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RESEARCH ARTICLE

A proof of the Erdős primitive set conjecture

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Dedicated to Carl Pomerance

Abstract

A set of integers greater than 1 is primitive if no member in the set divides another. Erdős proved in 1935 that the series $f(A) = \sum_{a\in A} 1/(a\log a)$ is uniformly bounded over all choices of primitive sets A. In 1986, he asked if this bound is attained for the set of prime numbers. In this article, we answer in the affirmative.

As further applications of the method, we make progress towards a question of Erdős, Sárközy and Szemerédi from 1968. We also refine the classical Davenport–Erdős theorem on infinite divisibility chains, and extend a result of Erdős, Sárközy and Szemerédi from 1966.

Contents

1. Introduction

A set of integers $A \subset \mathbb{Z}_{>1}$ is *primitive* if no member in A divides another. For example, the integers in a dyadic interval $(x, 2x]$ form a primitive set. Similarly, the set of primes is primitive, along with the set $\mathbb{N}_k$ of numbers with exactly k prime factors (with multiplicity), for each $k \geq 1$. Another well-known example is the set of perfect numbers.[1]

[1]Since Ancient Greece, a number n is classified as 'perfect', 'abundant' or 'deficient', depending on whether the sum of its proper divisors equals n, is greater than n or is less than n, respectively.

The study of primitive sets emerged in the 1930s as a generalization of one special problem. A classical theorem of Davenport asserts that the set of abundant numbers has a positive asymptotic density. This was originally proved by sophisticated analytic methods, but Erdős soon found an elementary proof by using primitive abundant numbers.[2] The proof ideas led people to introduce the abstract definition of primitive sets and study them for their own sake. See Hall [21] or Halberstam–Roth [20, §5] for detailed introductions to the subject.

There are a number of interesting and sometimes unexpected theorems about primitive sets. For instance, in 1934 Besicovitch [5] showed that the upper asymptotic density of a primitive set can be arbitrarily close to $1/2$, whereas in 1935 Behrend [4] and Erdős [13] proved the lower asymptotic density is always 0. In fact, Erdős proved the stronger result that

$$f(A) := \sum_{a\in A} \frac{1}{a\log a} \ < \ \infty,$$

uniformly over all primitive sets A. Later in 1986, Erdős [15, Conjecture 2.1] famously asked if the maximum is attained by the primes $\mathcal{P}$.

Conjecture 1.1 (Erdős primitive set conjecture). *For any primitive set A, $f(A) \le f(\mathcal{P})$.*

The prime sum is $f(\mathcal{P}) = \sum_p 1/(p\log p) = 1.6366\cdots$ after computations of Cohen [10]. In 1993, Erdős and Zhang [19] proved the bound $f(A) < 1.84$ for all primitive A. Recently in 2019, Lichtman and Pomerance [27] improved the bound to $f(A) < e^\gamma = 1.781\cdots$, where γ is the Euler–Mascheroni constant. Note the tail of the series for $f(\mathcal{P})$ converges quite slowly $O(1/\log x)$, and moreover there are sets $A \subset [x,\infty)$ for which $f(A) \sim 1$ as $x \to \infty$ (in this connection, see Conjecture 1.4 below). As such, Conjecture 1.1 is not susceptible to direct attack by computing partial sums up to x.

One potential strategy to approach Conjecture 1.1 is via integration. Namely,

$$f(A) = \sum_{a\in A} \frac{1}{a\log a} = \sum_{a\in A} \int_1^\infty a^{-t}\,\mathrm{d}t = \int_1^\infty f_t(A)\,\mathrm{d}t,$$

letting $f_t(A) = \sum_{a\in A} a^{-t}$. So Conjecture 1 would follow if for any $t > 1$, primitive set A,

$$f_t(A) \ \le \ f_t(\mathcal{P}). \tag{1}$$

However, it was shown in [2] that equation (1) holds if and only if

$$t \ge \tau := 1.1403\cdots,$$

where $t = \tau$ is the unique real solution to the equation

$$\sum_p p^{-t} = 1 + \Big(1 - \sum_p p^{-2t}\Big)^{1/2}.$$

The fact that τ is markedly larger than 1 gives some indication as to why the Erdős primitive set conjecture has remained open.

Similar analysis actually enables a disproof of a natural analogue of Conjecture 1.1 for the translated sum $f(A,h) = \sum_{a\in A} 1/a(\log a + h)$, in that there are primitive A for which $f(A,h) > f(\mathcal{P},h)$ once $h \ge 81$ [24], [23]. This was refined down to just $h \ge 1.04$ in [26] and suggests that the original conjecture (when $h = 0$), if true, is only 'barely' so.

Concerning equation (1), we also note Chan et al. [8] proved $f_t(A) \ \le \ f_t(\mathcal{P})$ for all $t \ge .7983$ for all 2-primitive sets A, thereby resolving Conjecture 1 in this special case (also see [9]). Here, a set A is 2-primitive if no member of A divides the product of two others.

[2]More precisely, 'primitive nondeficient numbers'

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A separate strategy for the problem is to split up A according to the smallest prime factor. That is, for each prime p let

$$A_p = \{n \in A : n \text{ has least prime factor } p\}.$$

As in [27], we say p is *Erdős strong* if the singleton set $\{p\}$ maximizes $f(A)$ among all primitive sets A all of whose elements have least prime factor p. That is, $f(A_p) \leq f(\{p\}) =: f(p)$ for all primitive A. Conjecture 1.1 would follow if every prime is Erdős strong since then $f(A) = \sum_p f(A_p) \leq f(\mathcal{P})$.

By a short argument in [27] (also see Lemma 2.3), a sufficient condition for a prime p to be Erdős strong is that

$$e^{\gamma} \prod_{q<p} \left(1 - \frac{1}{q}\right) \leq \frac{1}{\log p}. \tag{2}$$

Here, q runs over primes. Note the two sides of this inequality are asymptotically equal by Mertens' prime product theorem. By direct computation, equation (2) is satisfied by the first 10^8 odd primes but fails for $p = 2$ since $\log 2 > e^{-\gamma}$.

Moreover, 99.999973% of primes[3] satisfy equation (2), assuming the Riemann hypothesis and the linear independence hypothesis[4] [28]. This result is intimately related to the celebrated work of Rubinstein and Sarnak [30] on the prime number race between $\pi(x)$ and $\mathrm{li}(x)$. On the Riemann hypothesis alone, equation (2) fails for a positive proportion of primes p (in log density), and even unconditionally equation (2) is known to fail for infinitely many primes p. This perhaps suggests Conjecture 1.1 might be false, or at least beyond the reach of unconditional tools.

In this article, we establish Conjecture 1.1.

Theorem 1.2. *For any primitive set A, we have $f(A) \leq f(\mathcal{P})$.*

Moreover, we show that every odd prime is Erdős strong.

Theorem 1.3. *For any primitive set A and any prime $p > 2$, we have $f(A_p) \leq f(p)$.*

It remains an open question whether $p = 2$ is Erdős strong.

Another question related to Conjecture 1.1, in 1968 Erdős, Sárközy and Szemerédi posed the following [18, eq. (11)].

Conjecture 1.4 (Erdős–Sárközy–Szemerédi). *We have*

$$\lim_{x\to\infty} \sup_{\substack{A\subset[x,\infty) \\ A \text{ primitive}}} f(A) \leq 1.$$

This also appears in [31, p. 244] as Problem 2.2, and in [32, p. 224] as Problem 2.

Not much has been proven in this direction until very recently. Recall the set $\mathbb{N}_k$ of numbers with exactly k prime factors (with multiplicity) lies in $[2^k, \infty)$. Lichtman and Pomerance [27] proved $f(\mathbb{N}_k) \gg 1$, and in [25] it was shown $f(\mathbb{N}_k) \sim 1$ as $k \to \infty$. This means that if Conjecture 1.4 holds, then the limit must attain an *equality* of 1. We note [25, Theorem 4.1] gives for all $\epsilon > 0$,

$$f(\mathbb{N}_k) = 1 + O_\epsilon(k^{\epsilon-1/2}). \tag{3}$$

Moreover, computations up to $k = 20$ suggest the true rate of decay may be exponential $O(2^{-k})$; see [25].

The methods in this paper enable the following progress towards Conjecture 1.4.

[3]More precisely, the set of such primes has discrete, logarithmic density equal to $0.99999973\cdots$ within $\mathcal{P}$.

[4]Namely, the sequence of numbers $\gamma_n > 0$ such that $\zeta(\frac{1}{2} + i\gamma_n) = 0$ is linearly independent over $\mathbb{Q}$.

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Theorem 1.5. *We have*

$$\lim_{x\to\infty} \sup_{\substack{A\subset[x,\infty) \\ A \text{ primitive}}} f(A) \le e^{\gamma}\frac{\pi}{4} \approx 1.399.$$

Notation

Let $p(a), P(a)$ denote the smallest and largest prime factors of $a \in \mathbb{Z}_{>1}$, respectively, and denote $a^* = a/P(a)$. Let $\Omega(n)$ denote the number of prime factors of n (with multiplicity), and let $\mathbb{N}_k = \{n : \Omega(n) = k\}$. Define $f(a) = 1/(a \log a)$ and $f(A) = \sum_{a\in A} f(a)$ for $A \subset \mathbb{Z}_{>1}$. Let $\mathcal{P}$ be the set of prime numbers, whose elements we denote by p and q, unless otherwise stated. Also, $p^k \| n$ means $p^k \mid n$ and $p^{k+1} \nmid n$.

1.1. Proof outline of Theorem 1.2

The proof is a refinement of the argument of [27]. The key new idea is to exploit the fact that A cannot contain too many elements a with $P(a)$ just slightly less than a. This improves the critical case in the argument of [27] and ultimately leads to an improvement by a factor of $\pi/4$ from a contribution from each $a \in A$ which is not prime. Since $e^\gamma \pi/4 < f(\mathcal{P})$, this ultimately means that $f(A)$ is maximized when all elements are prime. (Additional care is needed for small numbers, using explicit bounds.)

Let us recall the rough argument of [27] (suppressing details for primes and small numbers). By Mertens' product theorem,

$$f(A) = \sum_{a\in A} \frac{1}{a\log a} < \sum_{a\in A} \frac{1}{a \log P(a)} \approx e^{\gamma} \sum_{a\in A} \frac{1}{a} \prod_{p<P(a)} \Big(1 - \frac{1}{p}\Big). \tag{4}$$

But $a^{-1}\prod_{p<P(a)}(1-p^{-1})$ is the natural density of $\mathrm{L}_a = \{ba \;:\; p \mid b \Rightarrow p \ge P(a)\}$, and these sets turn out to be disjoint by primitivity of A (Lemma 2.1). So the sum of densities in equation (4) is trivially at most 1, leading to the bound $f(A) < e^\gamma$ for primitive A. This is inspired by the original 1935 argument of Erdős [13].

There is a loss in the above argument when bounding a by $P(a)$, and this loss is largest when a is far from prime. We can save an additional factor of $\log P(a)/\log a$ for any individual $a \in A$, and this would be a significant improvement in the case $P(a)^2 < a$, say. Therefore, the critical case to handle is when $a \in A$ is composite with $P(a)$ close to a in size. The key new ingredient (Proposition 3.3) shows that if $P(a)^{1+v} > a$ uniformly for all $a \in A$ (so the savings factor is $\log P(a)/\log a > 1/(1+v)$), then we can bound the sum of densities in equation (4) by $\sqrt{v}$. This refines the trivial bound of 1 in the range $0 < v < 1$, and quantifies the earlier statement that A contains few elements a with $P(a)$ slightly less than a. As the savings $1/(1+v)$ improves with v, the worst-case scenario is when the subset of $a \in A$ with $P(a)^{1+v} \approx a$ contributes about $\frac{\mathrm{d}}{\mathrm{d}v}[\sqrt{v}] = 1/2\sqrt{v}$ to the sum of densities in equation (4). Combining these ingredients ultimately leads to a savings of $\int_0^1 \mathrm{d}v/2\sqrt{v}(1+v) = \pi/4$, as desired.

Lastly, the key Proposition 3.3 relies on the following observation (Lemma 3.1): Not only are the sets L_a disjoint but so too are L_{ac} for many choices of integers c (in fact, all choices of c with prime factors between $P_2(a)$ and $P_2(a)^{1/\sqrt{v}}$). Thus, the sum of densities of these L_{ac} must be at most 1. But these sets L_{ac} are self-similar to the L_a, and so the sum of their densities is roughly $1/\sqrt{v}$ times that of the L_a, giving the desired bound $\sqrt{v}$.

1.2. L*-primitive sets*

As outlined above, the subset of multiples of each $a \in A$,

$$\mathrm{L}_a := \big\{ba \in \mathbb{N} \;:\; p \mid b \implies p \ge P(a)\big\}, \tag{5}$$

arises naturally in our proof. As such, we shall introduce 'L' refinements of our common notions (here, L alludes to 'lexicographic'). Specifically, if $n \in \mathrm{L}_a$, we say n is an L-*multiple* of a, and a is an L-*divisor* of n. Most importantly, we introduce the following key definition.

Definition 1.6. A set $A \subset \mathbb{Z}_{>1}$ is L-*primitive* if $a' \notin \mathrm{L}_a$ for all distinct $a, a' \in A$.

That is, A is L-primitive if no member of A is an L-multiple of another. In particular, this definition is weaker than primitive.

One may apply the basic argument as in equation (4) more generally for L-primitive sets A, leading to the same bound $f(A) < e^\gamma$ (again ignoring small numbers). Moreover, L-primitive sets play a central role in the proof of Theorem 1.2. However, it turns out the bound e^γ is essentially best possible for L-primitive sets (see Proposition 5.3), which is markedly larger than $f(\mathcal{P})$. This further highlights the subtlety of Conjecture 1.1.

1.3. Density and divisibility chains

Recall the natural (asymptotic) density $\mathrm{d}(S) = \lim_{x\to\infty} |S \cap [1,x]|/x$ of a set $S \subset \mathbb{N}$. We also consider log density $\delta(S)$ and log log density $\Delta(S)$, given by

$$\delta(S) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\in S, n\le x} \frac{1}{n}, \qquad \text{and} \qquad \Delta(S) = \lim_{x\to\infty} \frac{1}{\log\log x} \sum_{n\in S, 1<n\le x} \frac{1}{n\log n}, \tag{6}$$

provided these limits exist. Recall the corresponding upper densities $\overline{\mathrm{d}}(S), \overline{\delta}(S), \overline{\Delta}(S)$ always exist, by replacing $\lim_{x\to\infty}$ with $\limsup_{x\to\infty}$ (and similarly $\liminf_{x\to\infty}$ for lower densities).

Taking an abstract view, a primitive set is an antichain for the partial ordering of integers by divisibility. As such, this naturally leads to the dual notion of a chain in this context. Namely, an infinite sequence of integers $1 < d_1 < d_2 < \cdots$ is a *divisibility chain* if $d_j \mid d_{j+1}$ for all $j \ge 1$. A classical 1937 theorem of Davenport and Erdos [11] asserts that if set $A \subset \mathbb{N}$ has upper log density $\overline{\delta}(A) > 0$, then it contains an infinite divisibility chain $D \subset A$.

Analogously, we introduce the following refinement.

Definition 1.7. An infinite sequence of integers $1 < d_1 < d_2 < \cdots$ is an L-*divisibility chain* if $d_{j+1} \in \mathrm{L}_{d_j}$ for all $j \ge 1$.

That is, d_{j+1} is an L-multiple of d_j for all $j \ge 1$. In particular, this definition is stronger than a (mere) divisibility chain.

We refine the Davenport–Erdos theorem to L-divisibility chains.

Theorem 1.8. *If a set $A \subset \mathbb{N}$ has upper* log *density $\overline{\delta}(A) > 0$, then A contains an infinite L-divisibility chain.*

In 1966, Erdős, Sárközy and Szemerédi [16, Theorem 1] quantified the Davenport–Erdős theorem by showing such a divisibility chain D satisfies $\limsup_{y\to\infty} \sum_{d\in D, d\le y} 1/\sqrt{\log\log y} > 0$ and proved such growth rate is best possible.

They also studied the analogous question for upper log log density, which they write 'seems more interesting to us'. Namely, in [16, Theorem 2] they established the following quantitative result.

Theorem 1.9 (Erdős–Sárközy–Szemerédi). *If $A \subset \mathbb{N}$ has upper* log log *density $\overline{\Delta}(A) > 0$, then there is an infinite divisibility chain $D \subset A$ of growth*

$$\limsup_{y\to\infty} \sum_{\substack{d\in D\\ d\le y}} \frac{1}{\log\log y} \ \ge\ \frac{\overline{\Delta}(A)}{e^\gamma}. \tag{7}$$

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Analogously, we quantify Theorem 1.8 in the case of log log density, thereby refining Theorem 1.9 of Erdős–Sárközy–Szemerédi to L-divisibility chains.

Theorem 1.10. *If $A \subset \mathbb{N}$ has upper* log log *density* $\overline{\Delta}(A) > 0$*, then there is an infinite L-divisibility chain $D \subset A$ of growth*

$$\limsup_{y\to\infty} \sum_{\substack{d\in D\\ d\le y}} \frac{1}{\log\log y} \ \ge\ \frac{\overline{\Delta}(A)}{e^\gamma}.$$

In view of Proposition 5.3, we believe that the lower bound $\overline{\Delta}(A)/e^\gamma$ above is best possible for L-divisibility chains, though we are unable to settle this. Notably, this contrasts the situation in Theorem 1.9, as Erdős–Sárközy–Szemerédi conjectured $\overline{\Delta}(A)/e^\gamma$ in equation (7) might be improved to $\overline{\Delta}(A)$, which would be best possible for divisibility chains, if true [16, eq. (5)].

2. Preliminaries on L-primitive sets

Recall the set of L-multiples $\mathrm{L}_a := \{ba \in \mathbb{N} : p \mid b \Rightarrow p \ge P(a)\}$ from equation (5). In particular, $a \in \mathrm{L}_a$ for $b = 1$, and $p(b) \ge P(a)$ for $b > 1$. For $A \subset \mathbb{N}$, define $\mathrm{L}_A := \bigcup_{a\in A} \mathrm{L}_a$. Also, let $A_a = A \cap \mathrm{L}_a$ so that $\mathbb{N}_a = \mathrm{L}_a$ and $A_q = \{a \in A : p(a) = q\}$ for prime q.[5]

Observe that $a \in \mathrm{L}_{a'}$ if and only if $\mathrm{L}_a \subset \mathrm{L}_{a'}$, as well as the following trichotomy.

Lemma 2.1. *For any integers $a, a' > 1$, if $\mathrm{L}_a \cap \mathrm{L}_{a'} \ne \emptyset$ then $a \in \mathrm{L}_{a'}$ or $a' \in \mathrm{L}_a$. Thus $\mathrm{L}_a \cap \mathrm{L}_{a'} = \emptyset$ or $\mathrm{L}_a \subset \mathrm{L}_{a'}$ or $\mathrm{L}_a \supset \mathrm{L}_{a'}$.*

Proof. Suppose $ba = b'a' \in \mathrm{L}_a \cap \mathrm{L}_{a'}$. If $b = 1$ or $b' = 1$, then $a \in \mathrm{L}_{a'}$ or $a' \in \mathrm{L}_a$. Otherwise, $b, b' > 1$, so $P(a) \le p(b)$ and $P(a') \le p(b')$ imply $b \mid b'$ or $b' \mid b$. Thus, $a' = a(b/b') \in \mathrm{L}_a$ or $a = a'(b'/b) \in \mathrm{L}_{a'}$ as well. □

As such, we see A is L-primitive if and only if the sets $\{\mathrm{L}_a\}_{a\in A}$ are pairwise disjoint.

Corollary 2.2. *If A is an* L*-primitive set, then L_a and $\mathrm{L}_{a'}$ are disjoint for distinct $a, a' \in A$.*

Recall L_a has natural density $\mathrm{d}(\mathrm{L}_a) = \frac{1}{a}\prod_{p<P(a)}(1 - \frac{1}{p})$. And by Mertens' product theorem $\prod_{p<x}(1 - \frac{1}{p}) \sim 1/e^\gamma \log x$, where $\gamma = .57721\cdots$ is the Euler–Mascheroni constant. By a show argument below, we relate $f(A)$ to density of L-multiples. This is essentially based on Erdős [13] (also see [17, Lemma 1], [27, Proposition 2.1]).

Lemma 2.3. *For an* L*-primitive set A and an integer $1 < n \notin A$, we have $f(A_n) < e^\gamma\, \mathrm{d}(\mathrm{L}_n)$.*

Proof. We may assume $A = A_n$ is finite since $f(A) = \lim_{x\to\infty} f(A \cap [1,x])$. As $n \notin A$, all elements of A are composite. Also, A is L-primitive so $\mathrm{d}(\mathrm{L}_A) = \sum_{a\in A} \mathrm{d}(\mathrm{L}_a)$ by Corollary 2.2. Next, Theorem 7 in [29] implies $\prod_{p<x} \frac{p}{p-1} < e^\gamma \log(2x)$ for all $x > 1$. Thus, for any composite integer $a > 1$, we have $a > 2P(a)$ so that

$$f(a) = \frac{1}{a\log a} \ \le\ \frac{1}{a\log 2P(a)} \ <\ \frac{e^\gamma}{a}\prod_{p<P(a)}\Big(1-\frac{1}{p}\Big) \ =\ e^\gamma\, \mathrm{d}(\mathrm{L}_a).$$

Hence, $f(A) = \sum_{a\in A} f(a) < e^\gamma\, \mathrm{d}(\mathrm{L}_A) \le e^\gamma\, \mathrm{d}(\mathrm{L}_n)$ since $A \subset \mathrm{L}_n$. □

We shall also need a technical refinement of Lemma 2.3. For this, we rewrite Mertens' product theorem as $\mu_x \sim 1$, where we denote

$$\mu_x := e^\gamma \log x \prod_{p<x}\Big(1-\frac{1}{p}\Big). \tag{8}$$

[5]Note the notation for A_q differs slightly from what is used in [19], [27]

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In particular, for a prime q we have

$$f(q) = \frac{1}{q\log q} = \frac{1}{q}\frac{e^\gamma}{\mu_q}\prod_{p<q}\left(1-\frac{1}{p}\right) = \frac{e^\gamma}{\mu_q}\mathrm{d}(\mathrm{L}_q). \tag{9}$$

We have the following explicit bounds for μ_x, which critically are monotonic. We give upper bounds which hold on real $x \in \mathbb{R}$, but for lower bounds it turns out it suffices to restrict to the subsequence of primes $q \in \mathcal{P}$.

Lemma 2.4 (Monotonic bounds). *For $q \in \mathcal{P}$ and $x \in \mathbb{R}$, define*

$$m_q := \inf_{\substack{p\ge q\\ p\in\mathcal{P}}} \mu_p, \quad \textit{and} \quad M_x := \sup_{\substack{y\ge x\\ y\in\mathbb{R}}} \mu_y.$$

Then we have

$$m_q \ge \begin{cases} \mu_7 = 0.9242\cdots & q \le 7\\ \mu_{19} = 0.9467\cdots & 7 < q \le 300\\ 1 - \frac{1}{2(\log q)^2} & q > 300. \end{cases} \quad \textit{and} \quad M_x \le \begin{cases} \mu_2 = 1.235\cdots & x \le 2\\ 1 + \frac{1}{2\log(2\cdot 10^9)^2} & 2 < x \le 2\cdot 10^9\\ 1 + \frac{1}{2(\log x)^2} & x > 2\cdot 10^9. \end{cases}$$

Proof. First, Rosser–Schoenfeld [29, Theorem 7] implies the product over primes $p < x$ is bounded in between

$$1 - \frac{1}{2(\log x)^2} \overset{(x>285)}{\le} e^\gamma \log x \prod_{p<x}\left(1-\frac{1}{p}\right) \overset{(x>1)}{\le} 1 + \frac{1}{2(\log x)^2}.$$

Note μ_x is increasing on $x \in (p, p']$ for consecutive primes p, p'. So the upper bound follows by computing $\mu_p < 1$ for the first 10^8 odd primes p (note $p_{10^8} \ge 2\cdot 10^9$). Hence, $\mu_x < 1$ for real $2 < x \le 2\cdot 10^9$. Below we display μ_q for the first few primes q, rounded to four significant digits.

q	μ_q	q	μ_q	q	μ_q	q	μ_q	q	μ_q
2	1.235	31	**0.9660**	73	0.9766	127	0.9902	179	0.9909
3	0.9784	37	0.9831	79	0.9809	131	0.9887	181	0.9874
5	0.9555	41	0.9836	83	0.9795	137	0.9902	191	0.9921
7	**0.9242**	43	0.9720	89	0.9829	139	0.9858	193	0.9889
11	0.9762	47	**0.9718**	97	0.9906	149	0.9925	197	0.9876
13	0.9492	53	0.9808	101	0.9890	151	0.9885	199	**0.9844**
17	0.9679	59	0.9883	103	0.9834	157	0.9896		
19	**0.9467**	61	0.9795	107	0.9818	163	0.9906		
23	**0.9551**	67	0.9854	109	0.9765	167	0.9892		
29	0.9811	71	0.9841	113	**0.9749**	173	0.9900		

The lower bound follows by identifying the primes q for which $\mu_q = \inf_{p\ge q}\mu_p$ (in bold above), and then computing $\mu_{199} < \mu_p$ for $199 < p \le 300$, as well as checking $\mu_{199} < 0.9846 < 1 - \frac{1}{2(\log x)^2}$ for $x > 300$. (In practice, we shall only need μ_q for $q = 7, 19$.) □

We may now prove a technical refinement of Lemma 2.3 using μ_q.

Lemma 2.5. *Let A be an* L*-primitive set. Take $v \ge 0$, an integer $n \notin A$ and denote $q = P(n)$. If $P(a)^{1+v} \le a$ for all $a \in A_n$, then*

$$f(A_n) = \sum_{a\in A_n}\frac{1}{a\log a} \le \frac{e^\gamma}{m_q}\frac{\mathrm{d}(\mathrm{L}_{A_n})}{1+v}.$$

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Proof. We may assume $A = A_n$ is finite, since $f(A) = \lim_{x\to\infty} f(A \cap [1,x])$. As $n \notin A$, all elements of A are composite. Also, A is L-primitive so $\mathrm{d}(\mathrm{L}_A) = \sum_{a\in A} \mathrm{d}(\mathrm{L}_a)$ by Corollary 2.2. Moreover, $(1+v)\log P(a) \le \log a$ for all $a \in A$. Thus, by definition of $\mu_{P(a)}$ in (8),

$$\frac{1}{a\log a} \le \frac{1}{1+v}\frac{1}{a\log P(a)} = \frac{e^\gamma}{\mu_{P(a)}}\frac{1}{(1+v)a}\prod_{p<P(a)}\Big(1-\frac{1}{p}\Big) = \frac{e^\gamma}{\mu_{P(a)}}\frac{\mathrm{d}(\mathrm{L}_a)}{1+v}.$$

By monotonicity $\mu_{P(a)} \ge m_{P(a)} \ge m_q$ for $a \in A \subset \mathrm{L}_n$. Hence, we conclude

$$f(A) = \sum_{a\in A}\frac{1}{a\log a} \le \frac{e^\gamma}{m_q}\frac{1}{1+v}\sum_{a\in A}\mathrm{d}(\mathrm{L}_a) = \frac{e^\gamma}{m_q}\frac{\mathrm{d}(\mathrm{L}_A)}{1+v}.$$

□

3. Primitive sets

Given $v \in (0,1)$, we shall be interested in elements $a \in A$ for which $P(a)^{1+v} > a$, and their multiples ac, where $c \in C_a^v$ for

$$C_a^v := \{c \in \mathbb{N} : p \mid c \implies p \in [P(a^*), P(a^*)^{1/\sqrt{v}})\}. \tag{10}$$

Note $c = 1 \in C_a^v$. Recall $a^* = a/P(a)$, so $P(a^*)$ is the second largest prime of a. Also if $1 < c \in C_a^v$ then $P(c) \le P(a^*)^{1/\sqrt{v}}$ is markedly smaller than $P(a) \ge P(a^*)^{1/v}$.

The following key lemma provides an upgrade to Corollary 2.2 in the case when A is primitive, not just L-primitive. Namely, the L_{ac} are disjoint, and so the larger set $\{ac : a \in A, c \in C_a^v\}$ is L-primitive.

Lemma 3.1. *Let A be a primitive set of composite numbers, and take $v \in (0,1)$. If $P(a)^{1+v} > a$ for all $a \in A$, then the collection of sets L_{ac}, ranging over $a \in A, c \in C_a^v$, are pairwise disjoint.*

Proof. Suppose $\mathrm{L}_{ac} \cap \mathrm{L}_{a'c'} \neq \emptyset$ for some $a, a' \in A$ and $c \in C_a^v$, $c' \in C_{a'}^v$. Without loss, by Lemma 2.1 we may assume $ac \in \mathrm{L}_{a'c'}$. Note if $c = 1$, then $a \in \mathrm{L}_{a'c'}$ implies $a' \mid a'c' \mid a$, which forces $a = a'$ and $c' = 1$ by primitivity of A. So assuming $(a,c) \neq (a',c')$, we deduce $c > 1$.

We factor $ac = p_1 \cdots p_k$ into primes $p_1 \ge \cdots \ge p_k$, so $ac \in \mathrm{L}_{a'c'}$ implies $a'c' = p_j \cdots p_k$ for some index $1 < j < k$. Since $P(a) > P(c), p(c) \ge P(a^*)$, we also have $a^* = p_i \cdots p_k$ for some $2 < i \le k$. If $i \le j$, then $a'c' \mid a^*$ so $a' \mid a$, contradicting A as primitive. Hence, $i > j$ so $a^* \mid a'c'$. Write $da^* = a'c'$, where $d = p_j \cdots p_{i-1}$, and note $P(d) = p_j = P(a')$. By definition of $1 < c \in C_a^v$, we have

$$p_j = P(d) \le P(c) < P(a^*)^{1/\sqrt{v}}. \tag{11}$$

Recall $P(a')^v > (a')^* \ge P((a')^*)$ for $a' \in A$. Now consider cases $c' > 1$ and $c' = 1$. When $1 < c' \in C_{a'}^v$, we have $P(c') = p_{j+1} \ge p_i = P(a^*)$. Thus,

$$p_j = P(a') > P((a')^*)^{1/v} > P(c')^{1/\sqrt{v}} \ge P(a^*)^{1/\sqrt{v}}. \tag{12}$$

But equation (12) contradicts equation (11), so L_{ac} and $\mathrm{L}_{a'c'}$ are disjoint.

Similarly, when $c' = 1$, we have $P((a')^*) = p_{j+1} \ge p_i = P(a^*)$ and so

$$p_j = P(a') > P((a')^*)^{1/v} \ge P(a^*)^{1/v}.$$

This also contradicts equation (11) (indeed $v < \sqrt{v}$). Hence, L_{ac} and $\mathrm{L}_{a'}$ are disjoint in both cases. □

Remark 3.2. The exponent $1/\sqrt{v}$ in the definition of C_a^v in equation (10) is chosen as large as possible, constrained by the final steps (11), (12) above. If one established a larger exponent in Lemma 3.1, this would improve the final savings factor $\int_0^1 \frac{\mathrm{d}}{\mathrm{d}v}[v^{1/2}]\frac{\mathrm{d}v}{1+v} = \pi/4$.

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In the following proposition, we use Lemma 3.1 in order to bound the density of L_{A_n} by essentially a savings factor $\sqrt{v}$ from the trivial bound $\mathrm{d}(\mathrm{L}_n)$, when $P(a)^{1+v} > a$ for all $a \in A_n$.

Proposition 3.3. *Let A be a finite primitive set. Take $v \in (0,1)$, an integer $n > 1$ with $n \notin A$, and denote $q = P(n)$. If $P(a)^{1+v} > a$ for all $a \in A_n$, then*

$$\mathrm{d}(\mathrm{L}_{A_n}) \;\le\; \sqrt{v}\, r_q\, \mathrm{d}(\mathrm{L}_n) \tag{13}$$

for the ratio $r_q := M_q/m_q$ when $q \ge 3$, and $r_2 := r_3$.

Proof. Without loss assume $A = A_n$. Then $ac \in \mathrm{L}_n$ for all $a \in A$, $c \in C_a^v$ (recall $p(ac) = p(a)$), and so $\mathrm{L}_{ac} \subset \mathrm{L}_n$. Note the condition $P(a)^{1+v} > a$ is equivalent to $P(a)^v > a^*$, and $v < 1$ implies $P(a) \nmid a^*$. By Lemma 3.1, we have the following (finite) disjoint union,

$$\mathrm{L}_n \;\supset\; \bigcup_{a\in A}\bigcup_{c\in C_a^v} \mathrm{L}_{ac}. \tag{14}$$

Thus, taking the density of equation (14), we obtain

$$\mathrm{d}(\mathrm{L}_n) \ge \mathrm{d}\Big(\bigcup_{a\in A}\bigcup_{c\in C_a^v}\mathrm{L}_{ac}\Big) = \sum_{a\in A}\sum_{c\in C_a^v}\mathrm{d}(\mathrm{L}_{ac}) = \sum_{a\in A}\mathrm{d}(\mathrm{L}_a)\sum_{c\in C_a}\frac{1}{c}, \tag{15}$$

noting $P(a) > P(c)$ for $1 < c \in C_a^v$, so $\mathrm{L}_{ac} = \{bac : p(b) \ge P(a)\} = c\cdot \mathrm{L}_a$. Then by definitions of C_a^v and μ_q in equations (10) and (8),

$$\begin{aligned}\sum_{c\in C_a^v}\frac{1}{c} &= \prod_{p\in[P(a^*),P(a^*)^{1/\sqrt{v}})}\Big(1-\frac{1}{p}\Big)^{-1} = \prod_{p<P(a^*)^{1/\sqrt{v}}}\Big(1-\frac{1}{p}\Big)^{-1}\prod_{p<P(a^*)}\Big(1-\frac{1}{p}\Big)\\ &= \frac{\log P(a^*)^{1/\sqrt{v}}}{\mu_{P(a^*)^{1/\sqrt{v}}}}\frac{\mu_{P(a^*)}}{\log P(a^*)} = \frac{\mu_{P(a^*)}}{\mu_{P(a^*)^{1/\sqrt{v}}}}\frac{1}{\sqrt{v}}.\end{aligned} \tag{16}$$

When $q \ge 3$, we use $\mu_{P(a^*)}/\mu_{P(a^*)^{1/\sqrt{v}}} \ge m_q/M_q = 1/r_q$, which follows by monotonicity of m_q, M_q in Lemma 2.4, and that $P(a^*), q \in \mathcal{P}$. Hence, plugging equation (16) back into equation (15),

$$\mathrm{d}(\mathrm{L}_n) \ge \frac{1}{\sqrt{v}\,r_q}\sum_{a\in A}\mathrm{d}(\mathrm{L}_a) = \frac{1}{\sqrt{v}\,r_q}\mathrm{d}(\mathrm{L}_A)$$

as desired.

The result similarly holds when $q = 2$: If $P(a^*) \ge 3$, then $\mu_{P(a^*)}/\mu_{P(a^*)^{1/\sqrt{v}}} \ge m_3/M_3 = 1/r_3$ as before. And if $P(a^*) = 2$, then $\mu_2/\mu_{2^{1/\sqrt{v}}} \ge 1$ also suffices. □

4. Deduction of Theorems 1.2, 1.3, 1.5

We now apply our analysis of the density of L-multiples to our original sum of interest $f(A) = \sum_{a\in A}\frac{1}{a\log a}$. First, we need a simple lemma on bounding certain monotonic sequences.

Lemma 4.1. *For $k \ge 1$, let $c_0 \ge c_1 \ge \cdots \ge c_k \ge 0$ and $0 = D_0 \le D_1 \le \cdots \le D_k$. If $d_1, \ldots, d_k \ge 0$ satisfy $\sum_{j\le i} d_j \le D_i$ for all $i \le k$, then we have*

$$\sum_{i\le k} c_i d_i \;\le\; \sum_{i\le k} c_i(D_i - D_{i-1}).$$

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Proof. By rearranging sums,

$$\sum_{i\le k} c_i d_i = \sum_{i\le k} c_i\Big(\sum_{j\le i} d_j - \sum_{j\le i-1} d_j\Big) = \sum_{i\le k-1}(c_i - c_{i+1})\sum_{j\le i} d_j \;+\; c_k\sum_{i\le k} d_i.$$

Since $c_i \ge c_{i+1}$ and $\sum_{j\le i} d_j \le D_i$, we conclude

$$\sum_{i\le k} c_i d_i \le \sum_{i\le k-1}(c_i - c_{i+1})D_i \;+\; c_k D_k = \sum_{i\le k} c_i(D_i - D_{i-1}).$$

□

To motivate the remainder of the proof, we offer a probabilistic interpretation of Proposition 3.3: For $v \ge 0$, consider $D(v) := \sup_A \mathrm{d}(\mathrm{L}_{A_n})/\mathrm{d}(\mathrm{L}_n)$, ranging over primitive sets A such that $P(a)^{1+v} > a$ for all $a \in A$. Note $D(v)$ may be viewed as a 'cumulative distribution function', since $D(0) = 0$ and $D(v) \to 1$ as $v \to \infty$. Now, Proposition 3.3 essentially bounds $D(v)$ by $\sqrt{v}$. Using the corresponding bound $1/2\sqrt{v}$ for the 'probability density function', we establish quantitative bounds below.

Proposition 4.2. *For any primitive set A, and any integer $n \notin A$ with $q = P(n) \ge 3$,*

$$f(A_n) \;\le\; \frac{\pi}{4}\frac{M_q}{m_q^2}\, e^{\gamma}\mathrm{d}(\mathrm{L}_n).$$

Proof. Without loss, we may assume $A = A_n$ is finite since $f(A) = \lim_{x\to\infty} f(A\cap[1,x])$. Also, $n \notin A$ implies all elements of A are composite.

Take $k \ge 1$ and any sequence $0 = v_0 < v_1 < \cdots < v_k = 1$, and partition the set $A = \bigcup_{0\le i\le k} A_{(i)}$, where $A_{(k)} = \{a \in A : P(a)^2 \le a\}$ and for $0 \le i \le k$,

$$A_{(i)} = \{a \in A : P(a)^{1+v_i} \le a < P(a)^{1+v_{i+1}}\}.$$

Then applying Lemma 2.5 to each $A_{(i)}$,

$$f(A) = \sum_{0\le i\le k} f(A_{(i)}) \;\le\; \frac{e^{\gamma}}{m_q}\sum_{0\le i\le k}\frac{\mathrm{d}(\mathrm{L}_{A_{(i)}})}{1+v_i}. \tag{17}$$

Note since A is primitive, $\{\mathrm{L}_{A_{(i)}}\}_{i\le k}$ are pairwise disjoint. Also, for each $j < k$, the first j components are $\bigcup_{0\le i\le j} A_{(i)} = \{a \in A : a < P(a)^{1+v_{j+1}}\} =: A^{(j)}$, so by Proposition 3.3 they have density

$$\sum_{0\le i\le j}\mathrm{d}(\mathrm{L}_{A_{(i)}}) = \mathrm{d}(\mathrm{L}_{A^{(j)}}) \le \sqrt{v_{j+1}}\, r_q\, \mathrm{d}(\mathrm{L}_n).$$

Also, for $j = k$ we have $\sum_{0\le i\le k}\mathrm{d}(\mathrm{L}_{A_{(i)}}) = \mathrm{d}(\mathrm{L}_A) \le \mathrm{d}(\mathrm{L}_n)$, which is trivially less than $r_q\mathrm{d}(\mathrm{L}_n)$. Let $c_i = \frac{1}{1+v_i}$, $d_i = \mathrm{d}(\mathrm{L}_{A_{(i)}})$, $D_i = \sqrt{v_{i+1}}\, r_q\, \mathrm{d}(\mathrm{L}_n)$ (here, we let $v_{k+1} = v_k$ so that $D_k - D_{k-1} = 0$). Thus, by Lemma 4.1 we have

$$\sum_{0\le i\le k}\frac{\mathrm{d}(\mathrm{L}_{A_{(i)}})}{1+v_i} = \sum_{0\le i\le k} c_i d_i \;\le\; \sum_{0\le i\le k} c_i(D_i - D_{i-1}) = r_q\,\mathrm{d}(\mathrm{L}_n)\sum_{0\le i\le k}\frac{\sqrt{v_{i+1}} - \sqrt{v_i}}{1+v_i}.$$

Hence, the weighted sum in equation (17) is bounded by

$$f(A) \le \frac{r_q}{m_q}\, e^{\gamma}\,\mathrm{d}(\mathrm{L}_n)\sum_{1\le i\le k}\frac{\sqrt{v_i} - \sqrt{v_{i-1}}}{1+v_{i-1}}. \tag{18}$$

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As equation (18) holds for any partition $0 = v_0 < v_1 < \cdots < v_k = 1$, we may set $v_i = \frac{i}{k}$ and obtain the corresponding integral,

$$\lim_{k\to\infty}\sum_{1\le i\le k}\frac{\sqrt{v_i}-\sqrt{v_{i-1}}}{1+v_{i-1}} = \lim_{k\to\infty}\sum_{1\le i\le k}\int_{v_{i-1}}^{v_i}\frac{\mathrm{d}}{\mathrm{d}v}\left[\sqrt{v}\right]\frac{\mathrm{d}v}{1+v_{i-1}} = \int_0^1\frac{\mathrm{d}v}{2\sqrt{v}(1+v)} = \frac{\pi}{4}.$$

Hence, we conclude $f(A) \le \frac{\pi}{4}\frac{r_q}{m_q}e^\gamma\,\mathrm{d}(\mathrm{L}_n)$ as desired. □

We illustrate the value of these bounds by deducing Theorem 1.3 in quantitative form.

Corollary 4.3. *Let A be a primitive set, and take an odd prime p. If $p \notin A$, then we have $f(A_p) < .901\,f(p)$, and moreover $f(A_p) \le (\frac{\pi}{4}+o(1))f(p)$ as $p \to \infty$. In addition, if $p > 23$ and $2p \notin A$, then $f(A_{2p}) < f(2p)$.*

Proof. For an odd prime q, define $b_q := \frac{\pi}{4}\frac{M_q}{m_q^2}\mu_q$. Then Proposition 4.2 shows that if $n \notin A$ we have

$$f(A_n) \le \frac{\pi}{4}\frac{M_q}{m_q^2}e^\gamma\mathrm{d}(\mathrm{L}_n) = \frac{q}{n}\,b_q f(q)$$

with $q = P(n) \ge 3$, recalling $\mathrm{d}(\mathrm{L}_n) = \frac{q}{n}\mathrm{d}(\mathrm{L}_q)$ and equation (9). In particular for $n = q, 2q$, we have $f(A_q) \le b_q f(q)$ and $f(A_{2q}) \le \frac{1}{2}b_q f(q)$. Note $\mu_q, m_q, M_q \sim 1$ implies $b_q \sim \frac{\pi}{4}$ as claimed. Also, the first few values of b_q are displayed below.

q	b_q	q	b_q
3	0.9006	23	0.8232
5	0.8795	29	0.8266
7	0.8507	31	0.8139
11	0.8564	37	0.8184
13	0.8327	41	0.8189
17	0.8491	43	0.8092
19	0.8305	47	0.8090

Observe for $q > 7$, we have

$$f(A_q) \le \frac{\pi}{4}\left(\frac{M_q}{m_{11}}\right)^2 f(q) \le \frac{\pi}{4}\left(\frac{1+1/2\log(2\cdot 10^9)^2}{\mu_{19}}\right)^2 f(q) < .879f(q).$$

In particular, with the table, we see $f(A_q) < .901f(q)$ for all $q > 2$ as claimed.

Finally, we note $f(A_{2q}) < f(2q)$ whenever $b_q < \frac{\log q}{\log(2q)}$. The result then follows since

$$b_q = \frac{\pi}{4}\frac{M_q}{m_q^2}\mu_q\ \frac{\log q}{\log(2q)} \qquad \text{iff} \quad q \le 23. \tag{19}$$

Indeed, this may be checked directly for $q < 47$. And for $q \ge 47$ we observe that $\log q/\log(2q) \ge \log 47/\log 94 \ge .847$ exceeds $b_q \le \frac{\pi}{4}(M_{2\cdot 10^9}/m_{47})^2 \le .834$. □

Importantly, $b_q < 1$ for all odd q, which means every odd prime is Erdős strong. However, it remains an open question whether $q = 2$ is Erdős strong. Now, if $2 \in A$ we immediately deduce $f(A) \le f(\mathcal{P}) = 1.6366\cdots$. Thus, to complete the proof of Theorem 1.2, it suffices to assume $2 \notin A$.

We achieve this in the result below. The argument is somewhat similar in spirit to that of Theorem 1.1 and Lemma 2.4 in [27].

Theorem 4.4. *For any primitive set A with $2 \notin A$, we have $f(A) < 1.60$.*

Proof. As $2 \notin A$, denote by $K \geq 2$ the exponent for which $2^K \in A$. Note K is unique by primitivity (Also, if $2^k \notin A$ for all k let $K = \infty$, in which case let $f(2^K) = 0$). Partition A into sets $A^0 = \{a \in A : 2 \nmid a\}$ and $A^k = \{a \in A : 2^k \| a\}$ for $k \geq 1$, and let $B^k = \{a/2^k : a \in A^k\}$. We have

$$\begin{aligned} f(A) &= f(2^K) + \sum_{p \in A} f(p) + \sum_{p \notin A} f(A_p) \\ &\leq f(2^K) + \sum_{\substack{p>2 \\ p \in A}} f(p) + \sum_{\substack{p>2 \\ p \notin A}} b_p f(p) + \sum_{\substack{p>2 \\ p \notin A}} \sum_{k=1}^{K-1} f((A^k)_{2^k p}), \end{aligned} \tag{20}$$

since $f((A^0)_p) \leq b_p f(p)$ if $p \notin A$ by Proposition 4.2. More generally, if $2^k p \notin A$, then

$$f((A^k)_{2^k p}) \leq 2^{-k} f((B^k)_p) \leq 2^{-k} b_p f(p).$$

By comparison, if $2^k p \in A$, then $f((A^k)_{2^k p}) = f(2^k p) \leq 2^{-k} f(p) \frac{\log p}{\log(2p)}$.

Observe that either $2^k p \notin A$ for all $k \geq 1$, or $2^J p \in A$ for a (unique) $J = J_p \in [1, K)$, in which case $(A^k)_{2^k p} = \emptyset$ for all $k > J$ by primitivity. Thus, by equation (19), it suffices to assume $2^k p \notin A$ for all $k \geq 1$ when $p \leq 23$, and $2^J p \in A$ for some $J \in [1, K)$ when $p > 23$, so

$$\begin{aligned} \sum_{\substack{p>2 \\ p \notin A}} \sum_{k=1}^{K-1} f((A^k)_{2^k p}) &\leq (1 - 2^{1-K}) \sum_{\substack{2<p \leq 23 \\ p \notin A}} b_p f(p) + \sum_{\substack{p>23 \\ p \notin A}} f(p) \Big((1 - 2^{1-J}) b_p + 2^{-J} \frac{\log p}{\log(2p)} \Big) \\ &\leq (1 - 2^{1-K}) \sum_{\substack{2<p \leq 23 \\ p \notin A}} b_p f(p) + \sum_{\substack{p>23 \\ p \notin A}} b_p f(p), \end{aligned}$$

since $2b_p > 1 > \frac{\log p}{\log(2p)}$ for all $p > 2$. Moreover, $(2 - 2^{1-K}) b_p \geq (2 - 1/2)\frac{\pi}{4} > 1.1$, so equation (20) becomes

$$\begin{aligned} f(A) &\leq f(2^K) + \sum_{\substack{p>2 \\ p \in A}} f(p) + (2 - 2^{1-K}) \sum_{\substack{2<p \leq 23 \\ p \notin A}} b_p f(p) + 2 \sum_{\substack{p>23 \\ p \notin A}} b_p f(p) \\ &\leq f(2^K) + (2 - 2^{1-K}) \sum_{2<p \leq 23} b_p f(p) + 2 \sum_{p>23} b_p f(p) \\ &=: f(2^K) + (2 - 2^{1-K}) C_1 + 2C_2. \end{aligned} \tag{21}$$

Now, we compute the constants C_1, C_2. First, let $M = M_{2 \cdot 10^9} = 1.001\cdots$. Recalling $\mu_p f(p) = e^\gamma \mathrm{d}(\mathrm{L}_p)$,

$$C_2 := \sum_{p>23} b_p f(p) = \frac{\pi}{4} e^\gamma \sum_{p>23} \frac{M_p}{m_p^2} \mathrm{d}(\mathrm{L}_p) \leq \frac{\pi}{4} \frac{M e^\gamma}{\mu_{23}^2} \prod_{p \leq 23} \big(1 - \tfrac{1}{p}\big) = 0.251135\cdots, \tag{22}$$

since $\sum_{p>q} \mathrm{d}(\mathrm{L}_p) = \prod_{p \leq q} (1 - \frac{1}{p})$. Similarly, we have

$$C_1 := \sum_{2<p \leq 23} b_p f(p) = \sum_{2<p \leq 23} \frac{\pi}{4} \frac{M}{m_p^2} e^\gamma \mathrm{d}(\mathrm{L}_p) = \frac{\pi}{4} M e^\gamma \cdot 0.39012\cdots = 0.5463\cdots \tag{23}$$

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Here, we computed

$$\sum_{2<p\le 23}\frac{1}{m_p^2}\mathrm{d}(\mathrm{L}_p) = \frac{1}{\mu_7^2}\sum_{2<p\le 7}\mathrm{d}(\mathrm{L}_p) + \frac{1}{\mu_{19}^2}\sum_{7<p\le 19}\mathrm{d}(\mathrm{L}_p) + \frac{1}{\mu_{23}^2}\mathrm{d}(\mathrm{L}_{23}) = 0.390126\cdots,$$

using $\sum_{q<p\le q'}\mathrm{d}(\mathrm{L}_p) = \prod_{p\le q}(1-\frac{1}{p}) - \prod_{p\le q'}(1-\frac{1}{p})$.

Hence, plugging equations (22) and (23) back into equation (21),

$$\begin{aligned} f(A) &\le f(2^K) + (2-2^{1-K})C_1 + 2C_2 \\ &\le 2^{-K}\Big(\frac{1}{\log 4} - 2C_1\Big) + 2(C_1+C_2) \le 2(C_1+C_2) \le 1.595. \end{aligned} \tag{24}$$

Here, we used $2C_1 > .722 > 1/\log 4$. This completes the proof. □

Remark 4.5. A similar argument as in Theorem 4.4 shows $f(A_2) < C_1 + C_2 < 0.80$ when $2 \notin A$. We leave this to the interested reader. Note this bound improves on $f(A_2) < e^\gamma/2 \approx 0.89$ from [27, Proposition 2.1] but unfortunately still exceeds $f(2) \approx 0.72$.

4.1. Proof of Theorem 1.5

Take $\epsilon > 0$. We shall introduce large parameters $y = y_\epsilon$, $k = k_{\epsilon,y}$ and $x = x_{\epsilon,k}$.

By Lemma 2.3, we have $f(A_n) \le e^\gamma \mathrm{d}(\mathrm{L}_n)$ for any integer $n \notin A$, $n > 1$ and when $y = y_\epsilon \in \mathbb{R}$ is sufficiently large by Proposition 4.2 we have the sharper bound

$$f(A_n) \;\le\; (\frac{\pi}{4}e^\gamma + \epsilon)\mathrm{d}(\mathrm{L}_n) \qquad \text{provided} \quad P(n) > y. \tag{25}$$

Next, by [?, Lemma 2], for $k = k_\epsilon = k_{\epsilon,y} \in \mathbb{N}$ sufficiently large,

$$\sum_{\substack{n\in\mathbb{N}_k\\ P(n)\le y}}\mathrm{d}(\mathrm{L}_n) \ll \frac{1}{k}\sum_{\substack{\Omega(n)=k\\ P(n)\le y}}\frac{1}{n} \ll (\log y)^2\, 2^{-k} < \epsilon. \tag{26}$$

Finally, since $f(\mathbb{N}_j) < 2$ crudely for all j there exists $x = x_{\epsilon,k} \in \mathbb{R}$ sufficiently large so that $f\big(\bigcup_{j\le k}\mathbb{N}_j \cap [x,\infty)\big) < \epsilon$.

Now, take a primitive set $A \subset [x,\infty)$, and consider the partition $A = A' \cup \bigcup_{n\in\mathbb{N}_k\setminus A} A_n$, where A' consists of elements $a \in A$ with at most k prime factors, and each other element $a \in A$ (with at least $k+1$ prime factors) then lies in $A_n = A \cap \mathrm{L}_n$, where $n \notin A$ is the product of the smallest k primes of a. Hence, we conclude

$$\begin{aligned} f(A) &= f(A') + \sum_{n\in\mathbb{N}_k\setminus A} f(A_n) \\ &\le f\Big(\bigcup_{j\le k}\mathbb{N}_j \cap [x,\infty)\Big) + \sum_{\substack{n\in\mathbb{N}_k\setminus A\\ P(n)\le y}} f(A_n) + \sum_{\substack{n\in\mathbb{N}_k\setminus A\\ P(n)> y}} f(A_n) \\ &\le \epsilon + e^\gamma \sum_{\substack{n\in\mathbb{N}_k\\ P(n)\le y}}\mathrm{d}(\mathrm{L}_n) + (\frac{\pi}{4}e^\gamma + \epsilon)\sum_{\substack{n\in\mathbb{N}_k\\ P(n)> y}}\mathrm{d}(\mathrm{L}_n) \;\le\; \epsilon + e^\gamma\,\epsilon + (\frac{\pi}{4}e^\gamma + \epsilon), \end{aligned}$$

by equations (25) and (26) and noting $\sum_{n\in\mathbb{N}_k, P(n)>y}\mathrm{d}(\mathrm{L}_n) \le 1$. Hence, letting $\epsilon \to 0$ completes the proof of Theorem 1.5.

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5. L-primitive sets revisited

5.1. Upper density

As mentioned in the introduction, one of the striking early results in the study of primitive sets was due to Besicovitch [5], who showed

$$\sup_{A \text{ primitive}} \overline{\mathrm{d}}(A) = \frac{1}{2}.$$

This came as quite a surprise at the time, in particular disproving a conjecture of Davenport. We shall extend this phenomenon further to L-primitive sets, in Proposition 5.2.

To proceed, we recall a result of Erdős [14], which bounds the density of the set of multiples of an interval. Also, see Hall–Tenenbaum [22, Theorem 21] for quantitatively stronger results. Denote the set of (all) multiples of $A \subset \mathbb{N}$ as $\mathrm{M}_A = \{na : n \in \mathbb{N}, a \in A\}$.

Proposition 5.1 (Erdős, 1936). *Let $\varepsilon(x)$ be any function with $\varepsilon(x) \to 0$ as $x \to \infty$. Then the upper density of $\mathrm{M}_{(x^{1-\varepsilon(x)},x]}$ tends to zero as $x \to \infty$.*

We prove a Besicovitch-type result for L-primitive sets, notably with full upper density.

Proposition 5.2. *We have $\sup_A \overline{\mathrm{d}}(A) = 1$ over L-primitive sets A.*

Proof. Take $h \in \mathbb{Z}_{>1}$, $\epsilon > 0$, and let $S = \{n \in \mathbb{N} : P(n) \le h\}$ be the set of h-smooth numbers. For a sequence of indices $k_1, k_2, \ldots$ to be determined, define intervals $I_i = (h^{k_i - 1}, h^{k_i}]$. Let $S_i := I_i \setminus S$, and note for $a \in S_i$ and $n \in \mathrm{L}_a$ we have $n \ge P(a)a > h^{k_i}$, so $n \notin I_i \supset S_i$. In particular, $a' \notin \mathrm{L}_a$ for distinct $a, a' \in S_i$, so each set S_i is L-primitive. Now, define the L-primitive set

$$A = \bigcup_{j \ge 1} \Big(S_j \setminus \bigcup_{1 \le i < j} \mathrm{M}_{I_i} \Big). \tag{27}$$

Note for each fixed $h > 1$ the set S has zero density, so $|S \cap [1,x]| < \epsilon x$ for $x \ge x_{h,\epsilon}$ sufficiently large. Also, by Proposition 5.1 we see $\overline{\mathrm{d}}(\mathrm{M}_{(x/h,x]}) \to 0$ as $x \to \infty$. So for k_i large enough, we may assume $\overline{\mathrm{d}}(\mathrm{M}_{I_i}) < \epsilon/2^i$.

For each i, the set of multiples M_{I_i} is a periodic set with period (dividing) $(h^{k_i})!$. So assuming $k_{i+1} \ge (h^{k_i})!$ the relative density of M_{I_i} inside I_{i+1} is at most $2\overline{\mathrm{d}}(\mathrm{M}_{I_i})$. Hence,

$$\begin{aligned} |A \cap [1, h^{k_j}]| &\ge |I_j| - \big|S \cap [1, h^{k_j}]\big| - 2h^{k_j} \sum_{1 \le i < j} \overline{\mathrm{d}}(\mathrm{M}_{I_i}) \\ &\ge (h^{k_j} - h^{k_j - 1}) - \epsilon h^{k_j} - 2\epsilon h^{k_j} \sum_{i \ge 1} 2^{-i}. \end{aligned}$$

Thus, dividing by $x = h^{k_j}$ we see $\overline{\mathrm{d}}(A) = \limsup_{x\to\infty} |A \cap [1,x]|/x \ge 1 - 1/h - 3\epsilon$. Taking $h \to \infty$ and $\epsilon \to 0$ completes the proof. □

5.2. The Erdős L-primitive set conjecture

Sets of L-multiples play a central role in our proof of Theorem 1.2, as the mathematical structures arising from a probabilistic interpretation of equation (4),[6] and implicit in the original 1935 argument of Erdős [13].[7] As such, it is natural to pose the L-primitive analogue of Conjecture 1.1, namely that $f(A) \le f(\mathcal{P})$ for all L-primitive sets A.

[6] A variant in [17], a set A 'possesses property I' if there is no solution to $a' = ba$ for $a, a' \in A$ with $p(b) > P(a)$. This is similar to A as L-primitive, but the latter imposes the inclusive inequality $p(b) \ge P(a)$, which arises naturally from a probabilistic viewpoint. This inclusivity leads to key structural properties, notably the trichotomy in Lemma 2.1.

[7] The author was also recently shown 'prefix-free sets' in [1], which coincides with L-primitive for sets of square-free numbers.

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However, this conjecture turns out to be false.

Proposition 5.3. *We have*

$$\sup_{\mathcal{A}\mathrm{L-primitive}} f(A) = \sum_p \max\{f(p), e^\gamma \mathrm{d}(\mathrm{L}_p)\}, \quad \textit{and} \quad \lim_{x\to\infty} \sup_{\substack{A\subset[x,\infty)\\ \mathcal{A}\mathrm{L-primitive}}} f(A) = e^\gamma. \tag{28}$$

Note the prime sum in equation (28) above is at least (and well approximated by) $f(\mathcal{P}) - f(2) + e^\gamma/2 \approx 1.805$. In particular, it exceeds $f(\mathcal{P}) \approx 1.636$. As such, Proposition 5.3, along with Conjecture 7.1 and related work in the literature, highlights how the Erdős primitive set conjecture is quite fragile under certain seemingly natural directions of generalization.

We now proceed to set up the proof of Proposition 5.3. First, the trichotomy in Lemma 2.1 leads to the following.

Lemma 5.4. *Every set $S \subset \mathbb{N}$ has a unique* L*-primitive subset $\langle S\rangle$ with $\mathrm{L}_{\langle S\rangle} = \mathrm{L}_S$. In particular, $\langle S\rangle = S$ if S is L-primitive.*

Proof. For any $s_1, s_2 \in S$, by Lemma 2.1 either $\mathrm{L}_{s_1} \cap \mathrm{L}_{s_2} = \emptyset$ or $\mathrm{L}_{s_1} \subset \mathrm{L}_{s_2}$ (or vice versa). Thus, each $s \in S$ has a (unique) smallest L-divisor $s' \in S$, inducing a map $S \to S : s \mapsto s'$. We define $\langle S\rangle$ as the image of this map. Explicitly, this is

$$\langle S\rangle := \{s \in S : s \notin \mathrm{L}_t \ \forall\, t < s, t \in S\}. \tag{29}$$

By minimality, $\mathrm{L}_{s_1} \cap \mathrm{L}_{s_2} = \emptyset$ for all $s_1, s_2 \in \langle S\rangle$, so $\langle S\rangle$ is L-primitive. Moreover, $\mathrm{L}_S = \bigcup_{s\in S} \mathrm{L}_s = \bigcup_{s'\in\langle S\rangle} \mathrm{L}_{s'} = \mathrm{L}_{\langle S\rangle}$, where the latter union over $\langle S\rangle$ is disjoint by L-primitivity. This completes the proof. □

Next, take $v > 0$, $n \in \mathbb{Z}_{>1}$, and consider the set $D_v(n)$ of prime divisors of n whose induced L-divisor is not smooth, that is,

$$D_v(n) = \Big\{p \mid n \ : \prod_{q^e \| n, q<p} q^e \le p^v\Big\}. \tag{30}$$

We cite the following result of Bovey, based on earlier work of Erdős [22, §1.2].

Proposition 5.5 (Bovey, 1977). *For each $v > 0$, there is a set $N_v \subset \mathbb{N}$ of full density with*

$$\frac{|D_v(n)|}{\log\log n} \to e^{-\gamma}\int_0^v \rho(x)\mathrm{d}x \tag{31}$$

as $n \to \infty$ on N_v. Here, ρ is the Dickman–de Bruijn function.

Remark 5.6. In probability, the right-hand side of equation (31) is called the Dickman distribution.

In particular, $|D_v(n)| \gg_v \log\log n$ for all $n \in N_v$. Now, we may define a map $\beta : N_u \to \mathbb{N}$ sending n to its L-divisor $\beta(n) = p\prod_{q^e\|n, q<p} q^e$, for the largest prime $p \in D_v(n)$.

Define the L-primitive generating set $B(v) := \langle\beta(N_v)\rangle$ as in Lemma 5.4. By construction, $\mathrm{L}_{B(v)} = \mathrm{L}_{\beta(N_v)} \supset N_u$ has full density. Also, by definition of β, D_v,

$$B(v) \subset \beta(N_v) \subset \{n \in \mathbb{N} : n \le P(n)^{1+v}\}. \tag{32}$$

We are now prepared to establish a local version of Proposition 5.3.

Proposition 5.7. *For each prime q, we have*

$$\lim_{y\to\infty} \sup_{\substack{A\subset[y,\infty)\\ \mathrm{L-primitive}A\not\ni q}} f(A_q) = \sup_{\mathrm{L-primitive}A\not\ni q} f(A_q) = e^\gamma \mathrm{d}(\mathrm{L}_q).$$

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Proof. By Lemma 2.3, we have $f(A_p) < e^{\gamma}\mathrm{d}(\mathrm{L}_p)$ for all L-primitive A not containing p. It now suffices to provide L-primitive sets $B \subset [y, \infty)$ with $f(B_q) \to e^{\gamma}\mathrm{d}(\mathrm{L}_q)$ as $y \to \infty$.

Fix $v > 0$. The L-primitive set $B(v)$ in equation (32) satisfies

$$f(B(v)_q) = \sum_{b \in B(v)_q} \frac{1}{b \log b} \geq \frac{1}{1+v} \sum_{b \in B(v)_q} \frac{1}{b \log P(b)}. \tag{33}$$

Next, for $x > e^{e^{e^y}}$ we may assume $N_v \subset [x, \infty)$ and retain full density. Observe then $B(v) \subset [y, \infty)$ is our candidate L-primitive set. Indeed, for each $n \in N_v$ by construction $\beta(n)$ is divisible by all primes $q \in D_v(n)$, so $\beta(n)$ is composite with $\beta(n) \geq |D_v(n)| \gg_v \log\log n \geq \log\log\log x > y$ for each $b \in B(v)$, for y sufficiently large. And note Mertens' product theorem gives

$$\mathrm{d}(\mathrm{L}_b) = \frac{1}{b} \prod_{p < P(b)} \Big(1 - \frac{1}{p}\Big) = \frac{e^{-\gamma} + o_y(1)}{b \log P(b)}.$$

Plugging back into equation (33), we obtain

$$f(B(v)_q) \geq \frac{e^{\gamma} + o_y(1)}{1+v} \sum_{b \in B(v)_q} \mathrm{d}(\mathrm{L}_b).$$

Recall $\mathrm{L}_{B(v)} = \mathrm{L}_{\beta(N_v)} \supset N_v$ has full density, which implies $(\mathrm{L}_{B(v)})_q = \mathrm{L}_{B(v)_q}$ has full relative density $\mathrm{d}(\mathrm{L}_{B(v)_q}) = \mathrm{d}(\mathrm{L}_q)$. Hence, by Corollary 2.2 this latter sum is

$$\sum_{b \in B(v)_q} \mathrm{d}(\mathrm{L}_b) = \mathrm{d}(\mathrm{L}_{B(v)_q}) = \mathrm{d}(\mathrm{L}_q).$$

Thus, taking $y \to \infty$ and $v \to 0$ gives $f(B(v)_q) \to e^{\gamma}\mathrm{d}(\mathrm{L}_q)$ as desired. □

Proof of Proposition 5.3. Take L-primitive $A \subset [x, \infty)$, so $p \geq x$ for all $p \in A$. Then by Lemma 2.3,

$$\begin{aligned} f(A) = \sum_p f(A_p) &= \sum_{\substack{p<x \\ \text{or } p \notin A}} f(A_p) + \sum_{p \in A, p \geq x} f(p) \\ &\leq e^{\gamma} \sum_{\substack{p<x \\ \text{or } p \notin A}} \mathrm{d}(\mathrm{L}_p) + \sum_{p \in A, p \geq x} f(p) \\ &\leq e^{\gamma} \sum_p \mathrm{d}(\mathrm{L}_p) + \big(e^{\gamma} + o_x(1)\big) \sum_{p \geq x} \mathrm{d}(\mathrm{L}_p) \;\leq\; e^{\gamma} + o_x(1) \end{aligned}$$

by Mertens' theorem, and noting $\sum_p \mathrm{d}(\mathrm{L}_p) = 1$. Thus, $\lim_x \sup_{A \subset [x,\infty)} f(A) \leq e^{\gamma}$. Equality in the limsup holds for the choice of $B = \bigcup_q B(v)_q$ and taking $v \to 0$ as in Proposition 5.7. Observe such B inherits L-primitivity from the $B(v)_q$. Note in general, a union $B = \bigcup_q B_q$ is L-primitive if each B_q is L-primitive. (By contrast, $B = \bigcup_q B_q$ is not necessarily primitive even if each B_q is primitive, e.g., $B = \{3, 6\}$.)

Next, consider the primes $\mathcal{Q} = \{q : f(q) > e^{\gamma}\mathrm{d}(\mathrm{L}_q)\} = \{q : 1/\log q > e^{\gamma}\prod_{p<q}(1 - \frac{1}{p})\}$. By Lemma 2.3 $f(A_q) < e^{\gamma}\mathrm{d}(\mathrm{L}_q)$ when $q \notin A$, so in general $f(A_q) < \max\{f(q), e^{\gamma}\mathrm{d}(\mathrm{L}_q)\}$ for all L-primitive A and all primes q. Hence,

$$f(A) = \sum_q f(A_q) < \sum_q \max\{f(q), e^{\gamma}\mathrm{d}(\mathrm{L}_q)\} = f(\mathcal{Q}) + e^{\gamma}\big(1 - \mathrm{d}(\mathrm{L}_{\mathcal{Q}})\big).$$

This bound is attained for the choice of $B' = \mathcal{Q} \cup \bigcup_{q \notin \mathcal{Q}} B(v)_q$, and taking $v \to 0$ as in Proposition 5.7. Again, B' inherits L-primitivity from the $B(v)_q$, as desired. □

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6. Deduction of Theorems 1.8, 1.10

Our study of sets of L-multiples leads to Theorem 1.8, refining Davenport–Erdős. This in turn enables the proof of Theorem 1.10, by a modification of the argument in [16, Theorem 2], with greater care given to the constants involved.

To proceed, we first establish some lemmas.

Lemma 6.1. *For any L-primitive $A \subset \mathbb{N}$, we have* $\underline{\mathrm{d}}(\mathrm{L}_A) \ge \sum_{a\in A} \mathrm{d}(\mathrm{L}_a)$. *Moreover, if* $\sum_{a\in A} 1/a < \infty$, *then the natural density* $\mathrm{d}(\mathrm{L}_A)$ *exists and equals* $\sum_{a\in A} \mathrm{d}(\mathrm{L}_a)$.

Proof. For each $a \in A$, we have $\underline{\mathrm{d}}(\mathrm{L}_a) = \mathrm{d}(\mathrm{L}_a)$. So taking the lower density of the finite (disjoint) union $\bigcup_{a\in A, a\le x} \mathrm{L}_a \subset \mathrm{L}_A$, we have $\sum_{a\in A, a\le x} \mathrm{d}(\mathrm{L}_a) \le \underline{\mathrm{d}}(\mathrm{L}_A)$ for all $x > 1$. Thus, $\sum_{a\in A} \mathrm{d}(\mathrm{L}_a) \le \underline{\mathrm{d}}(\mathrm{L}_A)$. Moreover, if $\sum_{a\in A} 1/a < \infty$, then for all $y > 1$

$$\frac{1}{x} \sum_{\substack{n\le x\\ n\in \mathrm{L}_{A\cap(y,\infty)}}} 1 \le \frac{1}{x} \sum_{a\in A, a>y} \left\lfloor \frac{x}{a} \right\rfloor \le \sum_{a\in A, a>y} \frac{1}{a} = o_y(1).$$

Thus, $\overline{\mathrm{d}}(\mathrm{L}_{A\cap(y,\infty)}) \to 0$ as $y \to \infty$, and so combining with $\mathrm{d}(\mathrm{L}_{A\cap[1,y]}) = \sum_{a\in A, a\le y} \mathrm{d}(\mathrm{L}_a)$ completes the proof. □

The following lemma shows that sets of L-multiples have a log density, refining Davenport–Erdős' elementary proof for sets of (all) multiples [12].

Lemma 6.2. *For any L-primitive $A \subset \mathbb{N}$, the* log *density* $\delta(\mathrm{L}_A)$ *exists and equals* $\sum_{a\in A} \mathrm{d}(\mathrm{L}_a)$.

Proof. In general, $\underline{\mathrm{d}}(S) \le \underline{\delta}(S) \le \overline{\delta}(S) \le \overline{\mathrm{d}}(S)$ for any $S \subset \mathbb{N}$. So for $S = \mathrm{L}_A$, by Lemma 6.1 it suffices to show

$$\sum_{a\in A} \mathrm{d}(\mathrm{L}_a) \ \ge\ \overline{\delta}(\mathrm{L}_A). \tag{34}$$

To this, for $y > 1$ let $A^y = \{a \in A : P(a) \le y\}$ and $L^y = \{n \in \mathrm{L}_A : P(n) \le y\}$. Note $L^y \subset \mathrm{L}_{A^y}$. Also $\sum_{a\in A^y} \frac{1}{a} \le \prod_{p\le y}(1-\frac{1}{p})^{-1} = O_y(1)$, so by Lemma 6.1 $\mathrm{d}(\mathrm{L}_{A^y})$ exists and equals $\sum_{a\in A^y} \mathrm{d}(\mathrm{L}_a)$ for all $y > 1$. In particular, $\mathrm{d}(\mathrm{L}_{A^y}) \to \sum_{a\in A} \mathrm{d}(\mathrm{L}_a)$ as $y \to \infty$.

Now, observe each $n \in \mathrm{L}_{A^y}$ is a L-multiple of a unique $a \in A^y$, so for $x \ge y > 1$ we have

$$\begin{aligned}\sum_{n\in L^x\cap \mathrm{L}_{A^y}} \frac{1}{n} &= \sum_{a\in A^y} \frac{1}{a} \prod_{P(a)\le p\le x} (1-\tfrac{1}{p})^{-1} = \sum_{a\in A^y} \frac{1}{a} \prod_{p<P(a)} (1-\tfrac{1}{p}) \prod_{p\le x}(1-\tfrac{1}{p})^{-1}\\ &= \mathrm{d}(\mathrm{L}_{A^y}) \prod_{p\le x}(1-\tfrac{1}{p})^{-1}.\end{aligned} \tag{35}$$

In particular, for $x = y$ we have $\sum_{n\in L^x} \frac{1}{n} = \mathrm{d}(\mathrm{L}_{A^x}) \prod_{p\le x}(1-\frac{1}{p})^{-1}$.

Then for all $x \ge y > 1$, by equation (35) and Mertens' theorem

$$\begin{aligned}\sum_{n\in L^x\setminus \mathrm{L}_{A^y}} \frac{1}{n} &= \sum_{n\in L^x} \frac{1}{n} - \sum_{n\in L^x\cap \mathrm{L}_{A^y}} \frac{1}{n}\\ &= \big(\mathrm{d}(\mathrm{L}_{A^x}) - \mathrm{d}(\mathrm{L}_{A^y})\big) \prod_{p\le x}(1-\tfrac{1}{p})^{-1} \ \ll\ (\log x)\big(\mathrm{d}(\mathrm{L}_{A^x}) - \mathrm{d}(\mathrm{L}_{A^y})\big).\end{aligned} \tag{36}$$

Recall the natural density $\mathrm{d}(\mathrm{L}_{A^y})$ exists, in which case equals the log density $\delta(\mathrm{L}_{A^y})$. Hence, by equation

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(36), for each $y > 1$ the upper log density is

$$\begin{aligned}\overline{\delta}(\mathrm{L}_A) &= \limsup_{x\to\infty} \frac{1}{\log x}\sum_{\substack{n\le x\\ n\in \mathrm{L}_A}}\frac{1}{n} \le \lim_{x\to\infty}\frac{1}{\log x}\sum_{\substack{n\le x\\ n\in \mathrm{L}_{A^y}}}\frac{1}{n} + \limsup_{x\to\infty}\frac{1}{\log x}\sum_{n\in L^x\setminus \mathrm{L}_{A^y}}\frac{1}{n}\\ &= \delta(\mathrm{L}_{A^y}) + \lim_{x\to\infty} O\big(\mathrm{d}(\mathrm{L}_{A^x}) - \mathrm{d}(\mathrm{L}_{A^y})\big)\\ &= \mathrm{d}(\mathrm{L}_{A^y}) + O\Big(\sum_{a\in A}\mathrm{d}(\mathrm{L}_a) - \mathrm{d}(\mathrm{L}_{A^y})\Big). \end{aligned} \tag{37}$$

Hence, $\mathrm{d}(\mathrm{L}_{A^y}) \to \sum_{a\in A}\mathrm{d}(\mathrm{L}_a)$ as $y\to\infty$ implies $\overline{\delta}(\mathrm{L}_A) \le \sum_{a\in A}\mathrm{d}(\mathrm{L}_a)$, giving (34). □

Theorem 1.8. *If* $\overline{\delta}(A) > 0$, *then* A *contains an infinite* L*-divisibility chain.*

Proof. We claim all such $A\subset\mathbb{N}$ contain an element $a\in A$ such that $A\cap \mathrm{L}_a$ has positive upper log density. (In other words, if $\overline{\delta}(A) > 0$, then there exists an element $a\in A$ such that $\overline{\delta}(A\cap \mathrm{L}_a) > 0$.)

Assume this claim holds. Letting $A^1 = A$, $a_1 = a$, and for $i\ge 1$ suppose $\overline{\delta}(A^i) > 0$. By the claim, there exists $a_i\in A^i$ such that $A^{i+1} := A^i\cap \mathrm{L}_{a_i}$ has positive upper log density. Hence, by induction, we obtain an L-divisibility chain $a_1, a_2, \cdots$, as desired.

Thus, it remains to establish the above claim. For sake of contradiction, suppose $A\cap \mathrm{L}_a$ has zero log density for all $a\in A$. Next, for the L-primitive generating set $B = \langle A\rangle$ by Lemma 6.1 $\delta(\mathrm{L}_B) = \sum_{b\in B}\mathrm{d}(\mathrm{L}_b)$ exists. Then for $z > 1$ large enough we have $\delta(\mathrm{L}_{B\cap(z,\infty)}) = \sum_{b\in B, b>z}\mathrm{d}(\mathrm{L}_b) < \overline{\delta}(A)$. Now, by assumption $\overline{\delta}(A\cap \mathrm{L}_b) = 0$ for all $b\le z$, $b\in B$, and so

$$\overline{\delta}(A) = \overline{\delta}(A\cap \mathrm{L}_{B\cap(z,\infty)}) \le \delta(\mathrm{L}_{B\cap(z,\infty)}) < \overline{\delta}(A),$$

a contradiction. Hence, there exists $a\in A$ such that $A\cap \mathrm{L}_a$ has positive upper log density. □

Theorem 1.10. *If* $\overline{\Delta}(A) > 0$, *then there is an infinite* L*-divisibility chain* $D\subset A$ *of growth*

$$\limsup_{y\to\infty}\sum_{\substack{d\in D\\ d\le y}}\frac{1}{\log\log y} \ge \frac{\overline{\Delta}(A)}{e^\gamma}.$$

Proof. Take $\epsilon > 0$. Without loss, we may suppose $A\subset [x_\epsilon,\infty)$ for x_ϵ sufficiently large so that by Proposition 5.3 $f(A') \le e^\gamma + \epsilon$ for all L-primitive subsets $A'\subset A$.

By definition of upper log log density $\Delta := \overline{\Delta}(A) > 0$, there exists an unbounded sequence $(x_j)_{j=0}^\infty \subset \mathbb{R}$ such that for all $j\ge 0$,

$$f(A\cap[1,x_j]) = \sum_{\substack{a\in A\\ a\le x_j}}\frac{1}{a\log a} > (\Delta-\epsilon)\log\log x_j. \tag{38}$$

Recall the L-primitive generating set $\langle S\rangle = \{s\in S : s\notin \mathrm{L}_t\ \forall t < s, t\in S\}$ of a set $S\subset\mathbb{N}$ from Lemma 5.4. We partition $A = \bigcup_{i\ge 0}A^i$ into a disjoint collection of L-primitive subsets, where $A^0 = \langle A\rangle$ and inductively $A^l = \langle A\setminus \bigcup_{i<l}A^i\rangle$. By construction, each $a = a_l\in A^l$ has a (finite) chain of L-divisors $a_i\in A^i$ with $\mathrm{L}_{a_0}\supset\cdots\supset \mathrm{L}_{a_l} = \mathrm{L}_a$. Also, note $f(A^i)\le e^\gamma + \epsilon$ by assumption, so in particular A^i has zero log log density. Hence, equation (38) implies each A^i in $A = \bigcup_{i\ge 0}A^i$ is nonempty. Next, define the subset $B = \bigcup_{j\ge 0}B_j$ for

$$B_j := A\cap[1,x_j]\setminus \bigcup_{1\le i<r_j}A^i, \qquad \text{where}\quad r_j := \frac{\Delta-2\epsilon}{e^\gamma+\epsilon}\log\log x_j.$$

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Note the sets B_j are pairwise disjoint: Indeed, since $A = \bigcup_{i\geq 0} A^i$, for each j we have $A \cap [1, x_j] \subset \bigcup_{i<s_j} A^i$ for some finite s_j, as determined by x_j. Then since $(x_j)_j$ is unbounded, (passing to a subsequence) we have $r_{j+1} > s_j$ and so $A \cap [1, x_j] \subset \bigcup_{i<r_{j+1}} A^i$. Thus, $B_j = A \cap [1, x_j] \setminus \bigcup_{i<r_j} A^i \subset \bigcup_{r_j \leq i < r_{j+1}} A^i$ inherits disjointness from the A^i, as claimed.

Since $B = \bigcup_{j\geq 0} B_j$ forms a disjoint union, for each $b \in B$ there is a unique index $J(b)$ such that $b \in B_{J(b)}$, that is,

$$b \leq x_{J(b)} \quad \text{and} \quad b \notin \bigcup_{i<r_{J(b)}} A^i. \tag{39}$$

In addition, B has positive upper log log density, since by definitions of B, r_j and equation (38),

$$\begin{aligned} f(B \cap [1, x_j]) \geq f(B_j) &\geq f(A \cap [1, x_j]) - \sum_{i<r_j} f(A^i) \\ &> (\Delta - \epsilon) \log\log x_j - r_j(e^\gamma + \epsilon) = \epsilon \log\log x_j. \end{aligned}$$

In particular, B has positive upper log density, so by Theorem 1.8 there exists an *infinite* L–divisibility chain $D \subset B$. Since $D := (d_k)_{k=0}^\infty$ is unbounded, (by passing to a subchain) we may assume $J(d_k) < J(d_{k+1})$ for all $k \geq 0$. Recall each $a \in A^i$ is at the end of an L-divisibility chain of length i. As $b \in B_{J(b)}$ and B_j is contained in $\bigcup_{r_j \leq i < r_{j+1}} A^i$, we infer each $d \in D \subset B$ is at the end of an L-divisibility chain of length (at least) $r_{J(d)}$. Write it as $c_0^{(k)} \mid c_1^{(k)} \mid \cdots \mid c_{r_{J(d_k)}}^{(k)} = d_k$, with

$$\mathrm{L}_{c_0^{(k)}} \supset \cdots \supset \mathrm{L}_{d_k}.$$

Now, let i_k be the least index such that $c_{i_k}^{(k)} > d_{k-1}$ and define

$$C := \{d_{k-1} < c_i^{(k)} \leq d_k \ : \ k, i \geq 0\} = \bigcup_{k\geq 0} \{c_i^{(k)} \ : \ i \in [i_k, r_{J(d_k)}]\}.$$

We may assume $(r_j)_j$ grows fast enough so that $\lfloor \epsilon\, r_{J(d_k)} \rfloor > d_{k-1}$. Then the trivial bound $c_i^{(k)} > i$ implies $c_{\lfloor \epsilon\, r_{J(d_k)} \rfloor}^{(k)} > d_{k-1}$, and so $\lfloor \epsilon\, r_{J(d_k)} \rfloor \geq i_k$. Thus,

$$\left|C \cap [1, x_{j(d_k)}]\right| \geq \left|C \cap [d_{k-1}, d_k]\right| \geq (1-\epsilon) r_{J(d_k)} = (1-\epsilon) \frac{\Delta - 2\epsilon}{e^\gamma + \epsilon} \log\log x_{j(d_k)}. \tag{40}$$

Hence, taking $\epsilon \to 0$ in equation (40) above gives $\limsup_{x\to\infty} \sum_{c\in C, c\leq x} 1/\log\log x \geq \Delta/e^\gamma$ as desired.

Finally, note C forms an infinite L-divisibility chain: For each k we have $c_j^{(k)} \in \mathrm{L}_{c_i^{(k)}}$ for all $i_k \leq i < j$, in particular $d_k \in \mathrm{L}_{c_i^{(k)}}$. Also, $d_k \in \mathrm{L}_{d_{k-1}}$ since D is an L-divisibility chain, so there exist factorizations

$$d_k = g c_i^{(k)} = h d_{k-1},$$

with $p(g) \geq P(c_i^{(k)})$ and $p(h) \geq P(d_{k-1})$. As $c_i^{(k)} > d_{k-1}$, we deduce $c_i^{(k)} \in \mathrm{L}_{d_{k-1}}$. Thus, the kth and $(k-1)$th pieces of C are linked together. Hence, C is indeed an L-divisibility chain. □

7. Closing remarks

In this discussion, we attempt to sample just a few of the multitude of open questions that have quickly arisen in connection with the Erdős primitive set conjecture. We have already described a few in the introduction, including Conjecture 1.4, as well as whether $p = 2$ is Erdős strong. We also note recent work has studied variants of the problem in function fields $\mathbb{F}_q[x]$; see [6], [7]. In addition, it would be interesting to further extend the classical study of sets of (all) multiples and of primitive sets,for example, see Hall [21] or Halberstam–Roth [20, §5], to sets of L-multiples and L-primitive sets.

We conclude with a related question of Banks and Martin, which offers a potential unified framework to view the results described in this article. For $k \geq 1$, recall $\mathbb{N}_k = \{n : \Omega(n) = k\}$, in particular $\mathbb{N}_1 = \mathcal{P}$. In 1993, Zhang [33] proved $f(\mathbb{N}_k) < f(\mathcal{P})$ for each $k > 1$. Later Bayless, Kinlaw and Klyve [3] showed that $f(\mathbb{N}_2) > f(\mathbb{N}_3)$. Banks and Martin [2] predicted $f(\mathbb{N}_k) < f(\mathbb{N}_{k-1})$ for each $k > 1$. In fact, they posed a vast generalization to Conjecture 1.1.

Conjecture 7.1 (odd Banks–Martin). *Let $k \geq 1$ and suppose A is a primitive set with $\Omega(n) \geq k$ for all $n \in A$. Then for any set of odd primes $\mathcal{Q}$, we have*

$$f(A(\mathcal{Q})) \leq f\big(\mathbb{N}_k(\mathcal{Q})\big). \tag{41}$$

Here, $A(\mathcal{Q})$ denotes the set of members of A composed of primes in $\mathcal{Q}$.

Banks and Martin managed to show equation (41) in the special case when the set of primes $\mathcal{Q}$ is quite sparse, namely $\sum_{p\in\mathcal{Q}} 1/p < 1.74$ (even when $2 \in \mathcal{Q}$). We note the original formulation of Conjecture 7.1 included the cases $2 \in \mathcal{Q}$, but this turns out to be false. Indeed, when $\mathcal{Q} = \mathcal{P}$ it was shown $f(\mathbb{N}_k) > f(\mathbb{N}_6)$ for each $k \neq 6$ [25]. In fact, numerical evidence suggests that in fact the reverse holds $f(\mathbb{N}_k) > f(\mathbb{N}_{k-1})$ for $k > 6$. Nevertheless, for $\mathcal{Q} = \mathcal{P} \setminus \{2\}$, the desired inequality $f(\mathbb{N}_k(\mathcal{Q})) < f(\mathbb{N}_{k-1}(\mathcal{Q}))$ holds up to at least $k = 20$.

Observe that Theorem 1.3 implies Conjecture 7.1 in the special case $k = 1$. Indeed, if $p \notin \mathcal{Q}$, then $A(\mathcal{Q})_p = \emptyset$, so we deduce $f(A(\mathcal{Q})) = \sum_{p\in\mathcal{Q}} f(A(\mathcal{Q})_p) \leq \sum_{p\in\mathcal{Q}} f(p) = f(\mathcal{Q})$. Moreover, if true, Conjecture 7.1 implies Conjecture 1.4 of Erdős–Sárközy–Szemerédi. This follows by an argument similar to Theorem 1.5, and using $f(\mathbb{N}_k(\mathcal{Q})) \to 1/2$ as $k \to \infty$ when $\mathcal{Q} = \mathcal{P} \setminus \{2\}$; see [25, Corollary 4.2]. We leave this to the interested reader.

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Competing Interest. The authors have no competing interest to declare.

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