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AUTOMORPHISMS OF FULL II_1 FACTORS, II

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The purpose of this note is to continue the author's study of the automorphisms of certain factors of type II_1. Namely, those factors arising from the left regular representation of a free nonabelian group. Our main result shows that the outer conjugacy classes of automorphisms of such a factor are not countably separated. This had previously been shown only when the number of free generators was assumed to be infinite.

Preliminaries. If A is a von Neumann algebra, we denote the group of *-automorphisms of A by Aut A and its normal subgroup of inner automorphisms by Int A. We let ϵ denote the canonical homomorphism, $\epsilon: \text{Aut } A \to \text{Aut } A/\text{Int } A$ and we denote the quotient group Aut A/Int A by Out A.

We let A_* denote the predual of A and endow Aut A with the topology of pointwise norm convergence in A_*. With this topology, Aut A is a topological group which is polish if A_* is separable [3]. A is called full if Int A is closed in this topology [1]. If A is a II_1 factor with canonical trace, tr, then the topology on Aut A is actually the topology of pointwise convergence in A, where A is given the norm $\|\times\|_2^2 = tr(x^*x)$, [3].

Let F_n be the free nonabelian group on n generators ($n=2, 3, \ldots, +\infty$). Then, F_n is a countable discrete group with infinite conjugacy classes. For each g in F_n, let $\lambda(g)$ be the unitary operator on $\ell^2(F_n)$ defined by:

$$(\lambda(g)f)(h) = f(g^{-1}h), f \quad \text{in} \quad \ell^2(F_2), h \quad \text{in} \quad F_n.$$

Let $U(F_n) = \{\lambda(g) \mid g \text{ in } F_n\}''$ be the left von Neumann algebra of F_n on $\ell^2(F_n)$. It is well-known that $U(F_n)$ is a factor of type II_1. Moreover, $U(F_n)$ is a full factor. To see this, one uses lemma 6.2.1 and 6.3.1 of [7] to show that $U(F_n)$ does not have property Γ; then by [1, corollary 3.8] we see that $U(F_n)$ is full.

Now, let $\mathbb{T}$ be the unit circle and let $\Lambda = (\gamma_1, \ldots, \gamma_n)$ be any sequence of elements of $\mathbb{T}$. Let $\{x_k\}_{k=1}^n$ be the free generators of F_n. Then, there is a unique automorphism α_Λ of $U(F_n)$ such that

$$\alpha_\Lambda(\lambda(x_k)) = \gamma_k \lambda(x_k), \qquad k = 1, \ldots, n.$$

Moreover, α_Λ is easily seen to be outer, if $\Lambda \neq (1, 1, \ldots, 1)$.

If X is a set and $\mathcal{B}$ is a σ-algebra of subsets of X, we call $(X, \mathcal{B})$ a Borel space and the sets in $\mathcal{B}$ are called Borel sets. If there is a countable family of

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sets in $\mathcal{B}$ which generate $\mathcal{B}$ as a σ-algebra and which separate the points of X, we say that $(X, \mathcal{B})$ is countably separated or smooth. For more on this subject, see [6]. If A is a II_1 factor, then Out A is a Borel space with the quotient Borel structure obtained from Aut A. If $\sim$ denotes the equivalence relation of conjugacy in the group Out A, it is the quotient space Out $A/\sim$ that is of interest to us.

1. The Main Theorem

1. THEOREM. For any $n>1$, the Borel space Out $(U(F_n))/\sim$ is not smooth.

Proof. Let $\{x_1, \ldots, x_n\}$ be the free generators of F_n and let γ be a fixed element of the unit circle, $\mathbb{T}$, such that $\gamma^k = 1 \Leftrightarrow k = 0$. For each t in $\mathbb{T}$ define α_t, an automorphism of $U(F_n)$, by:

$$\alpha_t(\lambda(x_1)) = t\lambda(x_1)$$
$$\alpha_t(\lambda(x_2)) = \gamma\lambda(x_2)$$
$$\alpha_t(\lambda(x_k)) = \lambda(x_k) \quad \text{for} \quad k>2.$$

Clearly the map $t \mapsto \alpha_t : \mathbb{T} \to \mathrm{Aut}\,(U(F_n))$ is continuous and therefore the composition $t \mapsto \epsilon(\alpha_t) : \mathbb{T} \to \mathrm{Out}\,(U(F_n))$ is also continuous. Moreover, the map $t \mapsto \epsilon(\alpha_t)$ is one-to-one and hence a homeomorphism onto the compact set $\{\epsilon(\alpha_t) \mid t \text{ in } \mathbb{T}\}$.

Let K be the countable subgroup of $\mathbb{T}$ consisting of all roots of all powers of γ. That is, $K = \{t \text{ in } \mathbb{T} \mid t^k = \gamma^m$ for some integers k, m with $k \neq 0\}$. We define an equivalence relation on $\mathbb{T}$ by $t_1 \simeq t_2 \Leftrightarrow$ either $t_1 t_2 \in \{\gamma^k \mid k \text{ in } \mathbb{Z}\}$ or $t_1 t_2^{-1} \in \{\lambda^k \mid k$ in $\mathbb{Z}\}$. Since $T/\{\gamma^k \mid k \text{ in } \mathbb{Z}\}$ is not smooth by [7.2 of 6] one deduces that $T/\simeq$ is not smooth. We demonstrate that the Borel spaces $\pi \setminus K/\simeq$ and $\{\epsilon(\alpha_t) \mid t$ not in $K\}/\sim$ are Borel isomorphic. This will prove the theorem.

Let t_1 and t_2 be in $\mathbb{T} \setminus K$ and suppose that $\epsilon(\alpha_{t_1})$ and $\epsilon(\alpha_{t_2})$ are conjugate in Out $(U(F_n))$. Then, the embeddings

$$m \mapsto \epsilon(\alpha_{t_1}^m)$$
$$m \mapsto \epsilon(\alpha_{t_2}^m)$$

define the same topology on $\mathbb{Z}$ since conjugation in a topological group is a homeomorphism. As in lemma 1.2 of [8], one shows that these topologies on $\mathbb{Z}$ are exactly the weak topologies on $\mathbb{Z}$ determined by the group of characters $\langle t_1, \gamma\rangle$ and $\langle t_2, \gamma\rangle$. Hence, by lemma 1.3 of [8], we conclude that $\langle t_1, \gamma\rangle = \langle t_2, \gamma\rangle$. Hence there are integers m, p, k, ℓ such that

$$t_1 = t_2^m \gamma^k$$
$$t_2 = t_1^p \gamma^\ell$$

so that $t_1 = t_1^{pm} \gamma^{m\ell+k}$. Therefore, $t_1^{1-pm} = \gamma^{m\ell+k}$ so that $1-pm=0$ because $t_1 \notin K$. Thus, either $m=p=1$ or $m=p=-1$ so that in either case $t_1 \simeq t_2$.

Conversely, suppose that $t_1 \simeq t_2$ and t_1, t_2 are not in K. Suppose that $t_1 t_2 = \gamma^k$. Define an automorphism σ of $U(F_n)$ via

$$\sigma(\lambda(x_1)) = \lambda(x_1^{-1} x_2^k)$$
$$\sigma(\lambda(x_j)) = \lambda(x_j) \quad \text{for} \quad j > 1.$$

Clearly, σ define a surjective homomorphism of $\lambda(F_n)$ onto $\lambda(F_n)$; but, since $\tau(\lambda(x_1)) = \lambda(x_2^k x_1^{-1})$, $\tau(\lambda(x_j)) = \lambda(x_j)$ for $j > 1$ defines an inverse homomorphism to σ, we have that σ defines an automorphism of $\lambda(F_n)$ which therefore extends to an automorphism of $U(F_n)$. We easily compute that $\sigma^{-1}\alpha_{t_1}\sigma = \alpha_{t_2}$. Thus, a fortiori, $\epsilon(\alpha_{t_1}) \sim \epsilon(\alpha_{t_2})$. Since $t \mapsto \epsilon(\alpha_t)$ is a homeomorphism which carries the equivalence relation, $\simeq$, on $\mathbb{T} \setminus K$ to the equivalence relation, $\sim$, on $\epsilon(\mathbb{T} \setminus K)$, the Borel spaces $\mathbb{T} \setminus K / \simeq$ and $\epsilon(\mathbb{T} \setminus K)/\simeq$ are Borel isomorphic.

2. Automorphisms generating discrete subgroups of Out A. On $U(F_\infty)$ it is easy to define automorphisms θ such that $\{\epsilon(\theta^n) \mid n \in \mathbb{Z}\}$ is a discrete copy of $\mathbb{Z}$ in Out $(U(F_\infty))$ (for example let θ be the "shift" on the generators of F_∞, see [8].) It appears to be a reasonable conjecture that if $\{\epsilon(\theta^n) \mid n \in \mathbb{Z}\}$ is discrete in Out A then the crossed product $W^*(\theta, A)$ is a full II_1 factor, if A is. In any case, we exhibit such an automorphism on $u(F_n)$ and prove that $W^*(\theta, U(F_n))$ is full.

2. Proposition. There exists an automorphism θ on $U(F_n)$ such that $\{\epsilon(\theta^n) \mid n \in \mathbb{Z}\}$ is a discrete copy of $\mathbb{Z}$ in Out $(U(F_n))$ and $W^*(\theta, U(F_n))$ is a full II_1 factor.

Proof. Define $\theta(\lambda(x_1)) = \lambda(x_1 x_2)$
and $\theta(\lambda(x_k)) = \lambda(x_k), \qquad k \geq 2.$

Then θ defines an automorphism of $\lambda(F_n)$ and so extends to an automorphism of $U(F_n)$. Since $\theta^m(\lambda(x_1))$ is not conjugate to $\lambda(x_1)$ by the test on p. 76 of [5], we have that θ^m is an outer automorphism of $\lambda(F_n)$ for all $m \neq 0$. Then, by an argument due to H. Behncke, [4], θ^m is an outer automorphism on $U(F_n)$ for each $m \neq 0$.

Now let G be the semidirect product of $\lambda(F_n)$ by $\mathbb{Z}$ where $\mathbb{Z}$ acts as powers of θ. Then $U(G)$ is isomorphic to $W^*(\theta, U(F_n))$ and to see that $U(G)$ is full, it suffices by [1] to see that $U(G)$ does not have property Γ. In the notation of lemma 6.2.1. of [7], let

$\mathfrak{F} = \{(\lambda(w), \theta^m) \mid m$ in $\mathbb{Z}$, and w ends in a nonzero power of $x_1\}$

$c_1 = (\lambda(x_1 x_2 x_1^{-1}), id)$

$c_2 = (\lambda(x_2), id)$.

One easily verifies that $\mathfrak{F}$, c_1, c_2 satisfy the requirements of lemma 6.2.1 of [7]. Therefore, $U(G)$ does not have property Γ.

As noted in [8], $W^*(\theta, U(F_n))$ being full, forces $\{\epsilon(\theta^m) \mid m \in \mathbb{Z}\}$ to be discrete in Out $(U(F_n))$.

3. REMARKS. (a) Clearly there are many other automorphisms having the property described in Proposition 2. Classifying them appears to be a difficult task, however. (b) Although it is easy to construct automorphisms on $U(F_n)$ with finite outer period, the author does not know if one can construct automorphisms on $U(F_2)$ with nontrivial obstruction, γ, to the lifting problem:

$$\begin{array}{ccc} & \nearrow & \mathrm{Aut}\,(U(F_2)) \\ & & \downarrow \\ \mathbb{Z}_n & \longrightarrow & \mathrm{Out}\,(U(F_2)). \end{array}$$

See [8, §4.6].

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