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SOME REMARKS ON ABSTRACT DIFFERENTIAL OPERATORS

BY
M. A. MALIK

1. Let A be a closed linear operator with domain D_A dense in a Banach space B; D_A is also a Banach space under the graph norm. By $\mathscr{L}(B; D_A)$ we represent the space of continuous linear mappings from B to D_A and $R(\lambda; A)=(\lambda I-A)^{-1}$ denotes the resolvent of A; $\lambda \in \mathbf{C}$ (complex plane). Let $\mathscr{D}(R)$ represent the space of test functions on R (real line) with Schwartz topology and let $\mathscr{D}'(\mathscr{L}(B; D_A)) = \mathscr{L}(\mathscr{D}; \mathscr{L}(B; D_A))$ denote the space of $\mathscr{L}(B; D_A)$-valued distributions. For $E \in \mathscr{D}'(\mathscr{L}(B; D_A))$ we define AE by the relation $\langle AE, \varphi\rangle = A\langle E, \varphi\rangle$ for all $\varphi \in \mathscr{D}(R)$. We also define $\delta \otimes I$ by the relation $\langle \delta \otimes I, \varphi\rangle = \varphi(0)I$ for all $\varphi \in \mathscr{D}(P)$; δ is the Dirac distribution and I the identity operator. $E \in \mathscr{D}'(\mathscr{L}(B; D_A))$ is called an elementary solution of the operator $L=(1/i)(d/dt)-A$ if $LE=\delta \otimes I$.

In this note, we study the support of the solution of $Lu=0$ and the nonexistence of an elementary solution of L by imposing conditions on the growth of the resolvent $R(\lambda; A)$. These results are related to a paper by S. Agmon and L. Nirenberg [1]. Throughout this note "const." may not always be the same constant.

2. We prove

THEOREM 1. *If* $u \in \mathscr{D}'(\mathscr{L}(B; D_A))$ *satisfies the equation* $Lu=0$ *and the resolvent* $R(\lambda; A)$ *exists on a ray* $\Gamma(\arg \lambda=\theta; 0<\theta<\Pi)$ *where it satisfies*

$$|R(\lambda; A)| \leq \text{const. } e^{-\rho|\operatorname{Im}\lambda|} \tag{1}$$

for some $\rho>0$ *then* $\operatorname{supp} u \subset (-\infty, 0]$.

Proof. Let $u \in \mathscr{D}'(\mathscr{L}(B; D_A))$ be a solution of $Lu=0$. Consider a sequence $\varphi_\varepsilon \in \mathscr{D}(R)$ such that $\varphi_\varepsilon \to \delta$ as $\varepsilon \to 0$. It is easy to verify that the convolution $v=u * \varphi_\varepsilon$ also satisfies the equation $Lv=0$. Choose a function $\xi \in \mathscr{D}(R)$ vanishing outside the interval $[-T, T]$; $T>\rho$. Then the support of ξv is contained in $[-T, T]$ and

$$\frac{1}{i}\frac{d}{dt}(\xi v)-A\xi v = -i\xi' v. \tag{2}$$

As ξv and $\xi' v$ are of compact support, their Fourier transforms are vector valued entire functions and satisfy the equation

$$(\lambda I-A)\widetilde{(\xi v)} = -i\widetilde{(\xi' v)}. \tag{3}$$

From the Paley–Wiener theorem [2] and the hypothesis on the resolvent, we obtain

(4) $$|\widetilde{\xi v(\lambda)}| \leq \text{const. } e^{(T-\rho)|\text{Im}\,\lambda|}$$

for $\lambda \in \Gamma$ while $|\widetilde{\xi v(\lambda)}|$ is bounded on the real axis. Applying the Phragmen–Lindelof theorem, we conclude that $|\widetilde{\xi v(\lambda)}| = O(e^{(T-\rho)|\text{Im}\,\lambda|})$ in the upper half plane. But then the Paley–Wiener theorem implies that ξv vanishes for $t > T-\rho$ and so $v(t)$ vanishes for $T-\rho < t < T$. Repeated applications of this technique imply that $v = u * \varphi_\varepsilon$ vanishes for $t > 0$. Making $\varepsilon \to 0$, we conclude that the supp $u \subset (-\infty, 0]$.

THEOREM 2. *If $R(\lambda; A)$ exists on rays Γ_1 (arg $\lambda = \theta_1$; $0 < \theta_1 < \Pi$) and Γ_2 (arg $\lambda = \theta_2$; $\Pi < \theta_2 < 2\Pi$) where it satisfies the inequality*

(5) $$|R(\lambda; A)| \leq \text{const. } e^{-\rho|\text{Im}\,\lambda|}$$

for some $\rho > 0$, then the operator $L = (1/i)(d/dt) - A$ has no elementary solution in $\mathscr{D}'(\mathscr{L}(B; D_A))$.

Proof. Suppose $E \in \mathscr{D}'(\mathscr{L}(B; D_A))$ satisfies the equation $LE = \delta \otimes I$. It can be verified that

(6) $$\frac{1}{i}\frac{d}{dt}(\xi v) - A(\xi v) = -i\xi' v + \varphi_\varepsilon I$$

where $v = E * \varphi_\varepsilon$ and φ_ε, ξ are same as in the proof of Theorem 1. The Fourier transform of (6) along with the hypothesis leads to

(7) $$|\widetilde{\xi v(\lambda)}| = |R(\lambda; A)|\,|\widetilde{\xi' v(\lambda)} + \widetilde{\varphi_\varepsilon(\lambda)} I| \leq \text{const. } e^{(T-\rho+\varepsilon)|\text{Im}\,\lambda|}$$

for $\lambda \in \Gamma_1 \cup \Gamma_2$. Using the arguments as in the proof of Theorem 1, one concludes that the supp $v \subset \{t: |t| \leq \varepsilon\}$. Making $\varepsilon \to 0$ we find that the supp E is concentrated at the origin. Therefore,

(8) $$E = \sum U_k \otimes \delta^k; \qquad U_k \in \mathscr{L}(B; D_A)$$

and so its Fourier transform is a polynomial. But for $\lambda \in \Gamma_1 \cup \Gamma_2$ one has

(9) $$|\widetilde{E(\lambda)}| \leq \text{const. } e^{-\rho|\text{Im}\,\lambda|}, \text{ from where } |\widetilde{E(\lambda)}| \to 0 \text{ as } |\lambda| \to \infty$$

along $\Gamma_1 \cup \Gamma_2$; contradiction. Hence L has no elementary solution.

THEOREM 3. *If there exists a region in the complex plane where the resolvent $R(\lambda; A)$ does not exist then the operator L has no elementary solutions with compact support.*

Proof. If E is a distribution with compact support and satisfies $LE = \delta \otimes I$, it is easy to verify that the Fourier transform $E(\lambda)$ is an entire function and satisfies $(\lambda I - A)\widetilde{E(\lambda)} = I$. This implies that the resolvant $R(\lambda; A) = \widetilde{E(\lambda)}$ exists throughout the plane contradicting the hypothesis.

References

1. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. **16** (1963).

2. L. Hormander, *Linear partial differential operators*, Academic Press, New York, 1963.

Sir George Williams University,
Montreal, Quebec