Canad. J. Math. Vol. **62** (4), 2010 pp. 870–888
doi:10.4153/CJM-2010-045-2

The Brascamp–Lieb Polyhedron

Stefán Ingi Valdimarsson

Abstract. A set of necessary and sufficient conditions for the Brascamp–Lieb inequality to hold has recently been found by Bennett, Carbery, Christ, and Tao. We present an analysis of these conditions. This analysis allows us to give a concise description of the set where the inequality holds in the case where each of the linear maps involved has co-rank 1. This complements the result of Barthe concerning the case where the linear maps all have rank 1. Pushing our analysis further, we describe the case where the maps have either rank 1 or rank 2.

A separate but related problem is to give a list of the finite number of conditions necessary and sufficient for the Brascamp–Lieb inequality to hold. We present an algorithm which generates such a list.

1 Introduction

The Brascamp–Lieb inequality unifies and generalises several of the most central inequalities in analysis, among others the inequalities of Hölder, Young, and Loomis–Whitney. It has the form

$$\int_H \prod_{j=1}^m f_j^{p_j}(B_j x)\,\mathrm{d}x \le C \prod_{j=1}^m \Big(\int_{H_j} f_j\Big)^{p_j} \tag{1.1}$$

where H and H_j are finite dimensional Hilbert spaces of dimensions n and n_j respectively, $B_j\colon H \to H_j$ are linear maps, p_j are nonnegative numbers, C is a finite constant and f_j are nonnegative functions. We shall refer to $((B_j),(p_j))$ as the Brascamp–Lieb datum for this inequality.

The inequality was first written down by Brascamp and Lieb in [5] where they posed two questions. The first one was how to find the necessary and sufficient conditions on the datum $((B_j),(p_j))$ for (1.1) to hold, and the second was to determine when the best constant for (1.1) is attained by a tuple of centred gaussian functions, $f_j(x) = e^{-\langle x, A_j x\rangle}$, with each A_j a symmetric and positive semi-definite linear transformation.

In [7] Lieb showed that gaussians exhaust the inequality in the following sense.

Theorem 1.1 **(Lieb's Theorem)** *Let $C((B_j),(p_j))$ be the smallest constant we can take in* (1.1) *so that it holds for all tuples (f_j) of integrable functions, and let $C_g((B_j),(p_j))$ be the smallest constant we can take so that it holds for tuples of centred gaussians. Then*

$$C((B_j),(p_j)) = C_g((B_j),(p_j)).$$

Received by the editors January 24, 2008; revised January 6, 2009.
Published electronically May 20, 2010.
AMS subject classification: **44A35**, 14M15, 26D20.
Keywords: Brascamp–Lieb inequality, Loomis–Whitney inequality, lattice, flag.

Brascamp and Lieb proved this theorem in the case when each B_j has rank one in [5]. With this theorem, the fundamental question of when $C((B_j),(p_j))$ is finite has been reduced to the question of when $C_g((B_j),(p_j))$ is finite. In [3] and [4] the question is further reduced by showing that the Brascamp–Lieb inequality (1.1) holds for the datum $((B_j),(p_j))$ if and only if we have

$$\dim V \leq \sum_j p_j \dim(B_j V) \tag{1.2}$$

for all subspaces V of H, the scaling condition

$$\dim H = \sum_j p_j \dim(H_j) \tag{1.3}$$

holds, and

$$p_j \geq 0 \tag{1.4}$$

for all j.

Let us fix the maps B_j. Then for which tuples (p_j) does the Brascamp–Lieb inequality hold, that is, which tuples satisfy (1.2), (1.3), and (1.4)?

Since each of the conditions is a linear inequality or equality in the variables (p_j) and since the coefficients in (1.2) are dimensions of spaces which can only range through a finite set, it is clear that the set of tuples (p_j) such that these conditions hold is a convex set in $\mathbb{R}^m$ whose boundary consists of a finite number of hyperplanes. It is thus a polyhedron, and we shall refer to it as the *Brascamp–Lieb polyhedron*, $\mathcal{S} = \mathcal{S}((B_j))$, for the m-transformation (B_j).

The scaling and positivity conditions (1.3) and (1.4) imply that this polyhedron lies in the intersection of a hyperplane and the first 2^m-tant in $\mathbb{R}^m$. What portion of this intersection the polyhedron occupies can vary greatly. In particular, for Hölder's inequality the conditions in (1.2) do not give any restrictions and the polyhedron is this whole intersection. On the other hand, (1.2) for the Loomis–Whitney inequality restricts the polyhedron to the one point set $(p_j)_{1\leq j\leq n} = (\frac{1}{n-1})_{1\leq j\leq n}$.

Conditions (1.2), (1.3), and (1.4) give a description of $\mathcal{S}((B_j))$ in the sense that if we want to check whether a particular point (p_j) belongs to $\mathcal{S}$, then we can do so by checking (p_j) against each one of these conditions and if it satisfies them all, then the point belongs to the polyhedron. However, it might be beneficial to give an alternative description for two reasons. First, the shape of the polyhedron can still seem quite unclear. In particular, we do not have a result that says that the point (p_j) lies in the polyhedron if and only if it is of some prescribed form. Secondly, there is the question of how many conditions are included in (1.2). Although, as we said above, it is only a finite number because the dimension of the spaces involved can only range through a finite set, it remains unclear how to get an exhaustive list of the conditions, as it would seem to require examining each subspace V of H. In this note, we will address both of these problems.

For the first problem, it is known by the Weyl–Minkowski theorem that a bounded polyhedron is a polytope, that is, the convex hull of a finite set of points, so each point in the polyhedron can be written as a convex combination of the vertices of the polyhedron. Here we say that a point (q_j) is a vertex of a polyhedron if there exists a hyperplane such that the intersection of the hyperplane and $\mathcal{S}$ is the singleton $\{(q_j)\}$, and by writing (p_j) as a convex combination of the vertices, we mean that (p_j) lies in the polyhedron if and only if we can write $p_j = \sum_{s=1}^{s_0} \lambda_s q_{s,j}$ for all j, where $\lambda_s \geq 0$, $\sum_s \lambda_s = 1$ and q_s for $s = 1, \dots, s_0$ is an enumeration of the vertices. For these standard results in convexity see, for example, [2].

The problem of determining the vertices of $\mathcal{S}$ has until now only been resolved in the rank-one case. There we have the following result.

Theorem 1.2 (Rank-one case, Barthe [1]) *Let $B_j x = \langle v_j, x\rangle$ for vectors v_j in H. Then (q_j) is a vertex of $\mathcal{S}$ if and only if $q_j = \chi_J(j)$ where χ_J denotes a characteristic function of an index set J such that $B = \{v_j | j \in J\}$ is a basis for H.*

This result is reproved in [6] and [4].

In Section 2 we present a new analysis of the properties of the vertices that has the benefit that, aside from yielding a new proof of the result of Barthe, it makes it possible to determine the form of the vertices in several other cases.

Theorem 1.3 (Rank $n-1$ case) *Assume B_j all have rank $n-1$, and for each j let v_j be a nonzero element in the kernel of B_j. Then (q_j) is a vertex of $\mathcal{S}$ if and only if $q_j = \frac{1}{n-1}\chi_J(j)$, where J is an index set such that $B = \{v_j | j \in j\}$ is a basis for H.*

In order to state the main tool for our treatment of these results, we give the following definition.

Definition 1.4 Let V be a proper subspace of H which is not the space $\{0\}$. As in [4] we say that V is a *critical subspace* if $\dim V = \sum_j p_j \dim(B_j V)$, that is, if there is equality in (1.3) for V.

We define a *critical flag* to be a flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_s$ of subspaces of H, where each space is critical or each space except V_s is critical and $V_s = H$.

Theorem 1.5 *Let (q_j) be a vertex of $\mathcal{S}$. Then the support of (q_j), $\{j \mid q_j \neq 0\}$, can have at most n elements where n is the dimension of H.*

Furthermore, there will exist a critical flag $\mathcal{U}$ and an index set J such that the equations

$$p_j = 0 \quad \text{for } j \notin J; \tag{1.5}$$

$$\dim U = \sum_j p_j \dim(B_j U) \quad \text{if } U \in \mathcal{U} \tag{1.6}$$

have a unique solution $(p_j) = (q_j)$.

Finally, we will also push the analysis further to give a description of the vertices in the case when each B_j has rank either 1 or 2.

In Section 3 we address the second problem mentioned above, that is, how we can know which conditions are included in (1.2). To state the result we give the following definition.

Definition 1.6 Let $(V_k)_{k\in K}$ be a family of subspaces of a common space. Then the *lattice* of (V_k), denoted $\mathcal{L}_{(V_k)}$, is defined as the smallest set of subspaces such that the following holds.

(i) $V_k \in \mathcal{L}_{(V_k)}$ for each $k \in K$;
(ii) $V_1 \cap V_2, V_1 + V_2 \in \mathcal{L}_{(V_k)}$ for any $V_1, V_2 \in \mathcal{L}_{(V_k)}$.

In other words, the lattice of a given family of spaces is the smallest set of spaces that contains each member of the family and is closed under the operations of set intersection and vector space addition. We say that the lattice is generated by the family.

We neither require $\{0\}$ nor the whole space to be elements of the lattice.

Definition 1.7 For the m-transformation (B_j), we let $\mathcal{L}_{(B_j)}$ denote $\mathcal{L}_{(\ker(B_j))}$, the lattice generated by the kernels of B_j.

In Section 3 we prove the following theorem.

Theorem 1.8 *Let $((B_j),(p_j))$ be a Brascamp–Lieb datum. Then a necessary and sufficient condition for the the Brascamp–Lieb constant $C((B_j),(p_j))$ to be finite is that* (1.3) *and* (1.4) *hold, and* (1.2) *holds for each subspace in $\mathcal{L}_{(B_j)}$.*

However, even with Theorem 1.8 there remain some questions. Firstly, do we know that the number of elements in $\mathcal{L}_{(B_j)}$ is finite? The answer to this seems to be no in general; see [8] for an overview discussion on lattice theory, to which this question belongs. However, it is clear that the number of elements is countable and it is straightforward to generate a list of elements on which we can check (1.2) in sequence. So for computational purposes, a more important variant of this question is: how do we know when to stop, that is, when can we be sure that we have a list of all the conditions included in (1.2)? We will address this question towards the end of Section 3.

Remark 1.9 Michael Christ comments via personal communication that by working through the induction proof of the Brascamp–Lieb inequality in an algorithm that gives necessary and sufficient conditions for $C((B_j),(p_j))$ to be finite can be found. The proof we give of Theorem 1.8 is along these lines. The proof also establishes that the lattice $\mathcal{L}_{(B_j)}$ is sufficient.

2 The Vertices of $\mathcal{S}$

Lemma 2.1 *Let U and W be critical subspaces of H for a Brascamp–Lieb datum $((B_j),(p_j))$. Then $U \cap W$ and $U + W$ are also critical, and for all j such that $p_j > 0$, we have that*

$$\dim(B_jU) + \dim(B_jW) = \dim(B_j(U\cap W)) + \dim(B_j(U+W)). \tag{2.1}$$

https://doi.org/10.4153/CJM-2010-045-2 Published online by Cambridge University Press

Proof Since U and W are critical, we get that

$$
\begin{aligned}
&\sum_j p_j \dim(B_jU) + \sum_j p_j \dim(B_jW)\\
&\quad = \sum_j p_j(\dim(B_jU) + \dim(B_jW))\\
&\quad = \sum_j p_j(\dim(B_jU \cap B_jW) + \dim(B_jU + B_jW))\\
&\quad \geq \sum_j p_j(\dim(B_j(U \cap W)) + \dim(B_j(U + W)))\\
&\quad \geq (\dim(U \cap W) + \dim(U + W))\\
&\quad = (\dim U + \dim W),
\end{aligned} \tag{2.2}
$$

where we have twice used the fact that $\dim E + \dim F = \dim(E+F) + \dim(E \cap F)$ for any subspaces E and F. Also for the first inequality, we have used that $\dim(B_jU + B_jW) = \dim(B_j(U + W))$ and $\dim(B_jU \cap B_jW) \geq \dim(B_j(U \cap W))$. The second inequality follows since (p_j) belongs to the polyhedron $\mathcal{S}((B_j))$, and therefore the condition (1.2) holds with (p_j) and both $U \cap W$ and $U + W$.

Since we are assuming that the beginning and the end of this chain are equal, we must in fact have equality all the way. This tells us that we have equality in inequality (1.2) for $U \cap W$ and $U + W$ and that (2.1) holds for all j such that $p_j > 0$. ■

Proof of Theorem 1.5 If (q_j) is a vertex of $\mathcal{S}$, then we will have a set of indices J such that

$$q_j = 0 \qquad \text{for } j \notin J \tag{2.3}$$

and a collection of subspaces $\mathcal{V}$ consisting of the critical subspaces together with H and $\{0\}$ such that

$$\dim V = \sum_j q_j \dim(B_jV) \qquad \text{if } V \in \mathcal{V}. \tag{2.4}$$

A vertex of a polyhedron is the unique solution of the set of linear equations which the facets adjacent to the vertex satisfy. Thus, the system (2.3), (2.4) of linear equations determines the vertex (q_j) uniquely.

Let us now apply row operations to this system to simplify it. By subtracting the appropriate multiples of (2.3) from (2.4), we can substitute (2.4) with

$$\dim V = \sum_{j \in J} q_j \dim(B_jV) \qquad \text{for } V \in \mathcal{V}. \tag{2.5}$$

Now, take $U, W \in \mathcal{V}$. By Lemma 2.1, we have $U \cap W, U + W \in \mathcal{V}$ as well. (This is obvious if either U or W is $\{0\}$ or H.) Furthermore, the equality for W can be deduced from the equality for $U \cap W$, U and $U + W$ as follows:

$$
\begin{aligned}
&\Big(\dim(U \cap W) = \sum_{j \in J} q_j \dim(B_j(U \cap W)) \Big) \\
+ &\Big(\dim(U + W) = \sum_{j \in J} q_j \dim(B_j(U + W)) \Big) \\
- &\Big(\qquad \dim U = \sum_{j \in J} q_j \dim(B_j U) \qquad \Big) \\
\hline
= &\Big(\qquad \dim W = \sum_{j \in J} q_j \dim(B_j W) \qquad \Big)
\end{aligned}
$$

We have used (2.1) to simplify the right-hand side. This shows that we may remove the equation coming from W from (2.5) by row operations and thus without affecting the solution set.

Let us try to remove as many equations from (2.5) as we can without affecting the solution set to the system (2.5) and (2.3). First of all, (2.5) is content free for $V = \{0\}$, so we may throw that space out of $\mathcal{V}$. Let us then take a $U_1 \in \mathcal{V}$ such that no proper subspace of U_1 is in $\mathcal{V}$. Clearly such a space exists as we cannot have an infinite chain of nested subspaces in H. Define $\mathcal{V}_{U_1} := \{W \in \mathcal{V} : U_1 \subset W\}$. Then all the equalities for the subspaces in $\mathcal{V}$ can be deduced from the equalities for the subspaces in $\mathcal{V}_{U_1}$. To see this we note that if $W \in \mathcal{V} \setminus \mathcal{V}_{U_1}$, then $W \cap U_1 = \{0\}$, so the equality for W can be deduced from the equalities for U_1 and $U_1 + W$ which are elements of $\mathcal{V}_{U_1}$.

Next, let $U_2 \in \mathcal{V}_{U_1}$, $U_2 \neq U_1$ be such that no subspace $W \in \mathcal{V}_{U_1}$ lies properly between U_1 and U_2. Then, as in the last paragraph, we see that all equalities for subspaces in $\mathcal{V}_{U_1}$ can be deduced from the equalities for the subspaces in $\mathcal{V}_{U_2}$ and the equality for U_1. Continuing this process, we get a critical flag $U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_s$ such that all the equalities for the subspaces in $\mathcal{V}$ can be deduced from the equalities for the spaces in this flag.

Thus we have seen that by using row operations we can remove all the equations from (2.5) except the ones coming from this flag, which we shall refer to as $\mathcal{U}$, and still have the linear system

$$q_j = 0 \quad \text{for } j \notin J; \tag{2.6}$$

$$\dim U = \sum_j q_j \dim(B_j U) \quad \text{if } U \in \mathcal{U} \tag{2.7}$$

which is equivalent to the original one. Since H is n-dimensional, $\mathcal{U}$ can have at most n elements, so the number of equations in (2.7) is at most n. However, since the system (2.6), (2.7) is a linear system which has a unique solution in $\mathbb{R}^m$, there must be at least m equations in the system. Therefore, there must be at least $m-n$ elements not in the set J, and so the solution to the system (q_j) can have at most n nonzero elements. ■

The next lemma can be useful when checking that the Brascamp–Lieb inequality is satisfied.

Lemma 2.2 *Let a Brascamp–Lieb datum $((B_j),(p_j))$ be given and assume that $\mathcal{U} = (U_1,\dots,U_s)$ is a critical flag in H and that $U_s = H$. Assume also that the inequality* (1.2) *holds for any space $\tilde{W}$ which can be added into the flag.*

Then inequality (1.2) *holds for any subspace W of H, so $(p_j) \in \mathcal{S}((B_j))$.*

Proof Take a subspace W of H. If we reexamine the calculations in (2.2), we see that if U is a subspace of H and we assume that (1.2) holds for $U \cap W$ and $U + W$ and it holds with equality for U, then we get that (1.2) holds for W.

Let us now define $t_0 \in \{0,\dots,s\}$ such that $U_{t_0} \subset W$ but $U_{t_0+1} \not\subset W$. To ensure that t_0 is well defined, we allow it to take the value 0 in which case we define $U_0 = \{0\}$. We see that if (1.2) holds for $W \cap U_{t_0+1}$ and $W + U_{t_0+1}$, then it holds for W. Since $U_{t_0} \subset W \cap U_{t_0+1} \subset U_{t_0+1}$, we see that (1.2) holds for $W \cap U_{t_0+1}$ by assumption. For $W + U_{t_0+1}$, we argue inductively. We note that $W + U_{t_0+1} \supset U_{t_0+1}$, so we can repeat this process for that space, that is, find a $t_1 > t_0$ such that $U_{t_1} \subset W + U_{t_0+1}$ but $U_{t_1+1} \not\subset W + U_{t_0+1}$, and then (1.2) for $W + U_{t_0+1}$ will follow from the condition for $(W+U_{t_0+1}) \cap U_{t_1+1}$ which lies between U_{t_1} and U_{t_1+1} and the condition for $W+U_{t_1+1}$. This process will terminate since all of the spaces are subspaces of H and equality in (1.2) holds for H. In the end we will get a flag $U_{t_0} \subset \cdots \subset U_{t_r}$ which is a subflag of the flag $\mathcal{U}$ and can therefore contain no more than s elements. Furthermore, this flag has the property that to confirm that (1.2) holds for W, we need only to check that (1.2) holds for spaces V such that $U_t \subset V \subset U_{t+1}$ with $t \in \{t_0,\dots,t_r\}$. Since W was arbitrary, we have proved the lemma. ■

Remark 2.3 To verify that a point $(q_j) \in \mathcal{S}(B_j)$ is a vertex, it is enough to determine that the facets of $\mathcal{S}((B_j))$ which (q_j) lies on have a unique point of intersection. In other words, for (q_j) some of the inequalities from (1.2) and (1.4) will be equalities and it is enough to show that those equalities together with the scaling condition (1.3) have a unique solution, namely (q_j).

We are now in a position to list all the possible vertices in several cases. First let us assume that all the maps B_j have the same rank and prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2 As before, we let (q_j) be a vertex of the polyhedron and J be the set of indices j such that $q_j > 0$. If the vectors v_j for $j \in J$ do not span H, then we do not have a solution to the system (1.2), (1.3), and (1.4). To see this, let V be a subspace of codimension 1 which contains v_j for all $j \in J$. Then, since $B_j = \langle v_j, x\rangle$, we see that $V^\perp$ lies in the kernel of all the relevant B_j. Therefore, testing (1.2) on $V^\perp$ gives $1 = \dim V^\perp \le \sum_j q_j \dim(B_j V^\perp) = 0$, which is impossible. This shows that $B = \{v_j | j \in J\}$ as defined in the theorem is a spanning set for H, and then Theorem 1.5 shows that $|J| = n$ so B is in fact a basis for H.

Furthermore, testing (1.2) on $\ker B_j$ gives that $n-1 \le \sum_{j' \in J\setminus\{j\}} q_{j'}$, and, together with the scaling condition (1.3) $\sum_{j'\in J} q_{j'} = n$, we get that $q_j \le 1$ for each $j \in J$, so considering that $|J| = n$ we see that in fact $q_j = 1$ for each $j \in J$. This shows that the vertices of $\mathcal{S}$ must have the form prescribed by the theorem.

Conversely, let (q_j) be a point of the form prescribed by the theorem. As before, let J be the set of indices such that

$$q_j = 0 \qquad \text{if } j \notin J. \tag{2.8}$$

For each $j \in J$, take a nonzero $u_j \in \cap_{j' \neq j} \ker B_{j'}$ and note that $B_j u_j \neq 0$ since otherwise u_j could not be a linear combination of the elements of B. Then $\{u_j | j \in J\}$ forms a basis, and if we define

$$U_j = \sum_{\substack{j' \in J \\ j' \leq j}} \operatorname{span}(u_{j'}),$$

then $\mathcal{U} = (U_j)_{j \in J}$ is a maximal flag in H. Let $s_j = |\{j' \in J | j' \leq j\}|$. Then $\dim U_j = s_j$, and for $j' \in J$ we get that $\dim B_{j'} U_j = 0$ if $j' > j$ and $\dim B_{j'} U_j = 1$ if $j' \leq j$. The inequality (1.2) for U_j thus becomes

$$s_j \leq \sum_{\substack{j' \in J \\ j' \leq j}} p_j, \tag{2.9}$$

so with the choice $p_j = q_j$, there is clearly equality here for each $U_j \in \mathcal{U}$. Thus, $\mathcal{U}$ is a critical maximal flag, so Lemma 2.2 says that $(q_j) \in \mathcal{S}$. Furthermore, (q_j) is the unique solution to the system (2.8), (2.9) with equality sign, so (q_j) is a vertex of $\mathcal{S}$. ■

Proof of Theorem 1.3 Let (q_j) be a vertex of the polyhedron and J as before. We first note that if the spaces $\ker B_j$ for $j \in J$ do not span H, then we do not have a solution to the system (1.2), (1.3), and (1.4) as can be seen from testing (1.2) on a space V such that $\sum_{j \in J} \ker B_j \subset V$ and $\dim V = n - 1$. This gives

$$n - 1 = \dim V \leq \sum_j q_j \dim(B_j V) = (n-2) \sum_j q_j,$$

whereas the scaling condition (1.3) gives $n = \sum_j q_j (n-1)$. From this and Theorem 1.5, we see that $|J| = n$. Thus, if we pick a nonzero vector v_j from each $\ker B_j$, then $B = \{v_j | j \in J\}$ is a basis for H.

Testing (1.2) on $\ker B_j$ gives that $1 \leq \sum_{j' \in J \setminus \{j\}} q_{j'}$, and, together with the scaling condition (1.3) $(n-1) \sum_{j \in J} q_j = n$, we get that $q_j \leq 1/(n-1)$ for each $j \in J$, so considering the scaling condition again and that $|J| = n$, we see that in fact $q_j = 1/(n-1)$ for each $j \in J$. This shows that the vertices of $\mathcal{S}$ must have the form prescribed by the theorem.

Conversely, let (q_j) be a point of the form prescribed. Define

$$U_j = \sum_{\substack{j' \in J \\ j' \leq j}} \ker B_{j'}$$

https://doi.org/10.4153/CJM-2010-045-2 Published online by Cambridge University Press

then $\mathcal{U} := (U_j)_{j\in J}$ is a maximal flag in H. The set of inequalities (1.2) for this flag becomes

$$s_j \leq \sum_{\substack{j'\in J\\ j'\leq j}} (s_j - 1)p_{j'} + \sum_{\substack{j'\in J\\ j'>j}} s_j p_{j'} \qquad j \in J \tag{2.10}$$

where $s_j := |\{j' \in J | j' \leq j\}|$. Since the number of terms in the first sum is s_j and the number of terms in the last sum is $n - s_j$, it is evident that with $q_j = \frac{1}{n-1}$ for $j \in J$, each inequality in (2.10) is satisfied with equality. Moreover, these resulting equalities together with the equations $p_j = 0$ for $j \notin J$ have (q_j) as a unique solution, so (q_j) is a vertex of the polyhedron. ■

2.1 Mixed Rank One and Two

We can push this analysis further and examine the mixed rank case when each B_j has rank 1 or 2.

***Theorem 2.4* (Mixed rank 1 and 2)** *The point (q_j) is a vertex of $\mathcal{S}$ if and only if the following holds. There is a set of indices J which can be decomposed as $J = J_1 \cup J_2$, where B_j for $j \in J_1$ is a rank 1 linear transformation from H and B_j for $j \in J_2$ is a rank 2 linear transformation such that the following hold:*

(i) *$q_j = 0$ for all $j \notin J$.*
(ii) *$q_j = 1$ for all $j \in J_1$.*
(iii) *The set J_2 can be divided into two sets $J_{2,1}$, and $J_{2,2}$ such that*
  - *$q_j = \frac{1}{2}$ for all $j \in J_{2,1}$,*
  - *$q_j = 1$ for all $j \in J_{2,2}$.*

(iv) *There exists a graph $G = (J_{2,1}, E)$ with each element of $J_{2,1}$ belonging to exactly two edges so that the graph consists of disjoint cycles which must furthermore be of odd length.*
(v) *There exists an ordering of the edges $E = \{e_1, \ldots, e_{s_1}\}$ with the following properties. Take any ordering of $J_{2,2} = \{j_1, \ldots, j_{s_2}\}$ and of $J_1 = \{i_1, \ldots, i_{s_3}\}$. Then*

$$\{0\} = U_0 \subsetneqq \cdots \subsetneqq U_{s_1} = V_0 \subsetneqq \cdots \subsetneqq V_{s_2} = W_0 \subsetneqq \cdots \subsetneqq W_{s_3} = H \tag{2.11}$$

is a critical flag, where
  - *$U_{k-1} = (U_k \cap \ker B_{j_1}) + (U_k \cap \ker B_{j_2})$ where $e_k = \{j_1, j_2\} \in J_{2,1}$ and $\dim(U_k/U_{k-1}) = 1$ for every $k = 1, \ldots, s_1$,*
  - *$V_{k-1} = V_k \cap \ker B_{i_k}$ and $\dim(V_k/V_{k-1}) = 2$ for every $k = 1, \ldots, s_2$,*
  - *$W_{k-1} = W_k \cap \ker B_{j_k}$ and $\dim(W_k/W_{k-1}) = 1$ for every $k = 1, \ldots, s_3$.*

Proof Again, we begin by assuming that (q_j) is a vertex of $\mathcal{S}$ and J and $\mathcal{U} = (U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_s)$ are such that (1.5) and (1.6) have a unique solution, namely $(p_j) = (q_j)$. Furthermore, it will be convenient to assume that no critical subspace can be added into the flag. Our first goal will be to determine what the equations of criticality can look like or rather the equations (2.12) below. Then we will convert this flag into the flag (2.11) showing that criticality is maintained at each step and that the solution set to the equations of criticality is unchanged.

By subtracting the equation for U_{k-1} from the equation for U_k, we see that we can replace (1.6) with

$$\dim(U_k/U_{k-1}) = \sum_j q_j(\dim(B_jU_k) - \dim(B_jU_{k-1})) \tag{2.12}$$

for $U_k \in \mathcal{U}$, $k \geq 1$ and with $U_0 = \{0\}$. In this set of equations we note that the coefficients multiplying q_j sum up to the rank of B_j and the constant coefficients sum up to $\dim H$. Therefore, if we let m_1 and m_2 be the number of elements in J_1 and J_2, then the sum of the elements in the coefficient matrix of (2.12) equals m_1+2m_2. Furthermore, since the set of equations (2.12) uniquely determines $(q_j)_{j\in J}$ and $|J| = m_1 + m_2$, we get that $s \geq m_1 + m_2$.

We note that the coefficients on the right hand side of (2.12) must all be non-negative integers and in each equation at least one must be non-zero since the left hand side is never zero.

There are now two cases. Either there is an equation in (2.12), all of whose coefficients are zero except one, for q_j with $j \in J_2$, which is 1, or we can give a bound on the number of equations in (2.12) as follows. For each $j \in J_1$ the coefficients of q_j in (2.12) must all be 0 except one, which must be 1. Let n_1 be the number of equations that contain a nonzero coefficient for an element q_j with $j \in J_1$. Then $n_1 \leq m_1$. Let t be the sum of the coefficients multiplying q_j for $j \in J_2$ in these n_1 equations. Say that there are n_2 equations remaining. Then the coefficients multiplying q_j for $j \in J_2$ in these remaining equations sum up to $2m_2 - t$. In the case we are looking at, each of these equations must contain at least two nonzero coefficients so we get that $n_2 \leq m_2 - t/2$. From this we get the chain of inequalities

$$s = n_1 + n_2 \leq m_1 + m_2 - t/2 \leq m_1 + m_2 \leq s,$$

so we must in fact have equality all the way, that is, there are exactly m_1 of the equations which have a nonzero coefficient for an element q_j with $j \in J_1$ and these equations have only these nonzero coefficients and there are exactly m_2 equations left which have all of the nonzero coefficients for the q_j with $j \in J_2$ which sum up to $2m_2$. Moreover, each of these m_2 equations must have either one coefficient equal to 2 and all others 0 or two coefficients equal to 1 and all others 0.

Let us show that we can pick off, one by one, the indices $j \in J_2$ that force us into the first case and be left with a residual index set $\tilde{J} = J_1 \cup \tilde{J}_2$ that fall into the second case.

So, assume one of the equations in (2.12) is of the form $1 = q_j$ with $j \in J_2$. Since the sum of the coefficients in front of q_j equals 2, there must be another equation with the term q_j. Let us show that this other equation also takes the form $1 = q_j$. If it does not, then it must be of the form $t = q_j + Q$, where $t > 1$ is an integer and Q stands for terms with $q_{j'}$, $j' \in J_2 \setminus \{j\}$. Assume that it comes from (2.12) with U_{k_j}/U_{k_j-1}, where the codimension of U_{k_j-1} in U_{k_j} is t. Since the coefficient multiplying q_j is 1, we get that there are $t-1$ independent vectors in the intersection

https://doi.org/10.4153/CJM-2010-045-2 Published online by Cambridge University Press

of $\ker B_j/U_{k_j-1}$ and U_{k_j}/U_{k_j-1}. Let $\tilde{U}$ denote the vector sum of the span of these and U_{k_j-1}. By testing (1.2) on $\tilde{U}$ and subtracting (1.2) on U_{k_j-1} which we know gives an equality, we get that $t-1 \leq Q'$, where Q' denotes the contribution to this sum from terms $q_{j'}$, $j' \in J_2 \setminus \{j\}$. Now we get the chain of inequalities

$$t = 1 + (t-1) \leq q_j + Q' \leq q_j + Q = t,$$

and so we must have equality all the way, and, in particular, this shows that $\tilde{U}$ is critical, contradicting our assumption that no critical subspace could be added to the flag.

Furthermore, we see that the equations determining the q_j discussed in the preceeding paragraph are completely separate from the equations determining $q_{j'}$ for $j' \in \tilde{J} = J \setminus \{j\}$. We can therefore repeat the preceding analysis with $\tilde{J}$ instead of J and with the two equations $1 = q_j$ removed from (2.12). The conclusion is that we will get a set of indices $J_{2,2}$ such that the equations in (2.12) involving q_j for $j \in J_{2,2}$ take the form $1 = q_j$ and a residual set $\tilde{J} = J_1 \cup J_{2,1}$.

Let us determine what the equations involving $j \in \tilde{J}$ look like. For each $j \in J_1$, the relevant equation from (2.12) takes the form $1 = q_j$ since the left-hand side must be 1, as we know that $0 < q_j \leq 1$ for each $j \in J$. The equations for q_j with $j \in J_{2,1}$ must all be of the form $t_{j,j'} = q_j + q_{j'}$.

If $j = j'$, then the equation must have the form $2 = 2q_j$. We see this since the left-hand side cannot be larger than 2 as q_j is at most 1 and since we must always have

$$\dim(U_k/U_{k-1}) \geq \dim(B_j U_k) - \dim(B_j U_{k-1}),$$

so the coefficient on the left-hand side must be as large as any coefficient on the right-hand side. However, if the equation $2 = 2q_j$ comes from U_k/U_{k-1} and $\tilde{U}$ is any subspace which fits into the flag between U_k and U_{k-1}, then $\tilde{U}$ is also a critical space contradicting the assumption that no critical space could be added to the flag.

If $j \neq j'$, then $t_{j,j'}$ can equal 1 or 2. If

$$2 = q_j + q_{j'}, \tag{2.13}$$

then we must have $q_j = q_{j'} = 1$ as neither can be greater than 1. Let us say that this is the equation in (2.12) coming from the quotient U_k/U_{k-1}. Then $\dim(U_k/U_{k_1}) = 2$ and $\dim B_j U_k = 1 + \dim B_j U_{k-1}$, so $(\ker B_j \cap U_k) \setminus U_{k-1}$ is nonempty. Take a vector v_j in this set and let $\tilde{U} = U_{k-1} + \langle v_j \rangle$. Then $\dim(\tilde{U}/U_{k-1}) = 1$, but $\dim B_j \tilde{U} = \dim B_j U_{k-1}$. Thus, testing (1.2) on $\tilde{U}$ and subtracting the equation coming from the criticality of U_{k-1} gives $1 \leq \sigma q_{j'}$, where $\sigma \in \{0,1\}$ since the coefficients on the right-hand side must be less than those of (2.13) and the coefficient of q_j must be 0 due to how $\tilde{U}$ is constructed. This inequality forces $\sigma = 1$, and thus, since $q_{j'} = 1$, we get that $\tilde{U}$ is a critical subspace contradicting our assumption that no critical space could be added to the flag.

Thus, we must in fact have that all the equations from (2.12) involving q_j with $j \in J_{2,1}$ are of the form $1 = q_j + q_{j'}$, where j, j' are distinct elements of $J_{2,1}$. Define a graph G on $J_{2,1}$ with j, j' connected by an edge if they appear together in an equation like this. Since each q_j will appear in exactly two equations, it is clear that this graph will consist of disjoint cycles. Let us examine one of these cycles. We can write all of the equations relating to the vertices in this cycle in the form

$$\begin{aligned} q_{j_1} + q_{j_2} \qquad\qquad\qquad\qquad &= 1 \\ q_{j_2} + q_{j_3} \qquad\qquad\quad &= 1 \\ \ddots \qquad\qquad &\ \ \vdots \\ q_{j_{l-1}} + q_{j_l} &= 1 \\ q_{j_1} \qquad\qquad\qquad\qquad + q_{j_l} &= 1. \end{aligned}$$

The number of equations in this list is the same as the number of variables. However, if there is an even number of equations, then the sum we get by adding the even numbered equations is the same as the sum we get by adding the odd numbered equations. So this system does not have a unique solution, contrary to our assumptions. Therefore, the number of equations in each cycle is odd and in that case the system has a unique solution, which is clearly $q_j = \frac{1}{2}$ for all $j \in J_{2,1}$.

With this, we have proved the first four parts in the statement of the theorem.

For the final part, we wish to rearrange the flag $\mathcal{U}$ into a flag of the form (2.11), but we must ensure that the flag remains critical at each step. So, consider i_1, the first element of J_1. Exactly one of the equations in (2.12) contains q_{i_1}, and it has no other nonzero coefficients for any q_j. So say that equality comes from subtracting the equality for U_k from the equality for U_{k-1}. Then we see that $U_{k-1} \subset \ker B_{i_1}$ and $U_k \cap \ker B_{i_1} = U_{k-1}$. Let us consider the flag

$$\tilde{\mathcal{U}} = (U_1 \subsetneq \cdots \subsetneq U_{k-1} \subsetneq \tilde{U}_k \subsetneq \cdots \subsetneq \tilde{U}_{s-1} \subsetneq U_s),$$

where we have defined $\tilde{U}_l = U_{l+1} \cap \ker B_{i_1}$ for $k \le l \le s-1$. We will show that this is a critical flag.

Note that $\dim(\tilde{U}_l) \ge \dim(U_{l+1}) - 1$ since the codimension of $\ker B_{i_1}$ in H is 1 and $\dim(\tilde{U}_l) \ne \dim(U_{l+1})$ since $U_k \subset U_l$ and $U_k \cap \ker B_{i_1} \ne U_k$. Thus, $\dim(\tilde{U}_l) = \dim(U_{l+1}) - 1$. We get the chain of inequalities

$$\begin{aligned} \dim(U_{l+1}) - 1 = \dim(\tilde{U}_l) &\le \sum_j q_j \dim(B_j \tilde{U}_l) \le \sum_{j \ne i_1} q_j \dim(B_j U_{l+1}) \\ &\le \Big(\sum_j q_j \dim(B_j U_{l+1})\Big) - q_{i_1} = \dim(U_{l+1}) - 1. \end{aligned} \tag{2.14}$$

Here the first inequality is simply (1.2) applied to $\tilde{U}_l$; the second follows from the inclusion $\tilde{U}_l \subset U_{l+1}$ together with $\dim(B_{i_1}\tilde{U}_l) = 0$, and the third follows from $\dim(B_{i_1} U_{l+1}) = 1$. We must therefore have equality all the way, and that implies that $\tilde{U}_l$ is critical. We also note that $\tilde{U}_{s-1} = \ker B_{i_1} \cap U_s$ and $\dim(U_s/\tilde{U}_{s-1}) = 1$.

Furthermore, the effect of replacing the system of equalities (2.12) based on $\mathcal{U}$ with the corresponding system based on $\tilde{\mathcal{U}}$ amounts to a reordering of the equalities.

The equality $1 = q_{i_1}$ is moved from the k-th place to the last. This follows from the equalities $1 + \dim(\tilde{U}_l) = \dim(U_{l+1})$ and $q_{i_1} + \sum_j q_j \dim(B_j \tilde{U}_l) = \sum_j q_j \dim(B_j U_{l+1})$ from (2.14).

By carrying out the above procedure for each B_{i_k} for $k = 2, \dots, s_3$, we can reorder the flag so that it becomes

$$\mathcal{U}_1 = (U_1 \subsetneq \cdots \subsetneq U_t = W_0 \subsetneq \cdots \subsetneq W_{s_3})$$

and $W_{k-1} = W_k \cap \ker B_{i_k}$ and $\dim(W_k/W_{k-1}) = 1$ for every $k = 1, \dots, s_3$.

The same analysis can be carried out for each $\ker B_{j_k}$ for the elements of $J_{2,2}$. Thus, consider j_1, the first element of $J_{2,2}$. Exactly two of the equations in (2.12) for $\mathcal{U}_0$ contain q_{j_1}, and they have no other nonzero coefficients for any q_j. This follows since the equations from $\mathcal{U}_1$ are simply a reordering of the equations from $\mathcal{U}$. Say that these two equalities come from subtracting the equatity for U_{k_1} from the equality for U_{k_1-1} and from subtracting the equality for U_{k_2} from the equality for U_{k_2-1}. Assume that $k_1 < k_2$. Then we see that $U_{k_1} \cap \ker B_{j_1} = U_{k_1-1}$ and $U_{k_2} \cap \ker B_{j_1} = U_{k_2-1} \cap \ker B_{j_1}$. We consider the flag

$$\tilde{\mathcal{U}}_1 = (U_1 \subsetneq \cdots \subsetneq U_{k_1-1} \subsetneq \tilde{U}_{k_1} \subsetneq \cdots \subsetneq \tilde{U}_{t-2}, \subsetneq U_t)$$

where we have defined $\tilde{U}_l = U_{l+1} \cap \ker B_{i_1}$for $k_1 \leq l \leq k_2 - 2$ and we have defined $\tilde{U}_l = U_{l+2} \cap \ker B_{i_1}$ for $k_2 - 1 \leq l \leq t - 2$. We will show that this is a critical flag.

For $k_1 \leq l \leq k_2 - 2$, we see that $\tilde{U}_l$ is critical in exactly the same way as above using the chain of inequalities (2.14). The only change is that instead of relying on the codimension of $\ker B_{i_1}$ in H being 1, we rely on the codimension of $\ker B_{j_1} \cap U_{k_2-1}$ in U_{k_2-1} being 1.

For $k_2 - 1 \leq l \leq t - 2$, we note that $\dim(\tilde{U}_l) \geq \dim(U_{l+2}) - 2$ since the codimension of $\ker B_{j_1}$ in H is 2, and $\dim(\tilde{U}_l) \leq \dim(U_{l+2}) - 2$ since U_{l+2} contains two linearly independent vectors which are not in $\ker B_{j_1}$, one from $U_{k_1} \setminus U_{k_1-1}$ and the other from $U_{k_2} \setminus U_{k_2-1}$. We get the chain of inequalities

$$\begin{aligned}\dim(U_{l+2}) - 2 = \dim(\tilde{U}_l) &\leq \sum_j q_j \dim(B_j \tilde{U}_l) \leq \sum_{j \neq j_1} q_j \dim(B_j U_{l+2}) \\ &\leq \Big(\sum_j q_j \dim(B_j U_{l+2})\Big) - 2q_{j_1} = \dim(U_{l+2}) - 2.\end{aligned}$$

As before, we deduce from this that $\tilde{U}_l$ is critical. We also note that $\tilde{U}_{t-2} = \ker B_{j_1} \cap U_t$ and $\dim(U_t/\tilde{U}_{t-2}) = 2$.

Furthermore, the effect of replacing the system of equalities (2.12) based on $\mathcal{U}_1$ with the corresponding system based on $\tilde{\mathcal{U}}_1$ amounts to grouping together the two equalities $1 = q_{j_1}$ and replacing them with $2 = 2q_{j_1}$, which is then placed last on the list.

By carrying out this procedure for each B_{j_k} for $k = 2, \dots, s_2$, we can reorder the flag so that it becomes

$$\tilde{\mathcal{U}}_1 = (U_1 \subsetneq \cdots \subsetneq U_{s_1} = V_0 \subsetneq \cdots \subsetneq V_{s_2} = W_0 \subsetneq \cdots \subsetneq W_{s_3})$$

and $V_{k-1} = V_k \cap \ker B_{j_k}$ and $\dim(W_k/W_{k-1}) = 2$ for every $k = 1, \dots, s_2$.

The spaces U_k in the flag $\tilde{\mathcal{U}}_1$ are s_1 in number since the equation in (2.12) associated with U_k/U_{k-1} is $1 = q_{j_1} + q_{j_2}$, where $e_k = \{j_1, j_2\}$ is an edge of the graph G. From the look of this equality and where it comes from, we see that $U_k \cap \ker B_{j_1} \subset U_{k-1}$ and $U_k \cap \ker B_{j_2} \subset U_{k-1}$. We also note that $\dim(U_k \cap \ker B_{j_1}) \geq \dim U_k - 2$ since the codimension of $\ker B_{j_1}$ in H is 2. The codimension of U_{k-1} in U_k is 1, so there are now two possibilites, either

$$U_k \cap \ker B_{j_1} + U_k \cap \ker B_{j_2} = U_{k-1} \tag{2.15}$$

or $U_k \cap \ker B_{j_1} = U_k \cap \ker B_{j_2} = K$, where K is a subspace of codimension 2 in U_k.

Assume the second possibility. Then $K \subsetneq U_{k-1} \subsetneq U_k$. Let r be the index such that $U_{r-1} \subset K$ but $U_r \not\subset K$. Then $r < k$ and $\dim(B_{j_\eta} U_{r-1}) = 0$, but $\dim(B_{j_\eta} U_r) > 0$ for $\eta = 1, 2$. We know from previous discussion that for U_r/U_{r-1}, (2.12) is of the form $1 = q_{\tilde{j}_1} + q_{\tilde{j}_2}$ for some $\{\tilde{j}_1, \tilde{j}_2\} \in E$. From this it is clear that $\{\tilde{j}_1, \tilde{j}_2\} = \{j_1, j_2\}$, but this contradicts our previous conclusion concerning the graph G. It would imply that G was not a proper graph but rather a multigraph where the edge $\{j_1, j_2\}$ was repeated, and this repeated edge would constitute a cycle of even length.

Therefore (2.15) must hold and that completes the proof of one direction of the theorem.

For the other direction, we note that if (q_j) is a point of the form prescribed and $\mathcal{U}$ is the flag (2.11), then each of the spaces of $\mathcal{U}$ is critical. To see this, note that

$$\begin{aligned}\dim(H) - 2 = \dim(W_{s_3-1}) &\leq \sum_j q_j \dim(B_j W_{s_3-1}) \leq \sum_{j \neq j_{s_3}} q_j \dim(B_j H)\\ &= \sum_j q_j \dim(B_j H) - 2q_{j_{s_3}} = \dim(H) - 2,\end{aligned}$$

so W_{s_3-1} is critical and the criticality of the other elements of $\mathcal{U}$ follows in the same way.

The only spaces which can be added into the flag are spaces of the form $\tilde{W} = W_{k-1} + \langle w_k \rangle$, where $w_k \in W_k \setminus W_{k-1}$. Then

$$\begin{aligned}\dim(W_k) - 1 = \dim(\tilde{W}) &\leq \sum_{j \neq j_k} q_j \dim(B_j \tilde{W}) + q_{j_k}\\ &\leq \sum_{j \neq j_k} q_j \dim(B_j W_k) + q_{j_k} = \sum_j q_j \dim(B_j W_k) - 1,\end{aligned}$$

and from this we see that $\tilde{W}$ is in fact critical. Therefore, (q_j) lies in the Brascamp–Lieb polyhedron $\mathcal{S}((B_j))$.

Finally, it is clear that the equations associated with the criticality of the flag $\mathcal{U}$ have (q_j) as a unique solution. This shows that (q_j) is a vertex of the polyhedron. ■

Remark 2.5 From the proof of the theorem it is clear that we may rearrange the flag so that the equations for q_j with $j \in J_{2,2} \cap J_1$ come in any order. However, this is not the case for U_{s_1}. In fact, there might be only one way of choosing this maximal flag

https://doi.org/10.4153/CJM-2010-045-2 Published online by Cambridge University Press

for U_{s_1}. An example of such a configuration is where $\dim H = 5$ and for $j = 1, \dots, 5$ B_j is the rank two projection onto $\langle e_1, e_2 + e_3\rangle$, $\langle e_1, e_4\rangle$, $\langle e_2 + e_1, e_4 + e_3\rangle$, $\langle e_2, e_5\rangle$ and $\langle e_3, e_5 + e_4\rangle$, respectively (here $\{e_i\}_{i=1,\dots,5}$ is an orthonormal basis for H and the angled brackets denote the span of the listed vectors). Then the only maximal flag for which we have equality is

$$\langle e_5\rangle \subset \langle e_4, e_5\rangle \subset \langle e_3, e_4, e_5\rangle \subset \langle e_2, e_3, e_4, e_5\rangle \subset \langle e_1, e_2, e_3, e_4, e_5\rangle.$$

Remark 2.6 In the cases we have looked at, all of the vertices have had associated with them critical flags of maximal length. However, this is not the case in general as can be seen from the following example. We take H of dimension 8 with an orthonormal basis $(e_i)_{i=1,\dots,8}$. For $j = 1, \dots 4$, we take B_j to be the orthogonal projections onto the spaces $\langle e_1, e_2, e_5\rangle$, $\langle e_2, e_4, e_7\rangle$, $\langle e_1 + e_2, e_6, e_8\rangle$, and $\langle e_3 + e_4, e_5 + e_6, e_7 + e_8\rangle$, respectively. Then we have the flag

$$\langle e_1, e_2\rangle \subset \langle e_1, e_2, e_3, e_4\rangle \subset \langle e_1, e_2, e_3, e_4, e_5, e_6\rangle \subset \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\rangle$$

for which (2.12) becomes

$$\begin{aligned} p_1 + p_2 + p_3 &= 2\\ p_1 + p_2 + p_4 &= 2\\ p_1 + p_3 + p_4 &= 2\\ p_2 + p_3 + p_4 &= 2 \end{aligned}$$

which has the solution $p_1 = p_2 = p_3 = p_4 = \frac{2}{3}$. It is straightforward to confirm that the inequality (1.2) is satisfied for any subspace V of H, as, from Lemma 2.2, we know that we need only to check it for subspaces which can be placed into the flag. However, no linear combination of the p_j with nonnegative integer coefficients can equal 1, so there can be no one-dimensional subspace of H which has equality in (1.2).

Remark 2.7 If all the maps B_j have rank k, then (1.3) gives that

$$\sum_j p_j = n/k, \tag{2.16}$$

and we can rewrite (1.2) as

$$\dim V \leq \sum_j p_j \dim(B_j V) = \sum_j p_j(\dim V - \dim(\ker B_j \cap V)),$$

which says

$$\sum_j p_j \dim(\ker B_j \cap V) \leq \frac{n-k}{k} \dim V. \tag{2.17}$$

We can carry out the analysis of this section with conditions (1.4), (2.16), and (2.17), and in particular, we can recover a theorem similar to Theorem 2.4 for the case when all B_j have rank $n - 2$.

3 The Facets of $\mathcal{S}$

We begin this section with a proof of Theorem 1.8.

Proof The necessity of the conditions follows immediately from [4] as they are a subset of the necessary conditions established there.

To show that the conditions are sufficient, we use induction on $n+m$, where $n = \dim H$ and m is the degree of multilinearity of the form. For the base case we consider $m = 1$. Then testing (1.2) on $\ker B_1$ gives that $\dim\ker B_1 = 0$ so B_1 is surjective and then the scaling condition gives $\dim H_1 = \dim H$ and $p_1 = 1$. We see then that the inequality evidently holds with equality if we take $C(B_1, p_1) = (\det B_1)^{-1}$.

For the inductive step we take a datum $((B_j),(p_j))$ and assume that the result holds for each datum for which the quantity $m+n$ is smaller.

As before, the conditions (1.3), (1.4) along with (1.2) for $V \in \mathcal{L}_{(B_j)}$ define a bounded convex polyhedron in $\mathbb{R}^m$. To show that the result holds everywhere in this polyhedron, by multilinear interpolation it is enough to establish it at each vertex. As we have already dealt with the case $m = 1$, we may assume $m > 2$ and then we get that at a vertex, aside from the scaling condition, at least one of the linear inequalities defining the polyhedron must be satisfied with equality.

There are now two cases. Either we have $p_{j_0} = 0$ for some j_0, or there is a space $U \in \mathcal{L}_{(B_j)} \setminus \{\{0\}, H\}$ such that $\dim U = \sum_j p_j \dim(B_j U)$. In the first case we see that we may write the Brascamp–Lieb inequality without referring to j_0, and the result follows from the induction hypothesis since the degree of multilinearity has been reduced.

In the second case we can factor the Brascamp–Lieb form. Define

$$\tilde{B}_j \colon U \to B_j U : x \mapsto B_j x$$
$$\tilde{\tilde{B}}_j \colon U^\perp \to (B_j U)^\perp : x \mapsto \Pi_{(B_j U)^\perp} B_j x$$
$$\Gamma_j \colon U^\perp \to B_j U : x \mapsto \Pi_{B_j U} B_j x,$$

where $\Pi_{(B_j U)^\perp}$ and $\Pi_{B_j U}$ denote the orthogonal projections onto the relevant spaces. Then we can calculate

$$\begin{aligned}
\int_H \prod_{j=1}^m f_j^{p_j}(B_j x)\,\mathrm{d}x &= \int_{U^\perp}\int_U \prod_{j=1}^m f_j^{p_j}(\tilde{B}_j\tilde{x} + B_j\tilde{\tilde{x}})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\tilde{x}} \\
&\le C((\tilde{B}_j),(p_j)) \int_{U^\perp}\prod_{j=1}^m \Big(\int_{B_j U} f_j(\tilde{y} + B_j\tilde{\tilde{x}})\,\mathrm{d}\tilde{y}\Big)^{p_j}\,\mathrm{d}\tilde{\tilde{x}} \\
&= C((\tilde{B}_j),(p_j)) \int_{U^\perp}\prod_{j=1}^m \Big(\int_{B_j U} f_j(\tilde{y} + \Gamma_j\tilde{\tilde{x}} + \tilde{\tilde{B}}_j\tilde{\tilde{x}})\,\mathrm{d}\tilde{y}\Big)^{p_j}\,\mathrm{d}\tilde{\tilde{x}} \\
&= C((\tilde{B}_j),(p_j)) \int_{U^\perp}\prod_{j=1}^m \Big(\int_{B_j U} f_j(\tilde{y} + \tilde{\tilde{B}}_j\tilde{\tilde{x}})\,\mathrm{d}\tilde{y}\Big)^{p_j}\,\mathrm{d}\tilde{\tilde{x}}
\end{aligned}$$

$$\leq C((\tilde{B}_j),(p_j))C((\tilde{\tilde{B}}_j),(p_j))\prod_{j=1}^{m}\Big(\int_{B_jU^\perp}\int_{B_jU} f_j(\tilde{y}+\tilde{\tilde{y}})\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{\tilde{y}}\Big)^{p_j}$$

$$= C((\tilde{B}_j),(p_j))C((\tilde{\tilde{B}}_j),(p_j))\prod_{j=1}^{m}\Big(\int_{H_j} f_j(y)\,\mathrm{d}y\Big)^{p_j}.$$

Here we have used for the first inequality that, for almost any $\tilde{\tilde{x}} \in U^\perp$, the tuple $(f_j(\cdot + B_j\tilde{\tilde{x}}))$ consists of non-negative integrable functions defined on B_jU, and we can therefore use the Brascamp–Lieb inequality for the datum $((\tilde{B}_j),(p_j))$. For the next equality we use the definitions of Γ_j and $\tilde{\tilde{B}}_j$, and for the one below that we use the translation invariance of the inner integral and the fact that $\Gamma_j\tilde{\tilde{x}} \in B_jU$ for any $\tilde{\tilde{x}} \in U^\perp$. For the second inequality we use the fact that for any j the inner integral defines a nonnegative function of $\tilde{\tilde{B}}_j\tilde{\tilde{x}}$ with domain $(B_jU)^\perp$, and we can therefore use the Brascamp–Lieb inequality for the datum $((\tilde{\tilde{B}}_j),(p_j))$.

Since we can perform this calculation for any tuple of nonnegative integrable functions (f_j) defined on H_j, we have established the inequality

$$C((B_j),(p_j)) \leq C((\tilde{B}_j),(p_j))C((\tilde{\tilde{B}}_j),(p_j)).$$

In particular this shows that if both $C((\tilde{B}_j),(p_j))$ and $C((\tilde{\tilde{B}}_j),(p_j))$ are finite, then $C((B_j),(p_j))$ is finite. Since $\dim U < \dim H$ and $\dim U^\perp < H$, we may use the induction hypothesis to establish that this is the case. The positivity condition (1.4) clearly holds since the tuple (p_j) is inherited unchanged from the original datum. The scaling condition (1.3) for $\tilde{B}$ holds by the assumption that U is critical and by subtracting that condition from the scaling condition for H we see that (1.3) holds for $\tilde{\tilde{B}}_j$.

So the only conditions that remain to be checked are (1.2) for any space in $\mathcal{L}_{(\tilde{B}_j)}$ and $\mathcal{L}_{(\tilde{\tilde{B}}_j)}$.

First of all, we note that $\mathcal{L}_{(\tilde{B}_j)}$ is a subset of $\mathcal{L}_{(B_j)}$. To see this we note that it is enough to show that the building blocks of $\mathcal{L}_{(\tilde{B}_j)}$, the sets $\ker\tilde{B}_j$, lie in $\mathcal{L}_{(B_j)}$. Since $\tilde{B}_j = B_j|_U$, we get that $\ker\tilde{B}_j = \ker B_j \cap U$ and the inclusion follows as both the sets on the right-hand side are elements of $\mathcal{L}_{(B_j)}$. Now, for any $W \in \mathcal{L}_{(\tilde{B}_j)}$, we have that $W \subset U$ and therefore $\dim\tilde{B}_jW = \dim B_jW$. Therefore, the inequality

$$\dim W \leq \sum_j p_j \dim \tilde{B}_jW$$

is in the list of inequalities coming from $\mathcal{L}_{(B_j)}$.

Secondly, we study $\mathcal{L}_{(\tilde{\tilde{B}}_j)}$. Let us take an element W from this set. Note that $W \subset U^\perp$. Our aim is to establish that the inequality $\dim W \leq \sum_j p_j \dim \tilde{\tilde{B}}_jW$ follows from the inequalities in $\mathcal{L}_{(B_j)}$ together with the hypothesis of criticality of U. Since U is critical and the elements in the pairs U, W, and B_jU, $\tilde{\tilde{B}}_jW$ are orthogonal to each other, we see that we may equivalently establish the inequality

$$\dim(W+U) \leq \sum_j p_j \dim(\tilde{\tilde{B}}_jW + B_jU).$$

We note that the sets $\tilde{\tilde{B}}_j W + B_j U$ and $B_j(\tilde{\tilde{W}} + U)$ are the same. To see this take an element x from the former set. Then x has the form $\Pi_{(B_j U)^\perp} B_j y + B_j z$ with $y \in W$ and $z \in U$. Now there is an element $y' \in U$ such that $\Pi_{(B_j U)^\perp} B_j y = B_j y + B_j y'$. Then $x = B_j(y + (y' + z))$ with $y \in W$ and $y' + z \in U$. For the other direction we take $x \in B_j(W + U)$. Then we can write $x = B_j(y + z)$ with $y \in W$ and $z \in U$. We take y' as before and then $x = \tilde{\tilde{B}}_j y + B_j(z - y')$ with $y \in W$ and $z - y' \in U$.

Therefore, it is enough to show that $W + U \in \mathcal{L}_{(B_j)}$. To establish this we note first of all that $\ker \tilde{\tilde{B}}_j + U = \ker B_j + U$. To see this take $x \in \ker \tilde{\tilde{B}}_j$. This means, by definition, that $B_j x \in B_j U$, so $x \in \ker B_j + U$. On the other hand, if we take $x \in \ker B_j$ and write $x = y + z$ with $y \in U$ and $z \in U^\perp$, then $B_j z = B_j x - B_j y = -B_j y \in B_j U$ so $\tilde{\tilde{B}}_j z = 0$ so $z \in \ker \tilde{\tilde{B}}_j$. We also note that for any $W_1, W_2 \in \mathcal{L}_{(\tilde{\tilde{B}}_j)}$ we have that $(W_1 + U) \cap (W_2 + U) = (W_1 \cap W_2) + U$ and $(W_1 + U) + (W_2 + U) = (W_1 + W_2) + U$. The first of these follows from the fact that both W_1 and W_2 lie in $U^\perp$, and the second is self-evident.

From this we see that if $W \in \mathcal{L}_{(\tilde{\tilde{B}}_j)}$, then $W + U$ lies in the lattice generated by $\{\ker B_j + U, j = 1, \dots, m\}$, and, since $U \in \mathcal{L}_{(B_j)}$, this is a sublattice of $\mathcal{L}_{(B_j)}$. This completes the proof of the theorem. ■

By examining the above proof we can give a procedure that tells us when we have found all the conditions included in (1.2).

We will need an enumeration of the elements of $\mathcal{L}_{(B_j)}$. Call the generators of $\mathcal{L}_{(B_j)}$, namely $\ker B_j$, level 0 elements. Then for each $s \geq 1$ let level s elements be those elements of $\mathcal{L}_{(B_j)}$ that are not of any lower level and which can be written as a vector space sum or as an intersection of two elements of level less than s. Clearly, this will assign a unique level to each element of $\mathcal{L}_{(B_j)}$, and there are only finitely many elements of any given level. Thus, we can enumerate the elements by first assigning numbers to the elements of level 0 then those of level 1 and so on.

We take this enumeration and look for necessary conditions by going through it and decide (arbitrarily) to pause when we have found the necessary conditions (1.2) for $V \in \mathcal{V}$, where $\mathcal{V} \subset \mathcal{L}_{(B_j)}$. At this stage we wish to determine whether we have found all the necessary conditions for the Brascamp–Lieb inequality to hold. Conditions (1.2) for $V \in \mathcal{V}$, together with conditions (1.3) and (1.4), restrict the set of tuples (p_j) for which the Brascamp–Lieb inequality holds to a polyhedron $\tilde{\mathcal{S}}_{(B_j)}$, and we wish to determine whether $\tilde{\mathcal{S}}_{(B_j)} = \mathcal{S}_{(B_j)}$ where $\mathcal{S}_{(B_j)}$ is the Brascamp–Lieb polyhedron for (B_j). This will be the case if and only if each vertex of $\tilde{\mathcal{S}}_{(B_j)}$ is in $\mathcal{S}_{(B_j)}$. There exists an algorithm that lists all of the vertices of $\tilde{\mathcal{S}}_{(B_j)}$. For each vertex (q_j) in this list we know that m of conditions (1.2) for $V \in \mathcal{V}$, (1.3), and (1.4) are satisfied with equality. If none of these equalities comes from (1.2), then the support of (q_j) can only contain one element q_{j_0}, and we know from above that the Brascamp–Lieb inequality holds at this vertex if and only if $q_{j_0} = 1$ and $\ker B_{j_0} = \{0\}$. Otherwise there is a space $U \in \mathcal{V}$ which lies strictly between $\{0\}$ and H such that (1.2) holds with equality for U. By the proof above we see that the Brascamp–Lieb inequality holds at (q_j) if and only if it holds for the data $((\tilde{B}_j), (q_j))$ and $((\tilde{\tilde{B}}_j), (q_j))$, that is, if $(q_j) \in \mathcal{S}_{(\tilde{B}_j)}$ and $(q_j) \in \mathcal{S}_{(\tilde{\tilde{B}}_j)}$.

To determine whether this is the case we run through the above algorithm for both $\mathcal{S}_{(\tilde{B}_j)}$ and $\mathcal{S}_{(\tilde{\tilde{B}}_j)}$. This recursion can only have n levels of depth and will therefore be completed in a finite number of steps. When it is completed we know whether (q_j) is in $\mathcal{S}_{(B_j)}$ in which case we move on to the next vertex, or whether (q_j) is not in $\mathcal{S}_{(B_j)}$ in which case we break the pause and continue looking for necessary conditions in the list of $\mathcal{L}_{(B_j)}$ until we decide again (arbitrarily) to pause and check whether we have now found all of the necessary conditions.

Acknowledgments This note forms part of the author's Ph.D. thesis from the University of Edinburgh. I would like to thank my supervisor Tony Carbery for his support and in particular for discussions relating to the material in Section 3.

Part of this work was completed during a stay at the University of Athens. I would like to thank Apostolos Giannopoulos for his hospitality and helpful comments.

References

[1] F. Barthe, *On a reverse form of the Brascamp–Lieb inequality.* Invent. Math. **134**(1998), no. 2, 335–361. doi:10.1007/s002220050267
[2] A. Barvinok, *A course in convexity.* Graduate Studies in Mathematics, 54, American Mathematical Society, Providence, RI, 2002.
[3] J. Bennett, A. Carbery, M. Christ, and T. Tao, *Finite bounds for Hölder–Brascamp–Lieb multilinear inequalities.* To appear, Math. Res. Lett.
[4] ———, *The Brascamp–Lieb inequalities: finiteness, structure and extremals.* Geom. Funct. Anal. **17**(2008), no. 5, 1343–1514. doi:10.1007/s00039-007-0619-6
[5] H. J. Brascamp and E. H. Lieb, *Best constants in Young's inequality, its converse, and its generalization to more than three functions.* Advances in Math. **20**(1976), no. 2, 151–173. doi:10.1016/0001-8708(76)90184-5
[6] E. A. Carlen, E. H. Lieb, and M. Loss, *A sharp analog of Young's inequality on S^N and related entropy inequalities.* J. Geom. Anal. **14**(2004), no. 3, 487–520.
[7] E. H. Lieb, *Gaussian kernels have only Gaussian maximizers.* Invent. Math. **102**, no. 1, 179–208. doi:10.1007/BF01233426
[8] G.-C. Rota, *The many lives of lattice theory.* Notices Amer. Math. Soc. **44**(1997), no. 11, 1440–1445.

UCLA Mathematics Department, Los Angeles, CA, U.S.A.

and

Science Institute, University of Iceland, Reykjavik, Iceland
e-mail: siv@hi.is

https://doi.org/10.4153/CJM-2010-045-2 Published online by Cambridge University Press