frac{1}{r}}$ both incommensurable, and $r < n$, then p must be of the form $\varpi^{n'}$ where n' is a factor of n, and ϖ is commensurable.

For, by (1), n cannot be prime to r. Hence we must have $n = \lambda n'$, and $r = \lambda r'$, where λ is the G.C.M. of n and r, and n' is prime to r'.

We must therefore have

$$p^{\frac{1}{\lambda n'}} = q^{\frac{1}{\lambda r'}};$$

whence

$$p^{\frac{1}{n'}} = q^{\frac{1}{r'}}.$$

Since n' is prime to r', $p^{\frac{1}{n'}}$ and $q^{\frac{1}{r'}}$ must be both commensurable, each $= \varpi$, say. Hence $p = \varpi^{n'}$.

3. Hence $p^{\frac{1}{n}}$ will be a surd of lowest possible order n, if, and not unless, p be not expressible as an exact n'^{th} power where n' is any factor of n.

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4. If $p^{\frac{1}{n}}$ be a surd of irreducible order n, then $p^{\frac{r}{n}}$, where $r<n$, is also a surd.

For, if $p^{\frac{r}{n}}$ were commensurable $=q$ say, then we should have $p^{\frac{1}{n}}=q^{\frac{1}{r}}$, where $r<n$. It would then follow by (2) and (3) that $p^{\frac{1}{n}}$ can be expressed as a surd of lower order than n.

5. If p be commensurable, then the necessary and sufficient condition for the irreducibility of x^n-p in the domain of real rational quantity is that $p^{\frac{1}{n}}$ be a surd of irreducible order n.

The condition is necessary; for, if $p^{\frac{1}{n}}$ can be expressed as a surd of lower order, then we must have $p=\varpi^{n'}$, where ϖ is commensurable, and n' is a factor of n. We should then have

$$x^n-p\equiv x^{\lambda n'}-\varpi^{n'}\equiv(x^{\lambda})^{n'}-\varpi^{n'}$$
$$\equiv(x^{\lambda}-\varpi)(x^{\lambda(n'-1)}+\varpi x^{\lambda(n'-2)}+\ldots\ldots+\varpi^{n'-1});$$

that is, x^n-p is reducible.

Also the condition is sufficient, for let us suppose that x^n-p is reducible. Let $p^{\frac{1}{n}}$ denote as usual the principal value of the n^{th} root of p; and let the n^{th} roots of unity be $1, \omega, \omega^2, \ldots, \omega^{n-1}$, so that the linear factors of x^n-p are $x-p^{\frac{1}{n}}, x-\omega p^{\frac{1}{n}}, \ldots, x-\omega^{n-1}p^{\frac{1}{n}}$. Since x^n-p is reducible, it must be possible to select a group of these factors whose product, say,

$$(x-\omega^{a_1}p^{\frac{1}{n}})(x-\omega^{a_2}p^{\frac{1}{n}})\ldots\ldots(x-\omega^{a_r}p^{\frac{1}{n}}),\ (r<n),$$

is rational. Hence, in particular, the absolute term of this product,

viz., $$(-1)^r\omega^{a_1+a_2+\ldots+a_r}p^{\frac{r}{n}}$$

must be real and commensurable. Since $\omega^{a_1+a_2+\ldots+a_r}$ must be real, its value must be either $+1$ or -1, and it is necessary that $p^{\frac{r}{n}}$ be commensurable $=q$, say. It follows that $p^{\frac{1}{n}}=q^{\frac{1}{r}}$ where $r<n$; that is, $p^{\frac{1}{n}}$ can be expressed as a surd of order lower than n.

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6. If $p^{\frac{1}{n}}$ be a surd of irreducible order n, then a relation of the form

$$a_{n-1}p^{\frac{n-1}{n}} + a_{n-2}p^{\frac{n-2}{n}} + \ldots\ldots + a_1 p^{\frac{1}{n}} + a_0 = 0 \quad - \quad - \quad (1),$$

where $a_0, a_1, \ldots, a_{n-1}$ are commensurable, and do not all vanish, is impossible.

Let $x = p^{1/n}$, then $x^n = p$; and we must have simultaneously

$$x^n - p = 0 \quad - \quad - \quad - \quad - \quad - \quad (2),$$

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 = 0 \quad - \quad - \quad (3).$$

Since the two equations (2) and (3) must have a root in common, their characteristic functions

$$x^n - p \text{ and } a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

must have a common factor, which, being determinable by purely rational operations, must have commensurable coefficients, since p, $a_0, a_1, \ldots, a_{n-1}$ are all commensurable. But this is impossible, for, by (5), $x^n - p$ is irreducible.

An exceedingly interesting proof of a particular case of this theorem, not involving the use of the imaginary roots of unity has recently been given to the Society by Mr D. B. Mair. I have several times tried without success to obtain a complete demonstration in the same manner.

(7) Two surds are said to be *similar* when their quotient is commensurable.

Two surds of unequal irreducible orders are necessarily dissimilar. For, if $p^{\frac{1}{r}}$ and $q^{\frac{1}{s}}$ $(r > s)$ were similar, we should have $p^{\frac{1}{r}} = tq^{\frac{1}{s}}$, where t is commensurable. Hence we could express $p^{\frac{1}{r}}$ in the form $(t^s q)^{\frac{1}{s}}$, that is, the order of $p^{\frac{1}{r}}$ is not irreducible as supposed.

8. The following theorem is an example of the consequences that follow from (6).

A root of any commensurable radicand cannot be the sum of a commensurable quantity and a surd.

If the root is commensurable, the theorem is at once obvious. If not, let the root be expressed as a surd of irreducible order r, say $p^{\frac{1}{r}}$; and let us suppose that

$$p^{\frac{1}{r}} = t + q^{\frac{1}{s}} \quad - \quad - \quad - \quad - \quad - \quad (1),$$

where $q^{\frac{1}{s}}$ is a surd of irreducible order s, and t is commensurable.

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First suppose that $r<s$. Then from (1) we derive

$$p=\left(t+q^{\frac{1}{s}}\right)^r$$
$$=t^r+{}_rC_1t^{r-1}q^{\frac{1}{s}}+\ldots+q^{\frac{r}{s}} \qquad (2).$$

Now $p \neq t^r$, and none of the coefficients ${}_rC_1t^{r-1}$, ${}_rC_2t^{r-2}, \ldots$ can vanish. But a relation of the form (2) is impossible by (6).

If $r=s$, a slight modification of the same proof will apply.

If $r>s$, we may consider the relation

$$q^{\frac{1}{s}}=-t+p^{\frac{1}{r}};$$

the impossibility of which may be proved as before.

9. A root of a commensurable radicand cannot be the sum of two dissimilar surds.

For, if possible, let

$$p^{\frac{1}{r}}=q^{\frac{1}{s}}+t^{\frac{1}{u}},$$

where $q^{\frac{1}{s}}$ and $t^{\frac{1}{u}}$ are dissimilar surds of irreducible orders s and u. Then we must have

$$p^{\frac{1}{r}}/q^{\frac{1}{s}}=1+t^{\frac{1}{u}}/q^{\frac{1}{s}}.$$

Now $p^{\frac{1}{r}}/q^{\frac{1}{s}}$ can be expressed as the root of a commensurable radicand, say in the form $(p^s/q^r)^{\frac{1}{rs}}$. Also, since $t^{\frac{1}{u}}$ and $q^{\frac{1}{s}}$ are dissimilar, their quotient is a surd, say the surd $v^{\frac{1}{w}}$ of irreducible order w. We should then have

$$(p^s/q^r)^{\frac{1}{rs}}=1+v^{\frac{1}{w}},$$

which is impossible by (9).

It is curious that it should be so easy to prove the impossibility of the relation $x^{\frac{1}{m}}=y^{\frac{1}{m}}+z^{\frac{1}{m}}$ (where it is impossible); and so difficult to establish the like for $x^m=y^m+z^m$, where x, y, z, m are integers, and $m>2$.

Many other applications and connected problems at once suggest themselves; but the treatment of most of them soon leaves purely elementary lines. The whole theory is, of course, a special, but peculiarly interesting, part of the theory of An Algebraic Close (Algebraischer Zahlkörper), an elegant presentation of which will be found in Weber's *Lehrbuch der Algebra.*