

AN EXAMPLE CONCERNING BOUNDED LINEAR REGULARITY OF SUBSPACES IN HILBERT SPACE

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Abstract

We study bounded linear regularity of finite sets of closed subspaces in a Hilbert space. In particular, we construct for each natural number $n \geq 3$ a set of n closed subspaces of ℓ^2 which has the bounded linear regularity property, while the bounded linear regularity property does not hold for each one of its nonempty, proper nonsingleton subsets. We also establish a related theorem regarding the bounded regularity property in metric spaces.

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1. Introduction

Infinite products of (linear and nonlinear) operators and, in particular, infinite products of retractions are often used for approximating points in the intersection of a finite number $m \geq 2$ of closed and convex sets via individual (best) approximations from each one of these sets. If all the sets are closed linear subspaces S_1, S_2, \dots, S_m (or, more generally, closed affine subspaces), of a given (real) Hilbert space H , then the best approximations to a given point $x \in H$ in each one of the subspaces S_i coincide with the orthogonal projections $P_{S_i}x$. Taking an arbitrary $x_0 \in H$ and defining the iterations $x_n = (P_{S_m}P_{S_{m-1}} \cdots P_{S_1})^n x_0$, $n = 1, 2, \dots$, we obtain a sequence $\{x_n\}$ which converges in norm to an element $x^* \in S := S_1 \cap \cdots \cap S_m$. This fundamental fact was first established for $m = 2$ by von Neumann in [3] and for the general case by Halperin in [2].

The theorems in [2, 3] did not state any rate of the above-mentioned convergence and in fact, it is now known that this convergence may be arbitrarily slow. Any uniform estimate of the convergence rate that was proved since then in some particular cases required special interplay between the subspaces S_i , for example, positivity of the angles between them and some of their intersections (see, for example, [4]). The properties of a given set of subspaces $\{S_1, S_2, \dots, S_m\}$ guaranteeing uniform

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convergence of infinite products were investigated by Bauschke and Borwein in [1], where they proved that this kind of convergence occurred if and only if the subspace $S_1^\perp + S_2^\perp + \dots + S_m^\perp$ is closed in H . This property was named *bounded linear regularity* of the given set of subspaces.

To provide a quantitative character to the property of regularity, one can use the *inclination* of a given set of subspaces $\{S_1, S_2, \dots, S_m\}$, where $m \geq 2$, which is defined by

$$\ell(S_1, S_2, \dots, S_m) = \inf_{x \notin S} \frac{\max_{1 \leq j \leq m} \|x - P_{S_j} x\|}{\|x - P_S x\|}.$$

Thus we see that the inclination compares the distances from an arbitrary point $x \in H$ to the subspaces S_i and to their intersection. When the inclination is positive, not only one obtains the uniform convergence of the infinite products, but also an estimate of the rate of this convergence. Many useful applications of the concept of inclination can be found in [5]. As a matter of fact, the positivity of the inclination (and of the inner inclination [6]) is equivalent to the above-mentioned bounded regularity property.

In the next section we exhibit for any natural number $n \geq 3$, a set $A = \{X_1, X_2, \dots, X_n\}$ of n closed linear subspaces of ℓ^2 with the following properties:

- (i) the bounded linear regularity property does not hold for each nonempty, proper nonsingleton subset of A (that is, its inclination is zero);
- (ii) the whole set A has the bounded linear regularity property (that is, its inclination is positive).

So we see that the bounded linear regularity property is *not* hereditary.

Property (i) is easily deduced from Proposition 2.1 below, while property (ii) follows from Proposition 2.2.

In Section 3 we turn our attention to the bounded regularity property in general metric spaces. We show (see Theorem 3.1 below) that if for each $j \in \{1, \dots, q\}$, $\mathcal{L}_j = \{C_{j,i} : i = 1, \dots, n_j\}$ is a finite collection of nonempty subsets of a metric space X such that $\bigcap_{j=1}^q (\bigcap_{i=1}^{n_j} C_{j,i}) \neq \emptyset$ and \mathcal{L}_j has the bounded regularity property for each $j \in \{1, \dots, q\}$, then the family

$$\mathcal{L} = \{C_{j,i} : j \in \{1, \dots, q\}, i \in \{1, \dots, n_j\}\}$$

possesses the bounded regularity property if and only if the family

$$\tilde{\mathcal{L}} = \left\{ \bigcap_{i=1}^{n_j} C_{j,i} : j = 1, \dots, q \right\}$$

has this property.

2. Bounded linear regularity of subspaces

Denote by $X = \ell^2$ the Hilbert space consisting of all the sequences of real numbers $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty x_i^2 < \infty$, with the inner product

$$\langle x, y \rangle = \sum_{i=1}^\infty x_i y_i, \quad x = \{x_i\}_{i=1}^\infty, \quad y = \{y_i\}_{i=1}^\infty \in X.$$

This inner product induces the norm

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{\infty} x_i^2 \right)^{1/2}, \quad x = \{x_i\}_{i=1}^{\infty} \in X.$$

Let $D_1, D_2 \subset X$ be nonempty sets. We say that D_1 and D_2 are orthogonal and write $D_1 \perp D_2$ if $\langle u, v \rangle = 0$ for all $u \in D_1$ and $v \in D_2$.

Let $n \geq 3$ be a natural number and let $e_i, i = 1, 2, \dots$, be the standard basis in X . Namely, for $j = 1, 2, \dots$, $e_j = \{e_{j,i}\}_{i=1}^{\infty}$ with

$$e_{j,j} = 1, \quad e_{j,i} = 0 \quad \text{for all integers } i \geq 1 \text{ such that } i \neq j.$$

For each $x \in X$ and each nonempty set $D \subset X$, set $\rho(x, D) = \inf\{\|x - y\| : y \in D\}$. Denote by L_1 the closed linear subspace generated by the vectors $\{e_{2i+1} : i = 1, 2, \dots\}$ and by L_2 the closed linear subspace generated by $\{e_{2i} : i = 1, 2, \dots\}$.

For each pair of nonempty sets $D_1, D_2 \subset X$, set

$$D_1 + D_2 = \{x + y : x \in D_1, y \in D_2\}.$$

Let L and \tilde{L} be two closed linear subspaces of L_1 such that

$$L + \tilde{L} = L_1 \quad \text{and} \quad L \perp \tilde{L}. \tag{2.1}$$

As we have already mentioned, we are going to construct a set $A = \{X_1, \dots, X_n\}$ of $n \geq 3$ closed linear subspaces of ℓ^2 with properties (i) and (ii). We construct them in such a way that the following property also holds:

(iii) $X_{j_1} \cap X_{j_2} = L$ for all $j_1, j_2 \in \{1, \dots, n\}$ such that $j_1 \neq j_2$.

Since L is an arbitrary linear closed subspace of L_1 , we can actually obtain many examples of sets A with properties (i)–(iii).

To this end, we first construct closed linear subspaces $X_{0,s} \subset L_2, s = 1, \dots, n$, such that $X_{s_1} \cap X_{s_2} = \{0\}$ for all $s_1, s_2 \in \{1, \dots, n\}$ satisfying $s_1 \neq s_2$ and then for all $s = 1, \dots, n$, we set $X_s = X_{0,s} + L$. Since $L_2 \perp L$, we conclude that property (iii) holds.

For each natural number k , each $j \in \{1, \dots, n\}$ and each $i \in \{1, \dots, n - 1\}$, set

$$\xi_{(n-1)n(k-1)+(j-1)(n-1)+i} = e_{2n^2(k-1)+2(j-1)n+2} + 2^{-k} e_{2n^2(k-1)+2(j-1)n+2(i+1)}. \tag{2.2}$$

Clearly, we have defined $\xi_i, i = 1, 2, \dots$.

Let $s \in \{1, \dots, n\}$. Denote by $X_{0,s}$ the closed linear subspace generated by the basis vectors

$$\begin{aligned} & \{\xi_{(n-1)n(k-1)+(j-1)(n-1)+s-1} : k \text{ is a natural number, } j \in \{1, \dots, n\} \text{ and } j < s\} \\ & \cup \{\xi_{(n-1)n(k-1)+(j-1)(n-1)+s} : k \text{ is a natural number, } j \in \{1, \dots, n\} \text{ and } j > s\}. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3),

$$X_{0,s} \perp L_1. \tag{2.4}$$

By (2.2), if k_1 and k_2 are natural numbers such that $j_1, j_2 \in \{1, \dots, n\}$, $i_1, i_2 \in \{1, \dots, n - 1\}$ and $(k_1, j_1) \neq (k_2, j_2)$, then

$$\langle \xi_{(n-1)n(k_1-1)+(j_1-1)(n-1)+i_1}, \xi_{(n-1)n(k_2-1)+(j_2-1)(n-1)+i_2} \rangle = 0.$$

By (2.2), the system of vectors ξ_i , $i = 1, 2, \dots$, is linearly independent. When combined with (2.3), this implies that for all $s_1, s_2 \in \{1, \dots, n\}$ satisfying $s_1 \neq s_2$,

$$X_{0,s_1} \cap X_{0,s_2} = \{0\}. \tag{2.5}$$

Now for each $s \in \{1, \dots, n\}$, set

$$X_s = X_{0,s} + L. \tag{2.6}$$

By (2.1) and (2.4), X_s is closed for all $s \in \{1, \dots, n\}$.

Our first proposition shows that the set $A = \{X_1, X_2, \dots, X_n\}$ has property (i).

PROPOSITION 2.1. *Let $\epsilon \in (0, 1)$ and $j \in \{1, \dots, n\}$. Then there is a point $z \in L_2$ such that $\|z\| = \rho(z, L) = 1$ and $\rho(z, X_i) \leq \epsilon$ for all $i \in \{1, \dots, n\} \setminus \{j\}$.*

PROOF. Choose a natural number k such that

$$2^{-k} < \epsilon/2. \tag{2.7}$$

There are two cases: $j = 1$ and $j > 1$.

Assume first that $j = 1$. Then by (2.3) and (2.6),

$$\xi_{(n-1)n(k-1)+s-1} \in X_{0,s} \subset X_s, \quad s = 2, \dots, n. \tag{2.8}$$

Let

$$z = e_{2n^2(k-1)+2}. \tag{2.9}$$

By (2.3) and (2.7)–(2.9), for all $s \in \{1, \dots, n\} \setminus \{1\}$,

$$\rho(z, X_s) \leq \|z - \xi_{(n-1)n(k-1)+s-1}\| = \|e_{2n^2(k-1)+2} - \xi_{(n-1)n(k-1)+s-1}\| \leq 2^{-k} < \epsilon/2$$

and the assertion of the proposition follows.

Assume next that $j > 1$. Then by (2.3) and (2.6), for all $s = 1, \dots, j - 1$,

$$\xi_{(n-1)n(k-1)+(j-1)(n-1)+s} \in X_{0,s} \subset X_s \tag{2.10}$$

and for all integers s satisfying $j < s \leq n$,

$$\xi_{(n-1)n(k-1)+(j-1)(n-1)+s-1} \in X_{0,s} \subset X_s. \tag{2.11}$$

Define

$$z = e_{2n^2(k-1)+2(j-1)n+2}. \tag{2.12}$$

By (2.9)–(2.12), (2.2) and (2.7), for all $s \in \{1, \dots, n\} \setminus \{j\}$,

$$\begin{aligned} \rho(z, X_s) &\leq \rho(z, X_{0,s}) = \rho(e_{2n^2(k-1)+2+2(j-1)n}, X_{0,s}) \\ &\leq \max\{\|e_{2n^2(k-1)+2(j-1)n+2} - \xi_{(n-1)n(k-1)+(j-1)(n-1)+i}\| : i = 1, \dots, n-1\} \\ &\leq 2^{-k} < \epsilon/2 \end{aligned}$$

and the assertion of the proposition follows once again.

This completes the proof of Proposition 2.1. □

Next we show that the set A has property (ii).

PROPOSITION 2.2. *Let $x \in X$ with*

$$\rho\left(x, \bigcap_{j=1}^n X_j\right) \geq 1. \tag{2.13}$$

Then there is $p \in \{1, \dots, n\}$ such that $\rho(x, X_p) \geq (2 + 2n)^{-1}$.

By (2.1) and (2.4)–(2.6),

$$\bigcap_{j=1}^n X_j = L. \tag{2.14}$$

Clearly, there is a vector $y \in X$ such that

$$y \in L, \quad \|x - y\| = \inf\{\|x - u\| : u \in L\}, \quad \{x - y\} \perp L. \tag{2.15}$$

There also is a point $z \in X$ such that

$$z \in \tilde{L}, \quad \|x - y - z\| = \inf\{\|x - y - u\| : u \in \tilde{L}\}, \quad \{x - y - z\} \perp \tilde{L}. \tag{2.16}$$

By (2.1), (2.15) and (2.16),

$$\{x - y - z\} \perp L_1.$$

By (2.13)–(2.15),

$$\|x - y\| \geq 1. \tag{2.17}$$

There are two cases:

$$\|z\| > 2^{-1} \quad \text{and} \quad \|z\| \leq 2^{-1}.$$

Assume that

$$\|z\| > 2^{-1}. \tag{2.18}$$

Let $j \in \{1, \dots, n\}$. By (2.16), (2.6), (2.1), (2.4) and the definition of L_1 ,

$$\{z\} \perp X_j. \tag{2.19}$$

Let

$$u \in X_j. \tag{2.20}$$

By (2.19) and (2.20),

$$\langle u, z \rangle = 0. \tag{2.21}$$

By (2.16) and (2.21),

$$\langle x - y - z - u, z \rangle = 0. \tag{2.22}$$

It follows from (2.22) that

$$\|x - y - u\|^2 = \|(x - y - z - u) + z\|^2 = \|x - y - z - u\|^2 + \|z\|^2 \geq \|z\|^2 > 4^{-1}.$$

Since u is an arbitrary element of X_j , we conclude that $\rho(x - y, X_j) \geq 2^{-1}$. Since $y \in L \subset X_j$ (see (2.6) and (2.15)), this implies that $\rho(x, X_j) \geq 2^{-1}$ for all $j \in \{1, \dots, n\}$ and the assertion of the proposition follows.

Assume now that

$$\|z\| \leq 2^{-1}. \tag{2.23}$$

By (2.17) and (2.23),

$$\|x - y - z\| \geq 2^{-1}. \tag{2.24}$$

By (2.1), (2.16) and (2.15),

$$\{x - y - z\} \perp L_1.$$

Thus

$$x - y - z \in L_2. \tag{2.25}$$

For any $j \in \{1, \dots, n\}$, denote by Y_j the closed linear subspace generated by the vectors

$$\{\xi_{(n-1)n(k-1)+(j-1)(n-1)+i} : k \text{ is a natural number, } i \in \{1, \dots, n-1\}\}.$$

By the definition of $X_j, Y_j, j = 1, \dots, n$, and $\xi_i, i = 1, 2, \dots$, we have, for any $j \in \{1, \dots, n\}$,

$$Y_j \subset L_2, \quad Y_j \perp L_1, \quad Y_j \perp X_j, \quad Y_{j_1} \perp Y_{j_2} \tag{2.26}$$

for all $j_1, j_2 \in \{1, \dots, n\}$ such that $j_1 \neq j_2$.

Clearly, $Y_1 + Y_2 + \dots + Y_n$ is a closed subspace of L_2 . Set

$$Y_0 = \{y \in L_2 : \{y\} \perp Y_i, i = 1, \dots, n\}. \tag{2.27}$$

Clearly, Y_0 is a closed linear subspace of L_2 (it may happen that $Y_0 = \{0\}$) and

$$Y_0 + Y_1 + \dots + Y_n = L_2. \tag{2.28}$$

By (2.25) and (2.28), there exist points

$$y_i \in Y_i, \quad i = 0, \dots, n, \tag{2.29}$$

such that

$$\sum_{i=0}^n y_i = x - y - z. \tag{2.30}$$

By definition,

$$X_{0,1} + X_{0,2} + \cdots + X_{0,n} \subset Y_1 + Y_2 + \cdots + Y_n.$$

By (2.24) and (2.30), there is an integer $j \in \{0, \dots, n\}$ such that

$$\|y_j\| \geq 2^{-1}(n+1)^{-1}. \quad (2.31)$$

By (2.6), (2.16), (2.1), (2.4), (2.29) and (2.27),

$$\{z + y_0\} \perp X_i, \quad i = 1, \dots, n. \quad (2.32)$$

There are two cases: $j = 0$ and $j \neq 0$.

Assume first that $j = 0$. Let

$$i \in \{1, \dots, n\} \quad \text{and} \quad u \in X_i. \quad (2.33)$$

By (2.31)–(2.33), (2.16), (2.1), (2.25), (2.29), (2.28), (2.30), (2.27) and (2.18),

$$\begin{aligned} \|x - y - u\|^2 &= \|x - y - z - y_0 - u + z + y_0\|^2 \\ &\geq \|z + y_0\|^2 + \|x - y - z - y_0 - u\|^2 \\ &\quad + 2\langle z + y_0, x - y - z - y_0 - u \rangle \\ &\geq \|z + y_0\|^2 + 2\langle z + y_0, x - y - z - y_0 \rangle \\ &= \|z + y_0\|^2 + 2\langle y_0, x - y - z - y_0 \rangle \\ &= \|z + y_0\|^2 = \|z\|^2 + \|y_0\|^2 \geq 4^{-1}(n+1)^{-2} \end{aligned}$$

and

$$\begin{aligned} \|x - y - u\| &\geq 2^{-1}(n+1)^{-1} \quad \text{for all } u \in X_i, \\ \rho(x - y, X_i) &\geq 2^{-1}(n+1)^{-1}. \end{aligned}$$

Since $y \in L$ (see (2.15)),

$$\rho(x, X_i) \geq 2^{-1}(n+1)^{-1} \quad \text{for all } i = 1, \dots, n$$

and the assertion of the proposition follows.

Assume now that $j \neq 0$. Let

$$u \in X_j. \quad (2.34)$$

Clearly,

$$\begin{aligned} \|x - y - u\|^2 &= \|x - y - z - y_j - u + z + y_j\|^2 \\ &\geq \|z + y_j\|^2 + 2\langle z + y_j, x - y - z - y_j - u \rangle \\ &\geq \|z + y_j\|^2 + 2\langle z + y_j, x - y - z - y_j \rangle - 2\langle z + y_j, u \rangle. \end{aligned} \quad (2.35)$$

By (2.16), (2.34), (2.6), (2.1), (2.2), (2.29) and (2.26),

$$\langle z + y_j, u \rangle = \langle z, u \rangle + \langle y_j, u \rangle = 0. \quad (2.36)$$

By (2.16), (2.28), (2.30), (2.26), (2.27) and (2.28),

$$\langle z + y_j, x - y - z - y_j \rangle = \langle y_j, x - y - z - y_j \rangle = 0. \tag{2.37}$$

By (2.35), (2.36), (2.37), (2.16), (2.29), (2.28), (2.1), (2.31) and (2.32),

$$\begin{aligned} \|x - y - u\|^2 &\geq \|z + y_j\|^2 = \|z\|^2 + \|y_j\|^2 \geq 4^{-1}(n + 1)^{-2}, \\ \|x - y - u\| &\geq 2^{-1}(n + 1)^{-1} \quad \text{for all } u \in X_j \end{aligned}$$

and

$$\rho(x - y, X_j) \geq 2^{-1}(n + 1)^{-1}.$$

Since $y \in L \subset X_j$ (see (2.6) and (2.15)), $\rho(x, X_j) \geq 2^{-1}(n + 1)^{-1}$ and the assertion of the proposition follows once again.

This completes the proof of Proposition 2.2. □

3. Bounded regularity in metric spaces

In this section we state and prove a theorem regarding bounded regularity in general metric spaces.

Let (X, ρ) be a metric space. For each point $x \in X$ and each subset $A \subset X$, set $\rho(x, A) = \inf\{\rho(x, y) : y \in A\}$. Fix a point $\theta \in X$.

Let $\mathcal{L} = \{X_1, \dots, X_n\}$ be a finite collection of nonempty subsets of X such that $\bigcap_{i=1}^n X_i \neq \emptyset$.

Recall that \mathcal{L} is said to have the bounded regularity property [1] if for each $\epsilon > 0$ and $M > 0$, there exists $\delta > 0$ such that:

(P1) if $x \in X$ satisfies $\rho(x, \theta) \leq M$ and $\rho(x, X_i) < \delta, i = 1, \dots, n$, then

$$\rho\left(x, \bigcap_{i=1}^n X_i\right) < \epsilon.$$

Suppose that \mathcal{L} possesses the bounded regularity property. Let $\epsilon > 0$ and $M > 0$ be given. Denote by $\mathcal{D}(M, \epsilon)$ the set of all numbers $\delta \in (0, \epsilon]$ for which (P1) holds and set

$$\phi_{\mathcal{L}}(\epsilon, M) = \sup(\mathcal{D}(\epsilon, M)).$$

It is clear that $\phi_{\mathcal{L}}(\epsilon, M) \leq \epsilon$ and (P1) holds with $\delta = \phi_{\mathcal{L}}(\epsilon, M)$.

THEOREM 3.1. *Suppose that q is a natural number. For each integer $j \in \{1, \dots, q\}$, let $\mathcal{L}_j = \{C_{j,i} : i = 1, \dots, n_j\}$ be a finite collection of nonempty subsets of X such that $\bigcap_{i=1}^{n_j} C_{j,i} \neq \emptyset$, \mathcal{L}_j possesses the bounded regularity property and*

$$\bigcap_{j=1}^q \left(\bigcap_{i=1}^{n_j} C_{j,i} \right) \neq \emptyset.$$

Let

$$\mathcal{L} = \{C_{j,i} : j \in \{1, \dots, q\}, i \in \{1, \dots, n_j\}\}$$

and

$$\tilde{\mathcal{L}} = \left\{ \bigcap_{i=1}^{n_j} C_{j,i} : j = 1, \dots, q \right\}.$$

Then \mathcal{L} has the bounded regularity property if and only if $\tilde{\mathcal{L}}$ has the bounded regularity property. If they possess the bounded regularity property, then for each $\epsilon, M > 0$,

$$\phi_{\tilde{\mathcal{L}}}(\epsilon, M) \geq \phi_{\mathcal{L}}(\epsilon, M) \geq \min\{\phi_{\mathcal{L}_j}(\phi_{\tilde{\mathcal{L}}}(\epsilon, M), M) : j = 1, \dots, q\}.$$

PROOF. Assume that \mathcal{L} possesses the bounded regularity property. To show that $\tilde{\mathcal{L}}$ possesses the bounded regularity property, let ϵ and M be positive. Let $x \in X, \rho(x, \theta) \leq M$,

$$\rho\left(x, \bigcap_{i=1}^{n_j} C_{j,i}\right) < \phi_{\mathcal{L}}(\epsilon, M), \quad j = 1, \dots, q. \tag{3.1}$$

By (3.1), the bounded regularity property of \mathcal{L} and the definition of $\phi_{\mathcal{L}}$,

$$\rho(x, C_{j,i}) < \phi_{\mathcal{L}}(\epsilon, M), \quad j = 1, \dots, q, i = 1, \dots, n_j,$$

and $\epsilon > \rho(x, \bigcap_{j=1}^q (\bigcap_{i=1}^{n_j} C_{j,i}))$. Thus $\tilde{\mathcal{L}}$ has the bounded regularity property and $\phi_{\tilde{\mathcal{L}}}(\epsilon, M) \geq \phi_{\mathcal{L}}(\epsilon, M)$.

In the other direction, assume that $\tilde{\mathcal{L}}$ has the bounded regularity property. To show that \mathcal{L} possesses the bounded regularity property, let ϵ and M be positive. Let $x \in X, \rho(x, \theta) \leq M$,

$$\rho(x, C_{j,i}) < \min\{\phi_{\mathcal{L}_j}(\phi_{\tilde{\mathcal{L}}}(\epsilon, M), M) : j = 1, \dots, q\}, \quad j = 1, \dots, q, i = 1, \dots, n_j. \tag{3.2}$$

Let $j \in \{1, \dots, q\}$. By (3.2) and the bounded regularity of \mathcal{L}_j ,

$$\rho\left(x, \bigcap_{i=1}^{n_j} C_{j,i}\right) < \phi_{\tilde{\mathcal{L}}}(\epsilon, M). \tag{3.3}$$

By (3.2), (3.3) and the bounded regularity of $\tilde{\mathcal{L}}$,

$$\rho\left(x, \bigcap_{j=1}^q \left(\bigcap_{i=1}^{n_j} C_{j,i}\right)\right) < \epsilon.$$

Thus \mathcal{L} possesses the bounded regularity property and

$$\phi_{\mathcal{L}}(\epsilon, M) \geq \min\{\phi_{\mathcal{L}_j}(\phi_{\tilde{\mathcal{L}}}(\epsilon, M), M) : j = 1, \dots, q\}.$$

This completes the proof of Theorem 3.1. □

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