J. Austral. Math. Soc. (*Series A*) 40 (1986), 261–266

ON A THEOREM OF FLEISCHER

G. MEHTA

(Received 10 October 1983; revised 31 August 1984)

Communicated by H. Rubinstein

Abstract

Fleischer proved that a linearly ordered set that is separable in its order topology and has countably many jumps is order-isomorphic to a subset of the real numbers. The object of this paper is to extend Fleischer's result and to prove it in a different way. The proof of the theorem is based on Nachbin's extension to ordered topological spaces of Urysohn's separation theorem in normal topological spaces.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 54 F 05, 90 A 10.

Introduction

Fleischer [4] proved that a linearly ordered set is isomorphic to a subset of the real numbers if and only if it is separable in its order topology and has countably many jumps. Fleischer's proof is based on a theorem of Birkhoff. (See Roberts [13, pages 111–116], for a proof of Birkhoff's theorem.)

Fleischer uses the sufficiency half of his result to derive the classic theorems of Debreu [1] on the existence of continuous order-preserving transformations on spaces that are connected and separable, or spaces that satisfy the second axiom of countability.

Fleischer's theorem assumes that the space is completely or linearly ordered. The object of this paper is to generalize the sufficiency half of Fleischer's theorem to spaces that may not be linearly ordered. The extension of Fleischer's result to

https://doi.org/10.1017/S1446788700027233 Published online by Cambridge University Press

partially ordered spaces is based on a separation theorem of Nachbin [10, page 30] which generalizes Urysohn's separation theorem in normal topological spaces to normally ordered topological spaces.

Preliminaries

A *preorder* $\leqslant$ on a topological space X is a reflexive transitive binary relation on X. We say that $x < y$ if and only if $x \leqslant y$ and not $y \leqslant x$. The preorder is said to be an *order* if it is antisymmetric. The preorder $\leqslant$ is said to be *decisive*, *complete* or *linear* if for two elements x, y belonging to X, either $x \leqslant y$ or $y \leqslant x$. It is said to be *continuous* if the sets $\{x \in X | x \leqslant y\}$ and $\{x \in X | y \leqslant x\}$ are closed in X for every y in X. Let X be a completely preordered set. The *order topology* of X is generated by the order intervals $\{x \in X | x < y\}$ and $\{x \in X | y < x\}$ for y in X.

Let x, y be two elements such that $x < y$. This pair of elements constitutes a *jump* if $(x, y) = \{a \in X | x < a < y\}$ is empty.

A subset of E of X is said to be *decreasing* if $b \in E$, $a \leqslant b$ imply that $a \in E$. Each subset E of X uniquely determine a *smallest decreasing subset* $d(E)$ containing E. Similarly, one defines the concept of an *increasing set* and the *smallest increasing subset* $i(E)$ containing E.

A topological space equipped with a preorder is said to be *normally preordered*, if, for every two disjoint closed subsets F_0 and F_1 of X, with F_0 decreasing and F_1 increasing, there exist two disjoint open subsets A_0 and A_1 such that A_0 includes F_0 and is decreasing, and A_1 includes F_1 and is increasing.

A preordered topological space $(X, \leqslant)$ is said to be *order-separable* if there exists a countable subset Z such that if x, y belong to X and $x < y$, then there exists a z in Z such that $x < z < y$.

Let E_1 and E_2 be two preordered sets. A function f from E_1 to E_2 is said to be *increasing* if x, y in E and $x \leqslant y$ imply that $f(x) \leqslant f(y)$. A function f on E_1 to E_2 is said to be *order-preserving* if it is increasing, and if $x < y$ implies that $f(x) < f(y)$. A *utility function* is an order-preserving real-valued function.

Utility functions

The generalization of Fleischer's result is based on the following theorem.

THEOREM 1. *Let $(X, \leqslant)$ be a normally preordered topological space and suppose that the preorder $\leqslant$ is continuous. Then if $(X, \leqslant)$ is order-separable, there exists a continuous order-preserving real function on $(X, \leqslant)$.*

https://doi.org/10.1017/S1446788700027233 Published online by Cambridge University Press

PROOF. This theorem is a consequence of Nachbin's separation theorem [10, page 30] in ordered topological spacees and, a proof can be found in Mehta [5, Theorem 1].

We are now ready to prove the following result.

THEOREM 2 (Fleischer). *A linearly ordered set X that is topologically separable in its order topology and has countably many jumps is order-isomorphic to a subset of the real numbers.*

PROOF. We note first that since X is linearly ordered and has the order topology, the preorder is continuous.

We prove next that X is normally preordered. Let F_0 and F_1 be disjoint closed subsets of X, with F_0 decreasing and F_1 increasing. If F_0 and F_1 exhaust X, then F_0 and F_1 are open, and X is normally pre-ordered. If they do not exhaust X, there is a point d not in F_0 or F_1. Since the preorder is decisive, we have $a < d < b$ for every a in F_0 and b in F_1. Hence, $\{x \in X | x < d\}$ and $\{x \in X | d < x\}$ are the required decreasing and increasing open sets containing F_0 and F_1, respectively. Thus $(X, \leqslant)$ is normally preordered.

Since X is topologically separable, it has a countable dense subset Z. Suppose first that there are only finitely many jumps (a_i, b_i), $i = 1, \ldots, n$. Interpose between each jump a copy of the open real interval $(i, i+1)$, $i = 1, \ldots, n$, with its usual ordering. If there are countably infinite jumps, interpose between each jump (a_n, b_n), $n = 1, 2, \ldots$, a copy of the open real interval $(n, n+1)$ with its usual ordering. Let Z' be the set of all non-integral, non-negative rational numbers. Enlarge the space X as indicated above and denote it by X'. The order on X' is obtained in a natural manner from the order in X and from the natural ordering of the real numbers. It is now a straightforward matter to verify that X' is order-separable in its extended order topology, with $Z \cup Z'$ being the countable order-dense subset.

X' is normally preordered because the extended preorder is decisive and continuous.

Hence, all the conditions of Theorem 1 are satisfied, and we conclude that there is a continuous utility function on X' whose restriction is continuous on X. It follows that X is order-isomorphic to a subset of the real numbers.

REMARK. The proof of Fleischer's theorem given above is based on Nachbin's separation theorem and is different from the one given by Fleischer.

In the above theorem, the assumption that X is linearly ordered is required only to prove that X is normally preordered. For normally preordered spaces that may not be linearly ordered, we have the following generalization of Fleischer's result.

THEOREM 3. *Let $(X, \leqslant, t)$ be a normally preordered topological space with a topology t such that $\leqslant$ is t-continuous. If X is toplogically separable and has countably many jumps, then there is a continuous order-preserving real-valued function f on X, provided that the order intervals $\{x \in X | x < y\}$ and $\{x \in X | y < x\}$, for y in X, belong to t.*

PROOF. Enlarge the space X to X' as in Theorem 2. The preordering on X is extended to X' in the following way. Consider a jump (a_n, b_n). The preordering on the open real interval $(n, n+1)$ that is interposed between (a_n, b_n) is defined to be the preordering induced by the natural preordering of the real numbers. Each element of the open real interval $(n, n+1)$ is defined to be not comparable with every other element of X', including points in other jumps. By doing this with each jump we obtain a preordering on X' that is an extension of the preordering on X. Note that X' has no jumps.

We now claim that X' is normally preordered. To prove this, let F and G be disjoint closed subsets of X' with F decreasing and G increasing. Let F_X be the intersection of F with X, and let F_J be the intersection of F with the jumps of X. G_X and G_J are defined similarly.

Since the topology on X' is the natural topology on the union of two spaces, F_X and G_X are disjoint closed subsets of X, with F_X decreasing and G_X increasing. Since X is normally preordered, there exist disjoint open sets A and B, with A decreasing and B increasing, such that A contains F_X and B contains G_X. Since the real line is normally preordered, we conclude that there exist disjoint open sets C and D, with C decreasing and D increasing, such that C contains F_J and D contains G_J. Hence $A \cup C$ is a decreasing open set containing F, and $B \cup D$ is an increasing open set containing G. Therefore X' is normally preordered.

As in Theorem 2, X' is order-separable. Thus all the conditions of Theorem 1 are satisfied, and we conclude that there is a continuous order-preserving function on X' whose restriction is continuous on X.

REMARK 1. The real-valued function f in Theorem 3 establishes a weak order-isomorphism between $(X, \leqslant)$ and the real numbers, in the sense that $x \leqslant y$ implies $f(x) \leqslant f(y)$. The converse implication may not hold, since $(X, \leqslant)$ is only partially preordered.

REMARK 2. Theorems 1 and 3 are useful in applications because every compact ordered space is normally ordered (see Nachbin, [10, page 48]).

REMARK 3. It is easily verified that Debreu's theorems [1] follow directly from Theorem 3, so that we have obtained a new generalization of these fundamental theorems based on Nachbin's theory of normal preorders. Observe that Theorem 3 is a common generalization of Fleischer's theorem and Debreu's theorems.

Conclusions

In Theorem 3 we proved the existence of a continuous utility function on a partially preordered topological space. We conclude by commenting on some related results in the literature.

Eilenberg [2] proved that a connected and separable completely preordered topological space is order-isomorphic to a subset of the real numbers. Debreu [1] proved the existence of a continuous utility function on a completely preordered space that is second countable or connected and topologically separable. Debreu's proof is based on the concept of a gap. A direct proof of Debreu's theorem based on Nachbin's theory of normal preorders is given in Mehta [6]. For further discussion of the post-Debreu developments, the reader is referred to Mehta [7].

Peleg [11] and Fishburn [3] have proved the existence of utility functions on strictly partially ordered topological spaces. Similar results have also been obtained by Richter [12]. Peleg's theorem is related to Debreu's theorems. It is proved in Mehta [8] that Peleg's theorem is a generalization of the Debreu theorems. Debreu's theorems, therefore, are a consequence of Peleg's theorem and the theorem of Fleischer. It is proved in Mehta [9] that Fleischer's theorem on linearly ordered sets is itself a consequence of Peleg's theorem. The exact nature of the relationship between the generalization of Fleischer's theorem proved in this paper and Peleg's theorem is an open question and certainly deserves further study.

Acknowledgements

I would like to thank an anonymous referee for his helpful comments. Some of the work for this paper was done while the author was visiting the University of Bonn, West Germany, and Professor Hildenbrand's hospitality is gratefully acknowledged.

References

[1] G. Debreu, "Representation of a preference ordering by a numerical function", pp. 159–166 *Decision processes*, R. Thrall, C. Coombs and R. Davis eds. (Wiley, New York, 1954).
[2] S. Eilenberg, "Ordered topological spaces", *Amer. J. Math.* **63** (1941), 39–45.
[3] P. Fishburn, *Utility theory for decision making* (Wiley, New York, 1970).
[4] I. Fleischer, "Numerical representation of utility", *Siam J.* **9** (1961), 48–50.
[5] G. Mehta, "Existence of an order-preserving function on normally preordered spaces", *Bull. Austral. Math. Soc.*, to appear.
[6] G. Mehta, "Toplogical ordered spaces and utility functions", *Internat. Econom. Rev.* **18** (1977), 779–782.
[7] G. Mehta, "Recent developments in utility theory", *Indian Econom. J.* **30** (1983), 103–125.
[8] G. Mehta, "Ordered topological spaces and the theorems of Debreu and Peleg", *Indian J. Pure and Appl. Math.* **14** (1983), 1174–1182.
[9] G. Mehta, "On a theorem of Peleg", *Indian J. Pure and Appl. Math.*, to appear.
[10] L. Nachbin, *Topology and order* (D. Van Nostrand Company, Princeton, New Jersey, 1965).
[11] B. Peleg, "Utility functions for partially ordered topological spaces", *Econometrica* **38** (1970), 93–96.
[12] M. Richter, "Revealed preference theory", *Econometrica* **34** (1966), 635–645.
[13] F. Roberts, *Measurement theory with applications to decision making, utility, and the social sciences* (*Encyclopedia of Mathematics and its Applications*, Volume 7, Addison-Wesley Publishing Company, 1979).

Departments of Economics and of Mathematics
University of Queensland
St. Lucia, Queensland, 4067
Australia