ON PÓLYA'S THEOREM

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1. Introduction. In 1927 J. H. Redfield **(9)** stressed the intimate interrelationship between the theory of finite groups and combinatorial analysis. With this in mind we consider Pólya's theorem **(7)** and the Redfield–Read superposition theorem **(8, 9)** in the context of the theory of permutation representations of finite groups. We show in particular how the Redfield–Read superposition theorem can be deduced as a special case from a simple extension of Pólya's theorem. We give also a generalization of the superposition theorem expressed as the multiple scalar product of certain group characters. In a later paper we shall give some applications of this generalization.

2. An extension of Pólya's theorem. Suppose that $D_1, D_2, \ldots, D_q$ is a partition of a set D into q subsets and D_i consists of the α_i elements $d_{i1}, d_{i2}, \ldots, d_{i\alpha_i}$, where

$$\sum_{i=1}^{q} \alpha_i = m.$$

Let $R_1, R_2, \ldots, R_q$ be a partition of a set R into q subsets. Elements of R are called *figures* and R_i is called the ith *figure range*. Let F be the set of functions of D into R with the restriction

$$f(D_i) \subseteq R_i, \qquad i = 1, 2, \ldots, q.$$

An element of F is called an $\mathfrak{F}$-*configuration*. Let $\mathfrak{G}$ be a permutation group of degree m and order g which permutes the elements of D and suppose the transitive constituents of $\mathfrak{G}$ are the q subsets $D_1, D_2, \ldots, D_q$. $\mathfrak{F}$-configurations f_1, f_2 are $\mathfrak{G}$-equivalent if there is a $\sigma \in \mathfrak{G}$ such that

$$f_1(d_{ij}) = f_2(d_{ij}\sigma), \qquad j = 1, 2, \ldots, \alpha_i;\ i = 1, 2, \ldots, q.$$

To each figure there is assigned a unique non-negative integer m called its *content*. (In the general case each figure of the ith figure range is assigned an ordered set of ω_i non-negative integers $(k_{i1}, k_{i2}, \ldots, k_{i\omega_i})$. However, without loss of generality and for simplicity, we consider the case when $\omega_i = 1$, $i = 1, 2, \ldots, q$.) If ϕ_{k_i} is the number of figures of content k_i belonging to R_i, then the polynomial

$$p(x_i) = \sum_{k_i=0}^{\infty} \phi_{k_i} x_i^{k_i}$$

is called the *figure counting series* of R_i. We now need to define the content of an $\mathfrak{F}$-configuration.

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Suppose $f \in F$ and f is defined by

$$f\colon d_{ij} \to f(d_{ij}) = \lambda_{ij} \in R_i, \qquad j = 1, 2, \ldots, \alpha_i;\ i = 1, 2, \ldots, q;$$

then, if λ_{ij} has content k_{ij}, the ordered set (k) of q non-negative integers

$$(1) \qquad (k) = \left(\sum_{j=1}^{\alpha_1} k_{1j}, \sum_{j=2}^{\alpha_2} k_{2j}, \ldots, \sum_{j=1}^{\alpha_q} k_{qj}\right)$$

is the *content* of f. Letting

$$k_i = \sum_{j=1}^{\alpha_i} k_{ij}, \qquad i = 1, 2, \ldots, q,$$

(1) may be written

$$(2) \qquad (k) = (k_1, k_2, \ldots, k_q).$$

Let $A_{(k)}$ be the number of $\mathfrak{G}$-inequivalent $\mathfrak{J}$-configurations of content (k). Then

$$(3) \qquad P(x_1, x_2, \ldots, x_q) = \sum\nolimits_{(k)} A_{(k)}\, x_1^{k_1} x_2^{k_2} \ldots x_q^{k_q}$$

is called the *$\mathfrak{J}$-configuration counting series.*

The object of this extension of Pólya's theorem is to express $P(x_1, x_2, \ldots, x_q)$ in terms of $\mathfrak{G}$ and $p(x_i)$, $i = 1, 2, \ldots, q$. This is accomplished using the $\mathfrak{J}$-cycle index $Z^I(\mathfrak{G})$ of $\mathfrak{G}$ defined below. This form of the cycle index of $\mathfrak{G}$ is mentioned in **(7)** but appears to have been overlooked.

A permutation $\sigma \in \mathfrak{G}$ is of $\mathfrak{J}$-type

$$(t) = (\{t_{11}, t_{12}, \ldots, t_{1\alpha_1}\};\ \{t_{21}, t_{22}, \ldots, t_{2\alpha_2}\}; \ldots;\ \{t_{q1}, t_{q2}, \ldots, t_{q\alpha_q}\})$$

if it contains t_{ij} disjoint cycles of length j of the elements of D_i. When no confusion arises, σ is simply said to be of $\mathfrak{J}$-type (t). The non-negative integers t_{ij} must satisfy the following equations:

$$(4) \qquad 1t_{i1} + 2t_{i2} + \ldots + \alpha_i\, t_{i\alpha_i} = \alpha_i, \qquad i = 1, 2, \ldots, q.$$

The number of permutations of $\mathfrak{J}$-type (t) belonging to $\mathfrak{G}$ is denoted by $h_{(t)}$. Let g_{ij} $(j = 1, 2, \ldots, \alpha_i; i = 1, 2, \ldots, q)$ be indeterminates. The $\mathfrak{J}$-cycle index $Z^I(\mathfrak{G})$ of $\mathfrak{G}$ is defined by

$$Z^I(\mathfrak{G}) = \frac{1}{g} \sum_{(t)} h_{(t)} g_{11}^{t_{11}} \ldots g_{1\alpha_1}^{t_{1\alpha_1}} g_{21}^{t_{21}} \ldots g_{2\alpha_2}^{t_{2\alpha_2}} \ldots g_{q1}^{t_{q1}} \ldots g_{q\alpha_q}^{t_{q\alpha_q}},$$

where the summation is over permutations of all $\mathfrak{J}$-types (t) belonging to $\mathfrak{G}$. We note here that when $\mathfrak{G}$ is transitive $Z^I(\mathfrak{G})$ is exactly the same **(4)** as the usual form of the cycle index $Z(\mathfrak{G})$ of $\mathfrak{G}$, and this will be assumed below.

Example. Suppose that $\mathfrak{G}$ is the permutation group which permutes the symbols a, b, c, d, e, f, and consists of:

$(a)(b)(c)(d)(e)(f)$; $(ab)(c)(d)(e)(f)$; $(a)(b)(cd)(e)(f)$; $(ab)(cd)(e)(f)$; $(ab)(c)(d)(ef)$; $(a)(b)(cd)(ef)$; $(ab)(cd)(ef)$; $(a)(b)(c)(d)(ef)$.

Then

$$Z^I(\mathfrak{G}) = \tfrac{1}{8}(g_{11}^2\, g_{21}^2\, g_{31}^2 + g_{12}\, g_{21}^2\, g_{31}^2 + g_{11}^2\, g_{22}\, g_{31}^2 + g_{11}^2\, g_{21}^2\, g_{32} + g_{12}\, g_{22}\, g_{31}^2 + g_{12}\, g_{21}^2\, g_{32} + g_{11}^2\, g_{22}\, g_{32} + g_{12}\, g_{22}\, g_{32}),$$

whereas

$$Z(\mathfrak{G}) = \tfrac{1}{8}(f_1^6 + 3f_1^4 f_2 + 3f_1^2 f_2^2 + f_2^3),$$

where f_1, f_2 are indeterminates.

Finally, for any set of power series $f(x_i)$, $i = 1, 2, \ldots, q$, let

$$Z^I[\mathfrak{G}, \{f(x_1), f(x_2), \ldots, f(x_q)\}]$$

denote the polynomial obtained from $Z^I(\mathfrak{G})$ by writing

$$g_{ij} = f(x_i^j), \qquad j = 1, 2, \ldots, \alpha_i;\ i = 1, 2, \ldots, q.$$

We are now able to state the theorem. However, the proof will be omitted since it is almost exactly the same as the proof of the *Hauptsatz* itself.

THEOREM 1 (Pólya's theorem, an extension). *The $\mathfrak{J}$-configuration counting series $P(x_1, x_2, \ldots, x_q)$ is obtained by substituting the set of figure counting series $p(x_i)$, $i = 1, 2, \ldots, q$, into the $\mathfrak{J}$-cycle index $Z^I(\mathfrak{G})$ of $\mathfrak{G}$. Symbolically,*

$$P(x_1, x_2, \ldots, x_q) = Z^I[\mathfrak{G}, \{p(x_1), p(x_2), \ldots, p(x_q)\}].$$

3. Representation theory.

Definitions. We must now examine certain properties of permutation representations of abstract finite groups. A group G of permutations is called a *permutation representation* of an abstract group P if there is a mapping μ of P onto G, $\sigma \rightarrow \mu(\sigma)$, $\sigma \in P$, $\mu(\sigma) \in G$ such that

$$\mu(\sigma_1)\mu(\sigma_2) = \mu(\sigma_1\sigma_2), \qquad \text{for all } \sigma_1, \sigma_2 \in P.$$

If the groups P and G are isomorphic, the permutation representation G of P is said to be *faithful*. Suppose P is of order p. The *characteristic* $\mathrm{X}(\sigma)$ of $\sigma \in P$ in G is the number of cycles of $\mu(\sigma)$ of length one. The set of p characteristics $\mathrm{X}(\sigma)$, $\sigma \in P$, is called the *character* of G and is denoted by X. Now if X and X' are the characters of permutation representations G and G' of P respectively, then

$$(\mathrm{X}, \mathrm{X}') = \frac{1}{p}\sum_{\sigma \in P} \mathrm{X}(\sigma)\mathrm{X}'(\sigma^{-1})$$

is called the *scalar product* of X and X'. The *multiple scalar product* is as defined in **(3)**. Let I be the character which has value unity for all $\sigma \in P$.

Now suppose G above is a permutation representation of P. Suppose G is of degree α and permutes the symbols $a_1, a_2, \ldots, a_\alpha$. The permutations induced on the homogeneous products, of degree q, of $a_1, a_2, \ldots, a_\alpha$ by G give a permutation representation of P **(7**, p. 300**)** called the *symmetrized Kronecker product representation* of dimension q, of P, denoted by $(G)^q$. X^q

denotes the character of $(G)^q$. Furthermore, we suppose, for simplicity of notation, that G is transitive. Let

$$(5)\qquad Z^I(G) = \frac{1}{p}\sum_{(j)} C_{j_1 j_2 \ldots j_\alpha} t_1^{j_1} t_2^{j_2} \ldots t_\alpha^{j_\alpha},$$

where the summation is over all $\mathfrak{J}$-types (j) belonging to G. $Z^I[G, \mathrm{X}(\sigma)]$, $\sigma \in P$, is the polynomial obtained from $Z^I(G)$ by writing

$$t_\lambda = \mathrm{X}(\sigma^\lambda), \qquad \lambda = 1, 2, \ldots, k.$$

Finally the group of order $n!$, which consists of all possible permutations on n elements, is called the *symmetric group of degree n* and is denoted by $\mathfrak{S}_n$. We now state as lemmas two well-known results. For Lemmas 1 and 2, see **(5)** and **(10**, p. 68**)** respectively.

Lemma 1. *If $\mu(\sigma)$, $\sigma \in P$ is of $\mathfrak{J}$-type $(j_1, j_2, \ldots, j_\alpha)$, then*

$$\mathrm{X}(\sigma^k) = \sum\nolimits_{t|k} t j_t,$$

where $\sum_{t|k}$ denotes that the summation is over all t that divide k (including $t = 1$) and σ^k denotes the kth power $(k \geqslant 1)$ of σ.

Lemma 2.

$$\mathfrak{L}(y^k) \exp\left(\frac{y}{1} t_1 + \frac{y^2}{2} t_2 + \ldots + \frac{y^p}{p} t_p + \ldots\right) = Z^I(\mathfrak{S}_k),$$

where $\mathfrak{L}(\ldots)$ denotes "the coefficient of . . . in."

Lemma 3.

$$\mathfrak{L}(y^k) Z^I[G, (1-y)^{-1}] = \frac{1}{p}\sum_{\sigma \in P} Z[\mathfrak{S}_k, \mathrm{X}(\sigma)] = \frac{1}{p}\sum_{\sigma \in P} \mathrm{X}^k(\sigma).$$

Proof.

$$\begin{aligned}
(6)\quad Z^I[G, (1-y)^{-1}] &= \frac{1}{p}\sum_{(j)} C_{j_1 j_2 \ldots j_\alpha} (1-y)^{-j_1} (1-y^2)^{-j_2} \ldots (1-y^\alpha)^{-j_\alpha} \\
&= \frac{1}{p}\sum_{(j)} C_{j_1 j_2 \ldots j_\alpha} \exp[-j_1 \log(1-y) - j_2 \log(1-y^2) \\
&\qquad - \ldots - j_\alpha \log(1-y^\alpha)] \\
&= \frac{1}{p}\sum_{(j)} C_{j_1 j_2 \ldots j_\alpha} \exp\left[j_1\left(y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\right)\right. \\
&\qquad + j_2\left(\frac{y^2}{1} + \frac{y^4}{2} + \frac{y^6}{3} + \ldots\right) + \ldots \\
&\qquad \left. + j_\alpha\left(\frac{y^{1.\alpha}}{1} + \frac{y^{2.\alpha}}{2} + \frac{y^{3.\alpha}}{3} + \ldots\right)\right] \\
&= \frac{1}{p}\sum_{(j)} C_{j_1 j_2 \ldots j_\alpha} \exp\left[j_1 y + (j_1 + 2j_2)\frac{y^2}{2} + \ldots\right. \\
&\qquad \left. + \left(\sum_{t|k} t j_t\right)\frac{y^k}{k} + \ldots\right].
\end{aligned}$$

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On regrouping under the summation and using Lemma 1, equation (6) becomes

$$(7)\quad Z^I[G, (1-y)^{-1}] = \frac{1}{p}\sum_{\sigma\in P} \exp\left[\mathrm{X}(\sigma)y + \mathrm{X}(\sigma^2)\frac{y^2}{2} + \ldots + \mathrm{X}(\sigma^k)\frac{y^k}{k} + \ldots\right].$$

Therefore, from Lemma 2 and equation (7),

$$(8)\qquad \mathfrak{L}(y^k)Z^I[G, (1-y)^{-1}] = \frac{1}{p}\sum_{\sigma\in P} Z^I[\mathfrak{S}_k, \mathrm{X}(\sigma)].$$

Finally, from (**6**, p. 300),

$$(9)\qquad \frac{1}{p}\sum_{\sigma\in P} Z^I[\mathfrak{S}_k, \mathrm{X}(\sigma)] = \frac{1}{p}\sum_{\sigma\in P} \mathrm{X}^k(\sigma).$$

This completes the proof of the lemma.

4. Generalization of the Redfield-Read superposition theorem. We now return to the discussion of § 2. Let P_H denote the transitive permutation representation of P (**2**, p. 233) induced by a subgroup H of P. Suppose $\mathfrak{G}$ is a permutation representation of P defined by a homomorphism π from P onto $\mathfrak{G}$. Let π_i be the homomorphism from P onto the permutation group $\mathfrak{G}_i$ (say), where $\pi_i(\sigma)$, $\sigma \in P$, is obtained by considering $\pi(\sigma)$ simply as a permutation on the elements of D_i. Thus $\mathfrak{G}_i$ permutes the elements of D_i. Clearly $\mathfrak{G}_i$ is a transitive permutation representation of P of degree α_i and is isomorphic as a permutation group (**2**, p, 236) to P_{H_i} for some subgroup H_i of P of index α_i. Suppose P_{H_i} has character X_i. In particular therefore

$$(10)\qquad Z(\mathfrak{G}_i) = Z^I(P_{H_i}), \qquad i = 1, 2, \ldots, q.$$

For all $\sigma \in P$ the monomial associated with $\pi_i(\sigma)$ in $Z^I(\mathfrak{G}_i)$ will be denoted by $z_i(\sigma)$. Then the notation $z_i[\sigma, p(x_i)]$ follows naturally from § 2.

LEMMA 4.

$$Z^I(\mathfrak{G}) = \frac{1}{p}\sum_{\sigma\in P}\prod_{i=1}^{q} z_i(\sigma).$$

Proof. This follows immediately from the above if $\mathfrak{G}$ is a faithful representation of P. If $\mathfrak{G}$ is not faithful, it is perhaps worth noting that, if $|\pi|$ is the order of the kernel of π, then

$$(11)\qquad Z^I(\mathfrak{G}) = \frac{|\pi|}{p}\sum_{\sigma\in P}\frac{1}{|\pi|}\prod_{i=1}^{q} z_i(\sigma)$$

$$(12)\qquad = \frac{1}{p}\sum_{\sigma\in P}\prod_{i=1}^{q} z_i(\sigma).$$

The equality of (11) and (12) has been tacitly assumed in equations (5) and (6).

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THEOREM 2.

$$\mathfrak{L}(y_1^{\theta_1} y_2^{\theta_2} \ldots y_q^{\theta_q}) Z^I[\mathfrak{G}, (1 - y_1)^{-1}, (1 - y_2)^{-1}, \ldots, (1 - y_q)^{-1}] = (\mathrm{X}_1^{\theta_1}, \mathrm{X}_2^{\theta_2}, \ldots, \mathrm{X}_q^{\theta_q})$$

(*we suppose that* $(\mathrm{X}_\omega^{\theta_\omega}) = (\mathrm{X}_\omega^{\theta_\omega}, I)$, $1 \leqslant \omega \leqslant q$).

Proof. Using Lemma 4,

$$Z^I[\mathfrak{G}, (1 - y_1)^{-1}, (1 - y_2)^{-1}, \ldots, (1 - y_q)^{-1}] = \frac{1}{p} \sum_{\sigma \in P} \prod_{i=1}^{q} z_i[\sigma, (1 - y_i)^{-1}].$$

Therefore, by an extension of Lemma 3,

$$\mathfrak{L}(y_1^{\theta_1} y_2^{\theta_2} \ldots y_q^{\theta_q}) Z^I[\mathfrak{G}, (1 - y_1)^{-1}, (1 - y_2)^{-1}, \ldots, (1 - y_q)^{-1}] = \frac{1}{p} \sum_{\sigma \in P} \prod_{i=1}^{q} \mathrm{X}_i^{\theta_i}(\sigma) = (\mathrm{X}_1^{\theta_1}, \mathrm{X}_2^{\theta_2}, \ldots, \mathrm{X}_q^{\theta_q}).$$

This completes the proof of the theorem.

5. Theorem 2 and the theory of graphs. We shall now show how Theorem 2 can be interpreted as a generalization of the Redfield–Read superposition theorem. We begin with a simple extension of R. C. Read's definition of a superposed graph **(8)** and our terminology will be that used in **(8)**.

Suppose $(\theta_\Omega) = (\theta_1, \theta_2, \ldots, \theta_q)$ is an ordered set of non-negative integers such that

$$\sum_{i=1}^{q} \theta_i = \Omega.$$

Let G_{ij} $(j = 1, 2, \ldots, \theta_i; i = 1, 2, \ldots, q)$ be a set of Ω unlabelled graphs, each on n nodes, such that the θ_i graphs $G_{i1}, G_{i2}, \ldots, G_{i\theta_i}$ are each topologically similar to some graph G_i (say), $i = 1, 2, \ldots, q$ (note that G_i and G_j $(1 \leqslant i, j \leqslant q; i \neq j)$ are not necessarily distinct). Thus the automorphism group of each of the graphs $G_{i1}, G_{i2}, \ldots, G_{i\theta_i}$ may be denoted by $\Gamma(G_i)$ (say), $i = 1, 2, \ldots, q$. The (θ_Ω)-superposition of the Ω graphs G_{ij} is defined as any graph which can be constructed by:

(i) labelling the nodes of each G_{ij} with the labels $A_1, A_2, \ldots, A_n$ in any manner;

(ii) identifying all nodes having the same labels.

Furthermore, if $C_1, C_2, \ldots, C_q$ are q "colours" and if, when the graphs G_{ij} have been labelled, we let $r_{ij}(A_\alpha, A_\beta)$ be the number of edges in G_{ij} which join A_α and A_β, then the (θ_Ω)-*superposition* of the Ω graphs G_{ij} corresponding to this labelling is defined as that graph on n nodes $A_1, A_2, \ldots, A_n$ for which A_α and A_β are joined by

$$\sum_{j=1}^{\theta_i} r_{ij}(A_\alpha, A_\beta)$$

edges coloured with "C_i" (thus each graph $G_{i\gamma}$, $\gamma = 1, 2, \ldots, \theta_i$, may be thought of as being coloured with the same colour C_i, $1 \leqslant i \leqslant q$). Two (θ_Ω)-superposed graphs are *similar as labelled graphs* if

$$\sum_{j=1}^{\theta_i} r_{ij}(A_\alpha, A_\beta)$$

is always the same for one graph as for the other ($i = 1, 2, \ldots, q$; $\alpha, \beta = 1, 2, \ldots, n$). Two (θ_Ω)-superposed graphs are *topologically similar* if, by relabelling one of the graphs, we can convert it into a graph which is similar as a labelled graph to the other. Otherwise they are said to be *distinct*. We shall denote the set of distinct (θ_Ω)-superposed graphs by $S(\theta_\Omega)$ and the cardinal of the set by $|S(\theta_\Omega)|$.

Before stating the next theorem we make the following assumptions about Theorem 2: (a) $P = \mathfrak{S}_n$; (b) $H_i = \Gamma(G_i)$, $i = 1, 2, \ldots, q$.

THEOREM 3. $|S(\theta_\Omega)| = (\mathrm{X}_1^{\theta_1}, \mathrm{X}_2^{\theta_2}, \ldots, \mathrm{X}_q^{\theta_q})$.

Proof. Let D_i consist of the α_i distinct labelled graphs obtained by labelling G_i with the labels $A_1, A_2, \ldots, A_n$ in all possible ways. Let R_i consist of figures $\phi_{i0}, \phi_{i1}, \phi_{i2}, \ldots, \phi_{i\beta}, \ldots$ of content $0, 1, 2, \ldots, \beta, \ldots$ respectively. Then

$$p(x_i) = (1 - x_i)^{-1}, \qquad i = 1, 2, \ldots, q.$$

Clearly:

(i) each $\mathfrak{J}$-configuration of content $(\theta_1, \theta_2, \ldots, \theta_q)$ corresponds uniquely to a (θ_Ω)-superposed graph and conversely:

(ii) two such graphs are topologically similar if and only if their corresponding $\mathfrak{J}$-configurations are $\mathfrak{G}$-equivalent.

The theorem follows immediately.

Remark 1. If $\theta_i = 1$ ($i = 1, 2, \ldots, q$), then Theorem 3 is an exact statement of Read's superposition theorem (**8**; **3**, p. 278). Also Dr. R. C. Read has pointed out to me that when $q = 1$ the formula of Theorem 3 is given in (**9**).

Remark 2. If, in Theorem 3, the elements of D_i are regarded as the cosets of $\Gamma(G_i)$ with respect to P and we drop the assumption that $P = \mathfrak{S}_n$, then it is clear that $|S(\theta_\Omega)|$ is equal to the number of transitive constituents of

$$(P_{H_1})^{\theta_1} \otimes (P_{H_2})^{\theta_2} \otimes \ldots \otimes (P_{H_q}{}^{\theta_q}),$$

where $\otimes$ denotes "Kronecker product" (**3**).

In a later paper we shall give some applications of Theorem 3.

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