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RESIDUALS OF THE JOIN OF ASCENDANT SUBGROUPS

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1. Introduction. 1.1. If G is a group, then we say H is an *ascendant subgroup* of G, and write H asc G, if there exists a sequence of subgroups $(G_\alpha)_{\alpha \leqq \rho}$ where ρ is some ordinal number, such that $G_0 = H$, $G_\rho = G$, $G_\alpha \lhd G_{\alpha+1}$ for all $\alpha < \rho$ and $G_\lambda = \bigcup \{G_\alpha | \alpha < \lambda\}$ for all limit ordinals $\lambda \leqq \rho$. $(G_\alpha)_{\alpha \leqq \rho}$ is said to be an *ascending series* from H to G. If $\rho < \omega$ where ω denotes the least infinite ordinal, then H is a subnormal subgroup of G and we write H sn G; if the index of subnormality is at most n, then we write $H \lhd^n G$.

Let $\mathfrak{X}$ be a class of groups. Then $\mathfrak{X}$ is called an *ascendant* (*subnormal*) *coalition class* if whenever H and K are ascendant (subnormal) $\mathfrak{X}$-subgroups of a group G then $J = \langle H, K \rangle$ is also an ascendant (subnormal) $\mathfrak{X}$-subgroup of G. The class of groups satisfying the minimal condition on subnormal subgroups is shown by Robinson [**2**] to be an ascendant coalition class. Denoting this class by Min-*sn*, the class of soluble groups by $\mathfrak{S}$, then a further result in [**2**] shows that $\mathfrak{S} \cap$ Min-*sn* is an ascendant coalition class.

1.2. *Main Results.* For a class of groups $\mathfrak{X}$ we denote by $G^{\mathfrak{X}}$ the $\mathfrak{X}$-residual of a group G, i.e. the intersection of all normal subgroups of G whose factor groups are $\mathfrak{X}$-groups. Here we generalize Robinson's result [**2**] and state:

THEOREM A. *Let* $G = \langle H, K \rangle$ *where* $H, K \in$ Min-*sn and* H, K asc G. *Then* $G/G^{\mathfrak{S}} \in \mathfrak{S}$ *and* $G^{\mathfrak{S}} = \langle H^{\mathfrak{S}}, K^{\mathfrak{S}} \rangle = H^{\mathfrak{S}} K^{\mathfrak{S}}$.

Two subgroups H and K of a group G are said to *permute* if $HK = KH$. An immediate corollary is then:

COROLLARY. *Let H be a perfect ascendant subgroup of a group G, and let K be an ascendant subgroup of G. Then if $H, K \in$ Min-sn, H permutes with K.*

We note that Wielandt has proved the result stated as Theorem A for groups with a composition series [**4**]. Our next result is similar. Let $L\mathfrak{N}$ denote the class of locally nilpotent groups; then:

THEOREM B. *Let* $G = \langle H, K \rangle$ *where* $H, K \in$ Min-*sn and* H, K asc G. *Then* $G/G^{L\mathfrak{N}} \in L\mathfrak{N}$ *and* $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle = H^{L\mathfrak{N}} K^{L\mathfrak{N}}$.

The equivalent result for nilpotent residuals is not true, since the join of two ascendant nilpotent subgroups which satisfy Min-*sn* need not be nilpotent. Let $G = \langle t, a; a^t = a^{-1}, t^2 = 1, a \in A \rangle$ where $A \cong C_{2^\infty}$. Denoting the class of nilpotent groups by $\mathfrak{N}$, then $T = \langle t \rangle$ and A are ascendant in G and belong to Min-*sn* $\cap\, \mathfrak{N}$, but G is not nilpotent.

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1.3. *Notation.* If $\mathfrak{X}$ is a class of groups then $S\mathfrak{X}$ is the class of subgroups of $\mathfrak{X}$-groups, $Q\mathfrak{X}$ is the class of epimorphic images of $\mathfrak{X}$-groups, $R\mathfrak{X}$ is the class of subdirect products of $\mathfrak{X}$-groups; and $L\mathfrak{X}$ is the class of locally $\mathfrak{X}$-groups. $\mathfrak{X} = R_0\mathfrak{X}$ if, whenever G/N_1 and $G/N_2 \in \mathfrak{X}$ where $N_1, N_2 \lhd G$, then $G/(N_1 \cap N_2) \in \mathfrak{X}$. $\mathfrak{F}$ denotes the class of finite groups.

If G is a group and $H \leqq G$ we denote the normalizer of H in G by $N_G(H)$, the largest normal subgroup of G contained in H by $\mathrm{Core}_G(H)$, and the smallest normal subgroup of G containing H by H^G.

2. Proofs of Theorems A and B.

2.1. *Preliminary Lemmas.* We first examine the $L\mathfrak{X}$-residuals of locally finite groups.

LEMMA 2.1. *Let $G \in L\mathfrak{F}$ and let $\mathfrak{X} = \langle S, R_0, Q\rangle\mathfrak{X} \leqq \mathfrak{F}$. Then $G/G^{L\mathfrak{X}} \in L\mathfrak{X}$ and $G^{L\mathfrak{X}} = \langle F^{\mathfrak{X}} | F$ is a finite subgroup of $G\rangle$.*

Proof. If F is a finite subgroup of $G/G^{L\mathfrak{X}}$ there is a normal subgroup intersecting F in 1 with quotient group in $L\mathfrak{X}$. Hence $F \in L\mathfrak{X}$ and by S-closure, $F \in \mathfrak{X}$ and $G/G^{L\mathfrak{X}} \in L\mathfrak{X}$.

Let $R = \langle F^{\mathfrak{X}} | F$ is a finite subgroup of $G\rangle$; then $R \lhd G$. Also, $F^{\mathfrak{X}} \leqq G^{L\mathfrak{X}}$ for all finite subgroups of G, whence $R \leqq G^{L\mathfrak{X}}$. However, if K/R is any finite subgroup of G/R there exists a finite subgroup F of G such that $FR/R = K/R$. From $FR/R \cong F/(F \cap R) \in Q\mathfrak{X} = \mathfrak{X}$ we obtain $K/R \in \mathfrak{X}$ and $G/R \in L\mathfrak{X}$. Hence $G^{L\mathfrak{X}} \leqq R$ and we have equality.

A subgroup H of a group G is termed *serial* if there is a series between H and G (see [**3**, p. 9]). We prove a further technical result involving $L\mathfrak{X}$-residuals.

LEMMA 2.2. *Let $G \in L\mathfrak{F}$ be generated by two serial subgroups A and B of G. Let $\mathfrak{X} = \langle R_0, S, Q\rangle\mathfrak{X}$ and suppose that for any two finite subgroups X and Y which are subnormally embedded in their join that $\langle X, Y\rangle^{\mathfrak{X}} = \langle X^{\mathfrak{X}}, Y^{\mathfrak{X}}\rangle = X^{\mathfrak{X}}Y^{\mathfrak{X}}$. Then $G^{L\mathfrak{X}} = \langle A^{L\mathfrak{X}}, B^{L\mathfrak{X}}\rangle = A^{L\mathfrak{X}}B^{L\mathfrak{X}}$.*

Proof. Let $N = G^{L\mathfrak{X}}$ and $M = A^{L\mathfrak{X}}B^{L\mathfrak{X}}$. We show that $M = N$. Obviously $M \leqq N$. Let F be a finite subgroup of G. Then there exists a finite subgroup F_1 such that $F \leqq F_1 = \langle F_1 \cap A, F_1 \cap B\rangle$. By hypothesis $A \cap F_1, B \cap F_1$ sn F_1, whence we may conclude that $F^{\mathfrak{X}} \leqq F_1^{\mathfrak{X}} = (F_1 \cap A)^{\mathfrak{X}}(F_1 \cap B)^{\mathfrak{X}} \leqq M$. By Lemma 2.1 we have $N \leqq M$ and hence equality.

We now consider groups satisfying the minimal condition on subnormal subgroups. For $G \in$ Min-*sn* let $F(G)$ be the smallest subgroup of finite index in G, and let $E(G) = F(G)'$. Then $E(G)$ is the smallest normal subgroup of G with Černikov factor group.

The following is proved by Hartley and Peng in [**1**]:

LEMMA 2.3. *Let $H, K \in$ Min-sn and suppose that H and K are ascendant subgroups of a group G. Then $E(H) \leqq N_G(K)$.*

We examine $E(G)$ in the following case:

LEMMA 2.4. *Let* $G = \langle H, K\rangle$ *where* H, K asc G *and* $H, K \in$ Min-*sn*. *Then* $G \in$ Min-*sn and* $E(G) = E(H)E(K)$.

Proof. The class Min-*sn* forms an ascendant coalition class (see [**2**]) and so $G \in$ Min-*sn*. Let $X = \langle E(H), E(K)\rangle$; then $X \leqq E(G)$. Clearly G/X^G is a Černikov group and so $X^G = E(G)$. Now $E(H)$ and $E(K)$ have no proper subgroups of finite index, so by results in [**2**], $X = E(H)E(K)$ and $X \lhd E(G)$. By 2.3 we have $[E(H), K] \leqq K$, whence $L = [E(H), K]X/X$ is a Černikov group; similarly $M = [E(K), H]X/X$ is a Černikov group. Now L and M are ascendant in $E(G)/X$ since $\langle E(H), K\rangle$ and $\langle H, E(K)\rangle$ are ascendant in G. Černikov groups form an ascendant coalition class by [**2**]; consequently $\langle L, M\rangle = X^G/X = E(G)/X$ is a Černikov group, which shows that $E(G) = X$.

2.2. *Proof of Theorem A*. Since Min-*sn* is an ascendant coalition class, $G \in$ Min-*sn* and $G/G^{\mathfrak{S}} \in \mathfrak{S}$.

Let $A = G^{\mathfrak{S}}$ and $B = H^{\mathfrak{S}} K^{\mathfrak{S}}$. Then $B \leqq A$. Now $G/E(G)$ is Černikov and hence is locally finite and $(G/E(G))^{L\mathfrak{S}} = (G/E(G))^{\mathfrak{S}}$. Since $E(G)$ is perfect, $E(G) \leqq G^{\mathfrak{S}}$. Therefore $(G/E(G))^{\mathfrak{S}} = G^{\mathfrak{S}}/E(G)$.

By Wielandt's results in [**4**] the Theorem is true in the finite case. Hence we may apply 2.2 to the group $G/E(G)$ to obtain $A = BE(G)$. Lemma 2.4 then gives $A = B$.

2.3 *Proof of Theorem B*. As in Theorem A, $G \in$ Min-*sn* and $G/G^{L\mathfrak{N}} \in L\mathfrak{N}$. Locally nilpotent groups satisfying Min-*sn* are Černikov and soluble (see [**3**, p. 154]). Hence by Theorem A, $G^{\mathfrak{S}} = H^{\mathfrak{S}} K^{\mathfrak{S}} \trianglelefteq H^{L\mathfrak{N}} K^{L\mathfrak{N}}$, and without loss of generality we may assume $G^{\mathfrak{S}} = 1$. Then G is soluble and locally finite. Again by [**4**] the result holds in the finite case. Applying 2.2, we have $G^{L\mathfrak{N}} = H^{L\mathfrak{N}} K^{L\mathfrak{N}}$.

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REFERENCES

1. B. Hartley and T. A. Peng, *Subnormality, ascendancy and the minimal condition*, J. Algebra *41* (1976), 58–78.

2. D. J. S. Robinson, *On the theory of subnormal subgroups*, Math. Z. *89* (1965), 30–51.

3. ——— *Finiteness conditions and generalized soluble groups, Part 1* (Springer, Berlin/Heidelberg/New York, 1972).

4. H. Wielandt, *Vertauschbare nachinvariante Untergruppen*, Abh. Math. Sem. Univ. Hamburg *21* (1957), 55–62.

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