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RADIAL FUNCTIONS AND MAXIMAL ESTIMATES FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

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Abstract

Maximal estimates are considered for solutions to an initial value problem for the Schrödinger equation. The initial value function is assumed to be radial in $\mathbb{R}^n$, $n \geq 2$.

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Let f belong to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and set

$$S_t f(x) = u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

where $a > 1$. Here $\widehat{f}$ denotes the Fourier transform of f, defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x) dx.$$

We then have $u(x,0) = f(x)$, and in the case $a = 2$, u is a solution to the Schrödinger equation $\Delta u = i\partial u/\partial t$. We set

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|, \quad x \in \mathbb{R}^n.$$

We also introduce Sobolev spaces H_s by setting

$$H_s = \{f \in \mathscr{S}'; \|f\|_{H_s} < \infty\}, \quad s \in \mathbb{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi\right)^{1/2}.$$

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We shall here study estimates of the type

$$(1)\qquad \Big(\int_{B(0;R)} |S^* f(x)|^2 dx\Big)^{1/2} \le C_R \|f\|_{H_s}, \quad f \in \mathscr{S}(\mathbb{R}^n),$$

where $B(0; R) = \{x \in \mathbb{R}^n;\ |x| \le R\}$. The inequality (1) has implications for the existence almost everywhere of $\lim_{t\to 0} u(x,t)$ for solutions u of the Schrödinger equation. These problems were first studied by Carleson [3]. Later the inequality (1) and related questions were studied in several papers: see for example Dahlberg and Kenig [5], Kenig and Ruiz [6], Carbery [2], Cowling [4], Sjölin [10], Vega [12] and Kenig, Ponce and Vega [7]. The following results are known. For $n = 1$, (1) holds with $s = 1/4$, and $1/4$ cannot be replaced by a smaller number. In one variable one also has the improvement

$$(2)\qquad \Big(\int_{B(0;R)} |S^* f(x)|^4 dx\Big)^{1/4} \le C_R \|f\|_{H_{1/4}}.$$

For $n = 2$, (1) holds with $s = 1/2$ and in the case $n = 2, a = 2$, Bourgain [1] also has a result for $H_s(\mathbb{R}^2)$ for some $s < 1/2$. In the case $n \ge 3$, (1) is known to hold for $s > 1/2$.

For radial functions in $\mathbb{R}^n$, $n \ge 2$, Prestini [9] has proved that

$$(3)\qquad \int_{B(0;R)} S^* f(x) dx \le C_R \|f\|_{H_{1/4}}$$

and here $1/4$ cannot be replaced by a smaller number.

The purpose of this paper is to improve the integrability in the left hand side of (3). For $n \ge 2$ we shall prove the following results.

THEOREM 1. *If $q = 4n/(2n-1)$, then for f radial,*

$$(4)\qquad \Big(\int_{B(0;R)} |S^* f(x)|^q\, dx\Big)^{1/q} \le C_R \|f\|_{H_{1/4}}.$$

If $q > 4n/(2n-1)$, then the estimate (4) does not hold for all radial functions f.

Theorem 1 is a direct consequence of the following theorem.

THEOREM 2. *Assume $2 \le q \le 4$. If $\alpha = q(2n-1)/4 - n$ and f is radial, then*

$$(5)\qquad \Big(\int_{B(0;R)} |S^* f(x)|^q |x|^\alpha\, dx\Big)^{1/q} \le C_R \|f\|_{H_{1/4}}.$$

If $\alpha < q(2n-1)/4 - n$ then (5) does not hold for all radial functions f.

In the proof of Theorem 2 we shall use Pitt's inequality for Fourier transforms, which states that

$$(6) \qquad \left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^q |\xi|^{-\gamma q} d\xi\right)^{1/q} \leq C\left(\int_{\mathbb{R}} |f(x)|^p |x|^{\alpha p}\, dx\right)^{1/p},$$

if $q \geq p, 0 \leq \alpha < 1 - 1/p, 0 \leq \gamma < 1/q$ and $\gamma = \alpha + 1/p + 1/q - 1$ (see for instance Muckenhoupt [8]). We take $q = 2$ and $\gamma = 1/4$ in (6) and then obtain

$$(7) \qquad \left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{-1/2} d\xi\right)^{1/2} \leq C\left(\int_{\mathbb{R}} |f(x)|^p |x|^{3p/4-1} dx\right)^{1/p}$$

for $4/3 \leq p \leq 2$.

Proof of Theorem 2. We assume $2 \leq q \leq 4$ and $1/p + 1/q = 1$ so that $4/3 \leq p \leq 2$. We let $t(x)$ be a measurable and radial function in $\mathbb{R}^n$ with $0 < t(x) < 1$ and set $Tf(x) = S_{t(x)}f(x)$, $f \in \mathscr{S}$. It is then sufficient to prove (5) with S^* replaced by T.

If f is radial we obtain $S_t f(s) = c_n s^{1-n/2} \int_0^\infty J_{n/2-1}(rs) e^{itr^a} \widehat{f}(r) r^{n/2} dr$, where $J_{n/2-1}$ denotes a Bessel function (see Stein and Weiss [11, p. 155]). Here we write $S_t f(s) = S_t f(x)$ if $s = |x|$ and $\widehat{f}(r) = \widehat{f}(\xi)$ if $r = |\xi|$.

Similarly, we obtain

$$Tf(s) = c_n s^{1-n/2} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} \widehat{f}(r) r^{n/2} dr.$$

To prove (5) we have to prove that

$$(8) \qquad \left(\int_0^R |Tf(s)|^q s^{q(2n-1)/4-1} ds\right)^{1/q} \leq C_R\left(\int_0^\infty |\widehat{f}(r)|^2 (1+r^2)^{1/4} r^{n-1} dr\right)^{1/2}.$$

We have

$$\begin{aligned} Tf(s)s^{(2n-1)/4-1/q} &= c_n s^{(2n-1)/4-1/q+1-n/2} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} \widehat{f}(r) r^{n/2} dr \\ &= c_n s^{3/4-1/q} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} g(r)(1+r^2)^{-1/8} r^{1/2} dr, \end{aligned}$$

where $g(r) = \widehat{f}(r)(1+r^2)^{1/8} r^{(n-1)/2}$. We set

$$Pg(s) = s^{3/4-1/q} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} g(r)(1+r^2)^{-1/8} r^{1/2} dr$$

and then have

$$Tf(s)s^{(2n-1)/4-1/q} = c_n Pg(s).$$

We have to prove that

$$(9) \qquad (\int_0^R |Pg(s)|^q ds)^{1/q} \le C_R(\int_0^\infty |g(r)|^2 dr)^{1/2}.$$

The basic idea in the proof of (9) is to estimate the adjoint of P by use of an inequality in our paper [10]. We set

$$P^*g(r) = (1+r^2)^{-1/8} r^{1/2} \int_0^R J_{n/2-1}(rs) e^{-it(s)r^a} s^{3/4-1/q} g(s)ds, \quad 0 < r < \infty,$$

if $g \in L^1(0, R)$. It is then easy to prove that

$$\int_0^\infty f(r)\overline{P^*g(r)}dr = \int_0^R Pf(s)\overline{g(s)}\,ds$$

if $g \in L^1(0, R)$, $f \in L^2(0, \infty)$ and f has a suitable decay at infinity. It is therefore sufficient to prove that

$$(10) \qquad (\int_0^\infty |P^*g(r)|^2 dr)^{1/2} \le C_R(\int_0^R |g(s)|^p ds)^{1/p}, \quad g \in L^p(0, R),$$

for $4/3 \le p \le 2$.

It is well-known that there exist constants b_1 and b_2 such that

$$|J_{n/2-1}(t) - (b_1 e^{it}/t^{1/2} + b_2 e^{-it}/t^{1/2})| \le C/t^{3/2}, \quad t > 1,$$

(see [11, p. 158]) and we therefore have

$$|t^{1/2} J_{n/2-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \le C/t, \quad t > 1.$$

It is also clear that

$$|t^{1/2} J_{n/2-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \le C, \quad 0 < t \le 1.$$

Setting $\gamma = 1/q - 1/4$ we have $s^{3/4-1/q} = s^{1/2}s^{-\gamma}$ and it follows that

$$\begin{aligned} P^*g(r) &= b_1(1+r^2)^{-1/8} \int_0^R e^{irs} e^{-it(s)r^a} s^{-\gamma} g(s)ds \\ &\quad + b_2(1+r^2)^{-1/8} \int_0^R e^{-irs} e^{-it(s)r^a} s^{-\gamma} g(s)ds + Q(r) \\ &= b_1 A(r) + b_2 B(r) + Q(r), \end{aligned}$$

where

$$(11) \qquad |Q(r)| \le C(1+r^2)^{-1/8} \int_0^R \min(1, 1/rs) s^{-\gamma} |g(s)| ds.$$

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We extend A to $\mathbb{R}$ by setting

$$A(\xi) = (1+\xi^2)^{-1/8} \int_0^R e^{i(\xi s - t(s)|\xi|^a)} s^{-\gamma} g(s)\,ds, \quad -\infty < \xi < 0.$$

Then $B(\xi) = A(-\xi), 0 < \xi < \infty$, and to estimate A and B it is therefore sufficient to prove that

$$(12) \qquad (\int_{-\infty}^{\infty} |A(\xi)|^2 d\xi)^{1/2} \le C_R \|g\|_p,$$

where

$$|g\|_p = (\int_0^R |g(s)|^p ds)^{1/p}.$$

Choose ρ real–valued in $C_0^\infty(\mathbb{R})$ such that $\rho(\xi) = 1, |\xi| \le 1$, and $\rho(\xi) = 0, |\xi| \ge 2$, and set $\rho_N(\xi) = \rho(\xi/N)$ for $N > 1$. Then set

$$A_N(\xi) = \rho_N(\xi)|\xi|^{-1/4} \int_0^R e^{i(s\xi - t(s)|\xi|^a)} s^{-\gamma} g(s)\,ds.$$

We shall prove that

$$(13) \qquad (\int_{\mathbb{R}} |A_N(\xi)|^2 d\xi)^{1/2} \le C_R \|g\|_p$$

with C_R independent of N, and (12) follows from this inequality.

We have

$$\begin{aligned}\int_{\mathbb{R}} |A_N(\xi)|^2 d\xi &= \int_{\mathbb{R}} A_N(\xi)\overline{A_N(\xi)} d\xi \\ &= \int_{\mathbb{R}} \rho_N(\xi)^2 |\xi|^{-1/2} (\int_0^R e^{i(s\xi - t(s)|\xi|^a)} s^{-\gamma} g(s)\,ds) \cdot (\int_0^R e^{-i(s'\xi - t(s')|\xi|^a)} s'^{-\gamma} \overline{g(s')}\,ds') d\xi \\ &= \int_0^R \int_0^R (\int_{\mathbb{R}} e^{i[(s-s')\xi - (t(s)-t(s'))|\xi|^a]} \rho_N(\xi)^2 |\xi|^{-1/2} d\xi) s^{-\gamma} g(s) s'^{-\gamma} \overline{g(s')}\,ds\,ds'.\end{aligned}$$

It is proved in [10, pp. 709–712], that the inner integral is bounded by $C|s-s'|^{-1/2}$ and we therefore obtain

$$(14) \qquad |A_N\|_2^2 \le C \int_{\mathbb{R}} \int_{\mathbb{R}} |s-s'|^{-1/2} s^{-\gamma} |g(s)| s'^{-\gamma} |g(s')|\,ds\,ds',$$

where we have extended g to $\mathbb{R}$ by setting $g(s) = 0$ outside $[0, R]$.

We shall now use the Riesz potential operator I_β, $0 < \beta < 1$, defined by

$$I_\beta f(x) = c_\beta \int_{\mathbb{R}} |x-y|^{-1+\beta} f(y)\,dy, \quad x \in \mathbb{R}.$$

Here c_β is chosen so that $(I_\beta f)\widehat{\ }(\xi) = |\xi|^{-\beta} \widehat{f}(\xi)$.

Using Fourier transforms one then has

$$\begin{aligned}\|A_N\|_2^2 &\le C\int_{\mathbb{R}} I_{1/2}(t^{-\gamma}|g|)(s)s^{-\gamma}|g(s)|ds\\ &= C\int_{\mathbb{R}} |\xi|^{-1/2}(s^{-\gamma}|g|)\widehat{\ }(\xi)\overline{(s^{-\gamma}|g|)\widehat{\ }(\xi)}d\xi\\ &= C\int_{\mathbb{R}} |\xi|^{-1/2}|(s^{-\gamma}|g|)\widehat{\ }(\xi)|^2 d\xi.\end{aligned}$$

This formula is justified since we may assume that g is bounded and vanishes close to the origin.

Invoking (7) one then obtains

$$\|A_N\|_2 \le C\Big(\int_{\mathbb{R}} |s^{-\gamma}g|^p|s|^{3p/4-1}ds\Big)^{1/p} = C\|g\|p,$$

since

$$-\gamma p + \frac{3p}{4} - 1 = -\Big(\frac{1}{q}-\frac{1}{4}\Big)p + \frac{3p}{4} - 1 = -\frac{p}{q} + p - 1 = 0.$$

It remains to prove that if $Q(r)$ satisfies (11), then

$$(15)\qquad \Big(\int_0^\infty |Q(r)|^2 dr\Big)^{1/2} \le C_R\|g\|_p.$$

For $0 < r < 1$ one has

$$|Q(r)| \le \int_0^R s^{-\gamma}|g|ds \le \Big(\int_0^R s^{-\gamma q}ds\Big)^{1/q}\|g\|_p \le C_R\|g\|_p,$$

since $\gamma q = 1 - q/4 < 1$. Hence

$$(16)\qquad \Big(\int_0^1 |Q(r)|^2 dr\Big)^{1/2} \le C_R\|g\|_p.$$

For $r > 1$ it follows from (11) that

$$|Q(r)| \le CQ_1(r) + CQ_2(r),$$

where $Q_1(r) = r^{-1/4}\int_0^{1/r} s^{-\gamma}|g|ds$ and $Q_2(r) = r^{-5/4}\int_{1/r}^R s^{-1-\gamma}|g|ds$ (here we assume $R > 1$).

Using a change of variable we obtain

$$\int_1^\infty Q_1(r)^2 dr = \int_0^1 M_1(t)^2 dt,$$

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where

$$M_1(t) = \frac{1}{t}Q_1(\frac{1}{t}) = \frac{1}{t}t^{1/4}\int_0^t s^{-\gamma}|g|ds$$
$$= t^{-3/4}\int_0^t s^{-\gamma}|g|ds \leq \int_0^t (t-s)^{-3/4}s^{-\gamma}|g|ds$$
$$\leq CI_{1/4}(s^{-\gamma}|g|)(t).$$

One has

$$(I_{1/4}(s^{-\gamma}|g|))\widehat{\ }|\xi| = |\xi|^{-1/4}(s^{-\gamma}|g|)\widehat{\ }(\xi)$$

and invoking Plancherel's theorem and arguing as above we obtain

$$\int_1^\infty Q_1(r)^2dr \leq C\int_{\mathbb{R}} |\xi|^{-1/2}|(s^{-\gamma}|g|)\widehat{\ }(\xi)|^2d\xi \leq C\|g\|_p^2.$$

It remains to estimate $Q_2(r)$. We have

$$\int_1^\infty Q_2(r)^2dr = \int_0^1 M_2(t)^2dt,$$

where

$$M_2(t) = \frac{1}{t}Q_2(\frac{1}{t}) = t^{1/4}\int_t^R s^{-1-\gamma}|g|ds \leq \int_t^R s^{-3/4}s^{-\gamma}|g|ds$$
$$\leq \int_t^R (s-t)^{-3/4}s^{-\gamma}|g|ds \leq CI_{1/4}(s^{-\gamma}|g|)(t),$$

and it follows as above that

$$(\int_1^\infty Q_2(r)^2dr)^{1/2} \leq C\|g\|_p.$$

Hence (15) is proved and the proof of (5) is complete.

We shall now prove that (5) does not hold if $\alpha < q(2n-1)/4 - n$. Therefore assume that (5) holds for $\alpha = q(2n-1)/4 - n - \varepsilon$, where $\varepsilon > 0$ is a small number. We shall prove that this leads to a contradiction.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be radial and non–negative. Assume that $\operatorname{supp}\varphi \subset \{\xi : 1 < |\xi| < 2\}$ and that $\varphi(\xi) = 1$ for $5/4 \leq |\xi| \leq 7/4$. Then set $\varphi_c(\xi) = \varphi(\xi/c), c > 1$, and choose f such that $\widehat{f} = \varphi_c$. It is then easy to see that

$$\|f\|_{H_{1/4}} \leq Cc^{n/2+1/4}. \qquad (17)$$

We have

$$S_tf(x) = c_n\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{it|\xi|^a}\varphi(\xi/c)d\xi$$
$$= c_n\int_{\mathbb{R}^n} e^{icx\cdot\eta}e^{it|c\eta|^a}\varphi(\eta)d\eta c^n$$

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and

$$S_0 f(x) = c_n c^n \int_{\mathbb{R}^n} e^{icx\cdot\eta}\varphi(\eta)d\eta = c_n c^n \widehat{\varphi}(cx).$$

It follows that

$$S^* f(x) \geq |S_0 f(x)| \geq c_0 c^n$$

for $|x| \leq \delta/c$, where c_0 and δ are positive constants. For $R > \delta$ we therefore obtain

$$\begin{aligned}
\Big(\int_{B(0;R)} |S^* f(x)|^q |x|^\alpha\, dx\Big)^{1/q} &\geq c_0\Big(\int_{|x|\leq\delta/c} c^{nq}|x|^\alpha dx\Big)^{1/q} \\
&= c_0 c^n \Big(\int_0^{\delta/c} t^{\alpha+n-1}dt\Big)^{1/q} \geq c_0 c^n (c^{-\alpha-n})^{1/q} \\
&= c_0 c^{n-(\alpha+n)/q}. \qquad (18)
\end{aligned}$$

Now

$$n - \frac{\alpha+n}{q} = n - \frac{2n-1}{4} + \frac{\varepsilon}{q} = \frac{n}{2} + \frac{1}{4} + \frac{\varepsilon}{q}$$

and combining (5) with (17) and (18) we obtain

$$c^{n/2+1/4+\varepsilon/q} \leq C c^{n/2+1/4}.$$

Taking c large we conclude that $\varepsilon \leq 0$, which gives a contradiction. The proof of Theorem 2 is complete.

We finally remark that the method which we used in the proof of Theorem 2 to show that (5) cannot be improved, can also be used to prove that the L^4 estimate in (2) cannot be replaced by an L^q estimate for $q > 4$.

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