EXTREMAL POINT AND EDGE SETS IN n-GRAPHS

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1. Introduction. A set of points (edges) of a graph is independent if no two distinct members of the set are adjacent. Gallai (**1**) observed that, if A_0 (B_0) is the minimum number of points (edges) of a finite graph covering all the edges (points) and A_1 (B_1) is the maximum number of independent points (edges), then:

$$A_0 + A_1 = B_0 + B_1 = m$$

holds, where m is the number of points of the graph.

The concepts of independence and covering are generalized in various ways for n-graphs. In this paper we establish certain connections between the corresponding extreme numbers analogous to the above result of Gallai.

Ray-Chaudhuri considered (**2**) independence and covering problems in n-graphs and determined algorithms for finding the minimal cover and some associated numbers. In the terminology of (**2**), this paper deals with relations between $(1, 1, \ldots, 1)$-covers and $(1, 1, \ldots, 1)$-matchings of complexes by taking also smaller faces of the simplices into account.

2. Definitions. The cardinal number of a set X is denoted by $|X|$. If X is a set of sets, then, as usual, $\bigcup X$ denotes the set union of all the members of X.

An n-graph ($n \geqq 2$) is an ordered pair of finite sets $G = (V_1, T_n)$, with $T_n \subset \{X | X \subset V; |X| = n\}$. Elements of V are the points of G and elements of T_n are the n-edges of G.

We assume throughout that: $m = |V| \geqq n$, and also that G has no isolated points, i.e.: $V \subset \bigcup T_n$.

If $X \subset Y \in T_n$ and $|X| = k \geqq 2$, we call X a k-edge of G. The set of all k-edges is denoted by T_k ($2 \leqq k \leqq n$). An edge of G is a k-edge for some k ($2 \leqq k \leqq n$).

A set of edges E is independent if whenever $X_1, Y \in E, X \neq Y$, then $X \cap Y = \emptyset$. A set of points is independent if it contains no 2-edge of G.

We write $X \in \mathscr{E}^i$ if X is an independent set of edges and

$$X \subset T_i \cup T_{i+1} \cup \ldots \cup T_n \qquad (2 \leqq i \leqq n).$$

Thus

$$(2.1) \qquad \mathscr{E}^2 \supset \mathscr{E}^3 \supset \ldots \supset \mathscr{E}^n,$$

and we simply write $X \in \mathscr{E}^2$ if X is an independent set of edges without restriction.

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The value of an independent set of edges $E \in \mathscr{E}^2$ is defined as:

$$v(E) = |\bigcup E| - |E| = \sum_{k=2}^{n} (k-1)n_k,$$

where n_k is the number of k-edges in E.

We observe that if $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E \in \mathscr{E}^2$, then

$$v(E) = v(E_1) + v(E_2).$$

Furthermore, if $e_1, e_2 \in E \in \mathscr{E}^2$, $e_1 \neq e_2$ and $e = e_1 \cup e_2 \in X \in T_n$, then

$$v(E') = v(E) + 1, \tag{2.2}$$

where $E' = (E - \{e_1, e_2\}) \cup \{e\}$. This follows from the fact that $\bigcup E' = \bigcup E$ and $|E'| = |E| - 1$.

If $E \in \mathscr{E}^2$, we define:

$$w(E) = \max\{v(X) |\ X \cap E = \emptyset; X \cup E \in \mathscr{E}^2\}.$$

In particular, $w(E) = 0$ if and only if E is a maximal independent set of edges. We define:

$$\alpha_i = \max\{v(E) |\ E \in \mathscr{E}^i\} \qquad (2 \leqq i \leqq n).$$

It follows from (1) that $\alpha_2 \geqq \alpha_3 \geqq \ldots \geqq \alpha_n$. Note, in particular, that if $v(E) = \alpha_2$, then $w(E) = 0$.

If $E \subset T_2 \cup T_3 \cup \ldots \cup T_n$ and $U \subset T_n$, we write:

$$E < U \quad \text{or} \quad U > E$$

if $E \subset \{x |\ x \subset y \in U\}$.

The set of edges E is said to cover the set of points $V' \subset V$ if $V' \subset \bigcup E$. The least number of n-edges which covers all the points V is denoted by a, i.e.:

$$a = \min\{|U| |\ U \in T_n; V = \bigcup U\}.$$

If $U \subset T_n$, $V = \bigcup U$, and $|U| = a$, we call U a minimal cover.

If $E \in \mathscr{E}^i$, $v(E) = \alpha_i$, and if there is a minimal cover $U > E$, we say that i is G-admissible and E is an admissible set of edges. We will show (Theorem 1) that if i is G-admissible and $E \in \mathscr{E}^i$ is any admissible set of edges, then

$$w(E) = m - a - \alpha_i = \beta_i.$$

Also (Theorem 1) we show that 2 is G-admissible and, of course $\beta_2 = 0$. Consequently, we may define the number:

$$z = \max\{i |\ 2 \leqq i \leqq n; i \text{ is } G\text{-admissible}; \beta_i = 0\}$$

which we call the covering number of G.

Let $g_j = |T_j|$ $(2 \leqq j \leqq n)$. If $1 \leqq r \leqq n$, $2 \leqq k_1 < k_2 < \ldots < k_r \leqq n$, $0 \leqq h_i \leqq g_{k_i}$ and $0 \leqq f_i \leqq k_i$ $(1 \leqq i \leqq r)$, then we will write:

$$[k_i, h_i f_i]_0^r \tag{2.3}$$

to denote the smallest integer p for which the following statement is true:

S_p: There is $P \subset V$ such that $|P| = p$ and there are sets $N_i \subset T_{k_i}$ $(1 \leqq i \leqq r)$ such that $|N_i| = h_i$ and

$$|x \cap P| \geqq f_i \qquad (x \in N_i;\, 1 \leqq i \leqq r). \tag{2.4}$$

Similarly, we denote by

$$[k_i, h_i f_i]_1{}^r \tag{2.5}$$

the largest integer p such that S_p' is true, where the statement S_p' is the same as S_p except that (2.4) is replaced by

$$|x \cap P| \leqq f_i \qquad (x \in N_i;\, 1 \leqq i \leqq r). \tag{2.6}$$

Note that the above definitions of (2.3) and (2.5) are meaningful since S_m holds trivially (with $P = V$) and S_0 is true (put $p = \emptyset$).

In the special case when $r = 1$, $k_1 = k$, $h_1 = g_k$, $f_1 = f$, we write $[k_1 f]_0$ and $[k_1 f]_1$ instead of (2.3) and (2.5).

We observe that if G is a 2-graph, then $A_0 = [2, 1]_0$ and $A_1 = [2, 1]_1$, where A_0 and A_1 are defined in the introduction in order to state Gallai's theorem.

3. Results.

THEOREM 1. (i) *If* $2 \leqq i \leqq n$, $E \in \mathscr{E}^i$, $v(E) = \alpha_i$, *then*

$$w(E) \leqq m - a - \alpha_i = \beta_i.$$

(ii) *If i is G-admissible and $E \in \mathscr{E}^i$ is admissible, then*

$$w(E) = m - a - \alpha_i = \beta_i.$$

(iii) 2 *is G-admissible and* $\beta_2 = 0$.

(iv) *If i is G-admissible and $i \leqq z$, then* $\beta_i = 0$.

Note in particular from (ii) and (iv) that $a + \alpha_z = m$. This corresponds to Gallai's theorem for 2-graphs. ($B_0 + B_1 = m_1$ mentioned in the introduction.)

THEOREM 2. *If* $2 \leqq k_1 < k_2 < \ldots < k_r \leqq n$; $0 \leqq f_i \leqq k_i$ *and*

$$0 \leqq h_i \leqq |T_{k_i}| \ (1 \leqq i \leqq r),$$

then

$$[k_i, h_i, f_i]_0{}^r + [k_i, h_i, k_i - f_i]_1{}^r = m.$$

Note in particular that $[2, 1]_0 + [2, 1]_1 = m$. This corresponds to $A_0 + A_1 = m$, Gallai's theorem for 2-graphs.

THEOREM 3. *If* $2 \leqq k' \leqq k \leqq n$ *and* $0 \leqq f \leqq k'$, *then*

$$[k, k - f]_0 = [k', k' - f]_0, \qquad [k, f]_1 = [k', f]_1.$$

4. Proofs. In order to prove Theorem 1 we require two lemmas.

LEMMA 1. *Let* $W \subset T_n$, $2 \leqq i \leqq n$, *and let* E *be a set of maximal value so that* $E < W$, $E \in \mathscr{E}^i$. *Also let* M *be a set of maximal value so that* $E \cap M = \emptyset$, $E \cup M \in \mathscr{E}^2$ *and* $M < W$. *Then there are sets* $F, N \subset W$ *such that:*
(a) $|F| = |E|, |N| = |M|, F \cap N = \emptyset$;
(b) $\cup F = \cup E, \cup M \subset \cup N \subset \cup M \cup \cup E$.

Proof. We first show that, if $e_1, e_2 \in E \cup M$, $e_1 \neq e_2$, then

$$e = e_1 \cup e_2 \not\subset w \in W. \tag{4.1}$$

Suppose that this is false and $e \subset w \in W$. If $e_1, e_2 \in E$, then

$$E' = (E - \{e_1, e_2\}) \cup \{e\} < W$$

and $E' \in \mathscr{E}^i$ and $v(E') = v(E) + 1$ by (2.2). This contradicts the maximality of $v(E)$.

If $e_1, e_2 \in M$, then

$$M' = (M - \{e_1, e_2\}) \cup \{e\} < W, \quad E \cap M' = \emptyset, \quad E \cup M \in \mathscr{E}^2,$$

and again $v(M') = v(M) + 1$ by (2.2). This contradicts the maximality of $v(M)$. Finally, if we assume that $e_1 \in E$ and $e_2 \in M$, then

$$E'' = (E - \{e_1\}) \cup \{e\} < W, \quad E'' \in \mathscr{E}^i \quad (\text{since } E \cup M \in \mathscr{E}^2)$$

and clearly $v(E'') > v(E)$ which again contradicts the maximality of $v(E)$. This proves (4.1).

It follows from (4.1) and the fact that $E \cup M < W$, that there is an injection $g: E \cup M \to W$ so that:

$$g(u) = w \Rightarrow u \subset w, \quad u \subset g(u) \in W \qquad (u \in E \cup M). \tag{4.2}$$

Put $g(E) = F$, $g(M) = N$; then (a) holds.

It follows from (4.2) that $\cup E \subset \cup F$ and $\cup M \subset \cup N$.

If there is a point $x \in \cup F - \cup E$, then there is some $e \in E$ so that $x \in g(e) - e$. Then $E' = (E - \{e\}) \cup \{e \cup \{x\}\} < W$, $E' \in \mathscr{E}^i$, and $v(E') > v(E)$, which is impossible. This proves that

$$\cup F = \cup E.$$

Similarly, if there is a point $x \in \cup N - \cup M \cup \cup E$, then there is $e' \in M$ so that $x \in g(E')$ and by putting $M' = (M - \{e'\}) \cup \{e' \cup \{x\}\}$, we contradict the maximality of $v(M)$. This shows that

$$\cup N \subset \cup M \cup \cup E.$$

This completes the proof of (b) and Lemma 1.

LEMMA 2. *If* $E \in \mathscr{E}^i$ *and* $v(E) = \alpha_i$, *then there is a set* $U' \subset T_n$ *which covers* V *such that*

$$|U'| = m - w(E) - \alpha_i.$$

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Proof. Let M be a set of maximum value so that $M \cap E = \emptyset$, $M \cup E \in \mathscr{E}^2$. Then $v(M) = w(E)$. It follows from the maximal property of $v(E)$ that if $e \in M$, then $e \in T_2 \cup T_3 \cup \ldots \cup T_{i-1}$. Hence, if u_k is the number of k-edges in $M \cup E$, then

$$|\bigcup M \cup E| = \sum_{k=2}^{n} k u_k.$$

By Lemma 1 there are $F, N \subset T_n$ such that Lemma 1(a) and (b) hold. Put $P = V - \bigcup E \cup M = V - \bigcup F \cup N$. Then

$$|P| = m - \sum_{k=2}^{n} k u_k.$$

P is an independent set of points for, if $e \subset P$ and $e \in T_2$, then

$$E \cup M \cup \{e\} \in \mathscr{E}^2$$

and this contradicts the maximality of $v(E)$ or $v(M)$.

Therefore, there is an injection $\psi: P \rightarrow T_n$ so that $x \in \psi(x)$ for $x \in P$. Let $L = \psi(P)$ and put $U' = F \cup N \cup L$. Then U' covers V and

$$|U'| = |L| + |N| + |F| = \left(m - \sum_{k=2}^{n} k u_k\right) + \sum_{k=2}^{n} u_k$$
$$= m - v(M) - v(E) = m - w(E) - \alpha_i.$$

This proves Lemma 2.

Proof of Theorem 1. (i) If $E \in \mathscr{E}^i$ and $v(E) = \alpha_i$, then by Lemma 2 there is a set $U' \subset T_n$ such that $|U'| = m - w(E) - \alpha_i$. The result follows since $|U'| \geqq a$.

(ii) Since E is admissible by hypothesis, then there is a minimal cover U such that $E < U$. Let M be a set of maximal value such that

$$M < U, \quad M \cap E = \emptyset, \quad M \cup E \in \mathscr{E}^2.$$

Then $v(M) \leqq w(E)$ by definition of $w(E)$. If $P = V - UM \cup E$, then there is no 2-edge $e \subset P$ such that $\{e\} < U$. Otherwise,

$$U > E \cup M \cup \{e\} \in \mathscr{E}^2,$$

and we contradict the maximality of $v(M)$. Therefore, since U covers V, it follows that there is an injection $\psi: P \rightarrow U$ so that $x \in \psi(x)$ for $x \in P$. Put $L = \psi(P)$. Then each element of P corresponds to a unique member of L. By Lemma 1, there are $F, N \subset U$ such that Lemma 1(a) and (b) hold. Clearly, L has no member in common with $F \cup N$ and thus

$$a = |U| \geqq |L| + |F| + |N| = m - |\bigcup E \cup M| + |E| + |M|$$
$$= m - v(M) - v(E) = m - w(E) - \alpha_i.$$

By Lemma 2, there is $U' \subset T_n$ such that $a \leqq |U'| \leqq m - w(E) - \alpha_i$. It follows that $a = m - w(E) - \alpha_i$ and this proves (ii).

(iii) Let $E' \in \mathscr{E}^2, v(E') = \alpha_2$. As we already observed, this implies that $w(E') = 0$. Hence, by Lemma 2, there is a set $U' \subset T_n$ which covers V so that

$$|U'| = m - \alpha_2.$$

Let U be any minimal cover of V and let E be a set of edges of maximal value so that $E < U$ and $E \in \mathscr{E}^2$. By Lemma 1 (with $M = N = \emptyset$), there is a set $F \subset U$ so that $|F| = |E|$ and $UF = UE$. The maximal condition on $v(E)$ ensures that the set $P = V - UE$ contains no 2-edge n with $\{n\} < U$. Therefore, since U covers V, there is a set of 2-edges $L \subset U$ so that $|L| = |P|$ and each element of P is a member of exactly one edge in L. Thus, the set of 2-edges $F \cup L$ covers V and, since U is minimal, $U = F \cup L$. Therefore,

$$|U| = |F| + |L| = |E| + (m - |\cup E|) = m - v(E) \geqq m - \alpha_2 = |U'|.$$

Since U is a minimal cover, it follows that $v(E) = \alpha_2$, and hence E is an admissible set and 2 is G-admissible.

(iv) If i is G-admissible and $i \leqq z$, then $\alpha_i \geqq \alpha_z$ and by (ii) and the definition of z,

$$0 \leqq \beta_i = m - a - \alpha_i \leqq m - a - \alpha_z = 0,$$

i.e. $\beta_i = 0$.

Proof of Theorem 2. Let P be a set of $p = [k_i, h_i, f_i]_1^r$ points so that S_p is true, i.e. there are sets $N_i \subset T_{k_i}$ $(1 \leqq i \leqq r)$ so that $|N_i| = h_i$ and (2.4) holds. Let $P' = V - P$, then

$$|x \cap P'| \leqq k_i - f_i \qquad (x \in N_i; 1 \leqq i \leqq r),$$

and therefore, by the definition of (2.5),

$$m - [k_i, h_i, f_i]_0^r = |P'| \leqq [k_i, h_i, k_i - f_i]_1^r = q. \tag{4.3}$$

Now let g be a set of q points so that S_q' is true, i.e. there are sets $N_i \subset T_{k_i}$ $(1 \leqq i \leqq r)$ so that $|N_i'| = h_i$ and

$$|x \cap g| \leqq k_i - f_i \qquad (x \in N_i'; 1 \leqq i \leqq r).$$

Then if $g' = V - g$, $|x \cap g'| \leqq f_i$ $(x \in N_i'; 1 \leqq i \leqq r)$, and hence

$$m - q = |g'| \geqq [k_i, h_i, f_i]_0^r. \tag{4.4}$$

The theorem follows from $|A|$ and $|B|$.

Proof of Theorem 3. Let P be a set of $p = [k, k - f]_0$ points so that every k-edge of G contains at least $k - f$ elements of P. Let x' be any k'-edge of G. Since $k' \leqq k$, there is a k-edge $x \supset x'$. Then

$$|x' - P| \leqq |x - P| \leqq f,$$

i.e.

$$|x' \cap P| \geqq k' - f.$$

It now follows from the definition that

$$[k', k' - f]_0 \leqq |P| = p.$$

Now let P_1 be a set of $p_1 = [k', k' - f]_0$ points such that every k'-edge contains at least $k' - f$ points of P_1. Suppose that there is $x \in T_k$ so that $|x \cap P_1| < k - f$. Then there is $y \subset x - P$ so that $|y| = f + 1$. Since $k' \geqq f + 1$, by hypothesis, it follows that there is $x' \in T_{k'}$, so that $y \subset x' \subset x$. Then $|x' \cap P| \leqq |x' - y| < k' - f$, a contradiction. This shows that

$$|x \cap P_1| \geqq k - f \qquad (x \in T_k),$$

and hence $p \leqq |P_1| = p_1$. This proves the first relation in Theorem 3.

By specializing Theorem 2 we obtain:

$$[k, k - f]_0 = m - [k, f]_1, \qquad [k', k' - f]_0 = m - [k', f]_1,$$

and by inspecting the first relation in Theorem 3, we have:

$$[k, f]_1 = [k', f]_1.$$

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