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BEYOND THE ENVELOPING ALGEBRA OF sl_3

DANIEL E. FLATH AND L. C. BIEDENHARN

0. Introduction. The problem which motivated the writing of this paper is that of finding structure behind the decomposition of the sl_3 representation spaces $V^* \otimes W = \text{Hom}(V, W)$ for finite dimensional irreducible sl_3-modules V and W. For sl_2 this extends the classical Clebsch-Gordon problem. The question has been considered for sl_3 in a computational way in [**5**]. In this paper we build a conceptual algebraic framework going beyond the enveloping algebra of sl_3.

For each dominant integral weight α let V_α be an irreducible representation of sl_3 of highest weight α. It is well known that, for weights α, μ, λ, the multiplicity of V_λ in $\text{Hom}(V_\alpha, V_{\alpha+\mu})$ is bounded by the multiplicity of μ in V_λ, with equality for generic α. This suggests the possibility of a single construction of highest weight vectors of weight λ in $\text{Hom}(V_\alpha, V_{\alpha+\mu})$ which is valid for all α.

In order to realize this possibility we introduce an analogue of a Weyl algebra, an algebra $\mathscr{A}$ of endomorphisms of $\oplus\ V_\alpha$ which is defined in Section 3 of this article. The construction referred to above amounts to the explicit decomposition of $\mathscr{A}$ as an sl_3-module. The principal technical tool in this program is Theorem 5.5. The main result, the decomposition, is stated as Theorem 6.6.

The analysis of $\mathscr{A}$ is facilitated by the fact that there is a generating set for $\mathscr{A}$ as an algebra which spans a lie algebra isomorphic to so_8. In Sections 7 and 8 of this article, we decompose $\mathscr{A}$ as an so_8-representation and use the result to show that $\mathscr{A}$ has no nonzero proper two-sided ideal.

1. Representations of sl_3. Let $\mathfrak{g}$ denote sl_3, the lie algebra of 3×3 traceless complex matrices, and denote by $\mathfrak{h}$ the subspace of diagonal matrices.

The group P of weights of $\mathfrak{g}$ will be identified with $\mathbf{Z}^3/\langle (1, 1, 1) \rangle$ as follows: For $\lambda = (a_1, a_2, a_3) = (a_1a_2a_3) \in P$ and $H = b_1E_{11} + b_2E_{22} + b_3E_{33} \in \mathfrak{h}$, define

$$\lambda(H) = \sum a_ib_i.$$

The group $\mathscr{S}_3$ of permutations on three letters acts on P through the formula:

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$$\sigma(a_1, a_2, a_3) = (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}).$$

A weight λ is *positive* if

$$2a_1 - a_2 - a_3 \geqq 0 \quad \text{and} \quad a_1 + a_2 - 2a_3 \geqq 0;$$

and it is *dominant* if $a_1 \geqq a_2 \geqq a_3$. We say $\lambda_1 \geqq \lambda_2$ of two weights if $\lambda_1 - \lambda_2$ is positive.

An element w of a $\mathfrak{g}$-module W is a λ *vector* of W for $\lambda \in \mathfrak{h}^*$ if $Hw = \lambda(H)w$ for all $H \in \mathfrak{h}$. We say that λ is a *weight* of W if there is a nonzero λ vector. If W is finite dimensional, this is only possible for $\lambda \in P$, and W is spanned by its weight vectors. The dimension of the space of λ vectors, the *multiplicity* of λ, will be denoted $\text{mult}_\lambda(W)$. If W is finite dimensional,

$$\text{mult}_{\sigma\lambda}(W) = \text{mult}_\lambda(W) \text{ for all } \sigma \in \mathscr{S}_3.$$

Every finite dimensional irreducible representation of $\mathfrak{g}$ has a unique highest weight. That weight is dominant and of multiplicity one; it determines the isomorphism class of the representation. Every dominant weight is the highest weight of a finite dimensional irreducible representation. The highest weight vectors in a simple $\mathfrak{g}$-module are those elements which are annihilated by both E_{12} and E_{23}. We shall write π_λ to denote an irreducible representation of highest weight λ.

For $\lambda = (pq0)$ dominant,

$$\dim \pi_\lambda = \frac{1}{2}(p - q + 1)(p + 2)(q + 1).$$

LEMMA 1.1. *Let λ, α, β be dominant weights. Then*

$$\dim \text{Hom}_{\mathfrak{g}}(\pi_\lambda, \text{Hom}_{\mathbf{C}}(\pi_\alpha, \pi_\beta)) \leqq \text{mult}_{\beta-\alpha}(\pi_\lambda),$$

with equality if $\alpha + (210) + \sigma\lambda$ is dominant for all $\sigma \in \mathscr{S}_3$.

Proof. This is a bit of folklore. One reference is [**1**]. For convenience, we quickly sketch a proof here.

We prove first the inequality.

Let u be a nonzero α vector of π_α, and let v^* be a nonzero $(-\beta)$ vector of π_β^*, the representation of $\mathfrak{g}$ contragredient to π_β. Let $\mathbf{C}(\beta - \alpha)$ be the one dimensional representation of $\mathfrak{h}$ on $\mathbf{C}$ defined by the formula

$$Hz = (\beta - \alpha)(H)z$$

for $H \in \mathfrak{h}$ and $z \in \mathbf{C}$.

Define a linear map f as follows:

$$f\colon \text{Hom}_{\mathfrak{g}}(\pi_\lambda, \text{Hom}_{\mathbf{C}}(\pi_\alpha, \pi_\beta)) \to \text{Hom}_{\mathfrak{h}}(\pi_\lambda, \mathbf{C}(\beta - \alpha))$$

$$A \mapsto f(A)\colon w \mapsto \langle (Aw)u, v^*\rangle.$$

By using the fact that v^* is a vector of lowest weight in π_β^* one easily shows that f is injective, which gives the desired inequality.

We can use a multiplicity formula to establish the equality clause of the lemma.

First note that dim $\mathrm{Hom}_{\mathfrak{g}}(\pi_\lambda, \mathrm{Hom}_{\mathbf{C}}(\pi_\alpha, \pi_\beta))$ equals the multiplicity of π_β as a subrepresentation of $\pi_\lambda \otimes \pi_\alpha$. Let $m(\beta)$ denote this multiplicity.

Let Q be the set of weights γ of π_λ for which there is $\sigma_\gamma \in \mathscr{S}_3$ (necessarily unique) such that

$$\sigma_\gamma(\alpha + (210) + \gamma) = \beta + (210).$$

In [**1**, **4**] the following formula is proved.

$$m(\beta) = \sum_{\gamma \in Q} \mathrm{sgn}(\sigma_\gamma)\ \mathrm{mult}_\gamma(\pi_\lambda).$$

The hypothesis of Lemma 1.1 is equivalent to the assertion that $\alpha + (210) + \gamma$ is dominant for all weights γ of π_λ. In that case, $Q = \{\beta - \alpha\}$ and $\sigma_{\beta-\alpha}$ is the identity.

If μ is a weight of π_λ, then $\lambda - \mu$ is in the subgroup of weights generated by $(1, -1, 0)$ and $(0, 1, -1)$, the roots. Thus every weight of $\pi_{(pq0)}$ can be written uniquely in the form (abc) with $a + b + c = p + q$.

LEMMA 1.2. *The weight* (abc) *with* $a + b + c = p + q$ *is a weight of* $\pi_{(pq0)}$ *if and only if there exists a partition* $a + b = b_1 + b_2$ *such that* $b_1 \geqq a \geqq b_2$ *and* $p \geqq b_1 \geqq q \geqq b_2 \geqq 0$. *Moreover, the multiplicity of* (abc) *in* $\pi_{(pq0)}$ *equals the number of such partitions of* $a + b$.

Proof. This is essentially equivalent to the branching law of [**6**].

The combinatorial meaning of the inequalities of the previous lemma is uncovered by arranging the various integers in a Gel'fand-Weyl pattern [**3**, **7**] as follows:

$$\begin{pmatrix} & & a & & \\ & b_1 & & b_2 & \\ p & & q & & 0 \end{pmatrix}.$$

LEMMA 1.3. *Let* (abc) *with* $a + b + c = p + q$ *be a dominant weight of* $\pi_{(pq0)}$. *Then its multiplicity is* $1 + \inf\{p - a, c, p - q, q\}$.

2. Construction of the representation V. Let

$$W = \mathbf{C}[a_1, a_2, a_3, a_{12}, a_{23}, a_{31}],$$

a polynomial ring in six independent commuting variables.

Let $\mathfrak{g}$ act on W as a lie algebra of derivations through the following formulas:

(2.1a) $E_{12} = a_1\partial_{a_2} - a_{31}\partial_{a_{23}}$

(2.1b) $E_{23} = a_2\partial_{a_3} - a_{12}\partial_{a_{31}}$

(2.1c) $E_{13} = a_1\partial_{a_3} - a_{12}\partial_{a_{23}}$

(2.1d) $E_{21} = a_2\partial_{a_1} - a_{23}\partial_{a_{31}}$

(2.1e) $E_{32} = a_3\partial_{a_2} - a_{31}\partial_{a_{12}}$

(2.1f) $E_{31} = a_3\partial_{a_1} - a_{23}\partial_{a_{12}}$

(2.1g) $E_{11} - E_{22} = a_1\partial_{a_1} - a_2\partial_{a_2} + a_{31}\partial_{a_{31}} - a_{23}\partial_{a_{23}}$

(2.1h) $E_{22} - E_{33} = a_2\partial_{a_2} - a_3\partial_{a_3} + a_{12}\partial_{a_{12}} - a_{31}\partial_{a_{31}}$.

Notice that a_1, a_2, a_3 span a space isomorphic to the defining representation of $\mathfrak{g}$ (highest weight (100)), and that a_{12}, a_{23}, a_{31} span a space isomorphic to its antisymmetric square (highest weight (110)): $a_{ij} = a_i \wedge a_j$.

Define three linear transformations M_+, M_-, M_0 on W:

(2.2a) $M_+ = -(\partial_{a_1}\partial_{a_{23}} + \partial_{a_2}\partial_{a_{31}} + \partial_{a_3}\partial_{a_{12}})$

(2.2b) $M_- = a_1a_{23} + a_2a_{31} + a_3a_{12}$

(2.2c) $M_0 = -(a_1\partial_{a_1} + a_2\partial_{a_2} + a_3\partial_{a_3} + a_{12}\partial_{a_{12}} + a_{23}\partial_{a_{23}} + a_{31}\partial_{a_{31}} + 3)$.

Let V be the kernel of M_+.

Each of M_+, M_-, M_0 commutes with $\mathfrak{g}$ above; because M_+ does so, V is itself a representation of $\mathfrak{g}$. Our next task is to decompose this representation.

For nonnegative integers j, let P^j be the space of homogeneous polynomials of degree j in W. Let H^j be the kernel of M_+ in P^j.

LEMMA 2.3. $P^j = H^j \oplus M_-P^{j-2}$.

Proof. By induction on j. The statement is trivial for $j = 0, 1$. Suppose it is true for integers $j \leqq k$. To establish its validity for $j = k + 2$ it will suffice to show that M_+ maps M_-P^k isomorphically onto P^k.

The inductive hypothesis implies that

$$P^k = \bigoplus_{0\leqq p\leqq k/2} M_-^pH^{k-2p}.$$

Thus all follows from

LEMMA 2.4. *M_+M_- acts as scalar multiplication by*

$$(p + 1)(p - k - 3) \neq 0 \quad \text{on } M_-^pH^{k-2p}.$$

Proof. Calculation shows that M_+, M_-, M_0 span a lie algebra isomorphic to sl_2:

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$$[M_+, M_-] = M_0 \quad [M_0, M_+] = 2M_+ \quad [M_0, M_-] = -2M_-.$$

Now establish by induction that for positive integers l,

$$M_+M_-^l = lM_-^{l-1}(M_0 - l + 1) + M_-^lM_+.$$

THEOREM 2.5. $H^j \simeq \bigoplus_{i=0}^{j} \pi_{(ji0)}$.

Proof. H^j contains a $\mathfrak{g}$-subrepresentation isomorphic to $\pi_{(ji0)}$, the one with highest weight vector $a_1^{j-i}a_{12}^i$. To show that these subrepresentations span H^j, we must check that

$$\sum_{i=0}^{j} \dim \pi_{(ji0)} = \dim H^j.$$

By Lemma 2.3,

$$\dim H^j = \dim P^j - \dim P^{j-2}.$$

The space of homogeneous polynomials of degree j in n variables has dimension $\binom{j+n-1}{j}$. Thus the formula we want is an easy induction:

$$\sum_{i=0}^{j} \frac{1}{2}(j - i + 1)(j + 2)(i + 1) = \binom{j+5}{j} - \binom{j+3}{j-2}.$$

COROLLARY 2.6. *The $\mathfrak{g}$-representation V is a multiplicity free sum of all finite dimensional irreducible representations of $\mathfrak{g}$.*

The algebra of operators on V generated by $\mathfrak{g}$ is isomorphic to the universal enveloping algebra of $\mathfrak{g}$.

Proof. Only the second assertion needs proof. It follows from the existence for every $x \neq 0$ in the enveloping algebra of a finite dimensional irreducible representation π of $\mathfrak{g}$ such that $\pi(x) \neq 0$.

We will denote by V_λ the subspace of V which is isomorphic to π_λ. A $(ji0)$-vector in $V_{(ji0)}$ is a $a_1^{j-i}a_{12}^i$. If λ is not dominant, write $V_\lambda = (0)$.

Let $\mathscr{S}_6$ be the group of permutations on the six symbols 1, 2, 3, 12, 23, 31. It acts linearly as ring automorphisms on the space W by $\sigma(a_k) = a_{\sigma(k)}$.

Let τ be the action of $\mathscr{S}_6$ on $\mathrm{End}_{\mathbf{C}}(W)$ given by

$$\tau(\sigma)T = \sigma \circ T \circ \sigma^{-1} \quad \text{for } T \in \mathrm{End}_{\mathbf{C}}(W).$$

In particular,

$$\tau(\sigma)a_k = a_{\sigma(k)} \quad \text{and} \quad \tau(\sigma)\partial_{a_k} = \partial_{a_{\sigma(k)}}.$$

Define subgroups K', K, and L of $\mathscr{S}_6$ by listing generators:

(2.7a) $K' = \langle (1\ 23)(2\ 31), (1\ 23)(3\ 12) \rangle.$

(2.7b) $K = \langle K', (1\ 23) \rangle.$

(2.7c) $L = \langle (1\ 2\ 3)(23\ 31\ 12), (1\ 3)(23\ 12) \rangle.$

The isomorphism classes of these groups are easily determined:

$$K' \simeq \mathbf{Z}_2 \times \mathbf{Z}_2,\ K \simeq \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2,\ L \simeq \mathscr{S}_3.$$

Because L normalizes K' and K, we can define subgroups G' and G of $\mathscr{S}_6$ as follows:

(2.8) $G' = K'L \qquad G = KL.$

It is not hard to see that $G' \simeq \mathscr{S}_4$ and that

$$G = G' \times \langle (1\ 23)(2\ 31)(3\ 12) \rangle.$$

LEMMA 2.9. *For each* $\sigma \in G$, $\sigma(V) = V$ *and* $\sigma(H^j) = H^j$.

Proof. Because $\tau(\sigma)M_+ = M_+$. In fact, G is the stabilizer in $\mathscr{S}_6$ of M_+.

We will henceforth use τ to denote the action of G on $\mathrm{End}_{\mathbf{C}}(V)$ given by

$$\tau(\sigma)T = \sigma \circ T \circ \sigma^{-1}.$$

3. Construction of the algebra $\mathscr{A}$. Define six operators on W by the formulas below.

(3.1a) $$\begin{pmatrix}100\\100\end{pmatrix} = 2a_1 + a_1^2\partial_{a_1} + a_1a_2\partial_{a_2} + a_1a_3\partial_{a_3} + a_1a_{12}\partial_{a_{12}} + a_1a_{31}\partial_{a_{31}} - a_2a_{31}\partial_{a_{23}} - a_3a_{12}\partial_{a_{23}}$$

(3.1b) $$\begin{pmatrix}010\\100\end{pmatrix} = a_{12}\partial_{a_2} - a_{31}\partial_{a_3}$$

(3.1c) $$\begin{pmatrix}001\\100\end{pmatrix} = \partial_{a_{23}}$$

(3.1d) $$\begin{pmatrix}110\\110\end{pmatrix} = 2a_{12} + a_{12}^2\partial_{a_{12}} + a_{12}a_{23}\partial_{a_{23}} + a_{12}a_{31}\partial_{a_{31}} + a_1a_{12}\partial_{a_1} + a_2a_{12}\partial_{a_2} - a_1a_{23}\partial_{a_3} - a_2a_{31}\partial_{a_3}$$

(3.1e) $$\begin{pmatrix}101\\110\end{pmatrix} = -a_1\partial_{a_{31}} + a_2\partial_{a_{23}}$$

(3.1f) $$\begin{pmatrix}011\\110\end{pmatrix} = \partial_{a_3}.$$

Calculations show that each of these operators carries the subspace V into itself. Henceforth they will be viewed as linear transformations on V, not W. The auxiliary space W will appear no more in this paper.

Define twelve more operators on V.

For e = 100, 010, 001:

$$(3.2a)\quad \begin{pmatrix} e \\ 010 \end{pmatrix} = \left[E_{21}, \begin{pmatrix} e \\ 100 \end{pmatrix}\right], \begin{pmatrix} e \\ 001 \end{pmatrix} = \left[E_{32}, \begin{pmatrix} e \\ 010 \end{pmatrix}\right].$$

For f = 110, 101, 011:

$$(3.2b)\quad \begin{pmatrix} f \\ 101 \end{pmatrix} = -\left[E_{32}, \begin{pmatrix} f \\ 110 \end{pmatrix}\right], \begin{pmatrix} f \\ 011 \end{pmatrix} = -\left[E_{21}, \begin{pmatrix} f \\ 101 \end{pmatrix}\right].$$

The algebra of operators on V generated by the nine $\begin{pmatrix} e \\ e' \end{pmatrix}$ and the nine $\begin{pmatrix} f \\ f' \end{pmatrix}$ will be denoted $\mathscr{A}$.

Observe that $\mathscr{A}$ contains $\mathfrak{g}$ and hence also the enveloping algebra of $\mathfrak{g}$.

$$(3.3)\quad E_{12} = \left[\begin{pmatrix} 101 \\ 101 \end{pmatrix}, \begin{pmatrix} 010 \\ 100 \end{pmatrix}\right] \quad E_{21} = \left[\begin{pmatrix} 101 \\ 011 \end{pmatrix}, \begin{pmatrix} 010 \\ 010 \end{pmatrix}\right]$$

$$E_{23} = \left[\begin{pmatrix} 101 \\ 110 \end{pmatrix}, \begin{pmatrix} 010 \\ 010 \end{pmatrix}\right] \quad E_{32} = \left[\begin{pmatrix} 101 \\ 101 \end{pmatrix}, \begin{pmatrix} 010 \\ 001 \end{pmatrix}\right].$$

We can therefore view $\mathscr{A}$ as the space of a $\mathfrak{g}$-representation ρ through the formula

$$\rho(x)a = [x, a] \text{ for } x \in g,\ a \in \mathscr{A}.$$

The analysis of the $\mathfrak{g}$-representation $\mathscr{A}$ is the principal object of this paper.

Each of the eighteen generators of $\mathscr{A}$ is written in the form $\begin{pmatrix} h \\ h' \end{pmatrix}$. We refer to h and h' as the *upper* and *lower labels*. These labels are interpreted as $\mathfrak{g}$-weights and have the following significance. The operator $\begin{pmatrix} h \\ h' \end{pmatrix}$ is an h'-vector in the $\mathfrak{g}$-representation ρ on $\mathscr{A}$. For each irreducible subrepresentation V_λ of V,

$$\begin{pmatrix} h \\ h' \end{pmatrix}(V_\lambda) \subset V_{\lambda+h}.$$

The next important proposition assures us that $\mathscr{A}$ is large enough for the study of all spaces $\mathrm{Hom}_{\mathbf{C}}(V_\mu, V_\lambda)$.

PROPOSITION 3.4. *Let U be a finite dimensional vector subspace of V and let $T \in \mathrm{End}_{\mathbf{C}}(U)$. Then there exists an element of $\mathscr{A}$ whose restriction to U equals T.*

Proof. By enlarging U we may assume that U is a sum of V_λ. Choose a basis B of U compatible with the decomposition $U = \oplus V_\lambda$, and choose v, $w \in B$, say

$$v \in V_{(ji0)} \quad \text{and} \quad w \in V_{(lk0)}.$$

We show that there is $a \in \mathscr{A}$ such that $av = w$ and $av' = 0$ for all $v' \neq v \in B$.

Indeed, given endomorphisms T_λ of V_λ there is an S in the enveloping algebra of $\mathfrak{g}$ such that S agrees with T_λ on each of the (finitely many) V_λ. So there exists $S \in \mathscr{A}$ such that

$$Sv = a_1^{j-i}a_{12}^{i} \quad \text{and} \quad Sv' = 0 \text{ for } v' \neq v \in B.$$

Now

$$R = \begin{pmatrix} 110 \\ 110 \end{pmatrix}^k \begin{pmatrix} 100 \\ 100 \end{pmatrix}^{l-k} \begin{pmatrix} 001 \\ 001 \end{pmatrix}^i \begin{pmatrix} 011 \\ 011 \end{pmatrix}^{j-i}$$

maps $a_1^{j-i}a_{12}^{i}$ to a nonzero multiple of $a_1^{l-k}a_{12}^{k}$. Finally there is Q in the enveloping algebra of $\mathfrak{g}$ such that $QRSv = w$. We take $a = QRS$.

COROLLARY 3.5. i) *If $T \in \mathrm{End}_{\mathbf{C}}(V)$ commutes with $\mathscr{A}$ then T is a scalar multiplication.*

ii) *The center of $\mathscr{A}$ is* $\mathbf{C}$, *the scalar multiplications.*

iii) *V is a simple $\mathscr{A}$-module.*

4. so_8. Calculation with the eighteen generators of $\mathscr{A}$ shows that the following three useful and easily remembered rules hold.

4.1.) The three operators with a given upper label commute.

4.2.) The three operators with a given lower label commute.

4.3a.) The three $\begin{pmatrix} 001 \\ \cdots \end{pmatrix}$ commute with the three $\begin{pmatrix} 011 \\ \cdots \end{pmatrix}$ and the three $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$.

b.) The three $\begin{pmatrix} 010 \\ \cdots \end{pmatrix}$ commute with the three $\begin{pmatrix} 011 \\ \cdots \end{pmatrix}$ and the three $\begin{pmatrix} 110 \\ \cdots \end{pmatrix}$.

c.) The three $\begin{pmatrix} 100 \\ \cdots \end{pmatrix}$ commute with the three $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$ and the three $\begin{pmatrix} 110 \\ \cdots \end{pmatrix}$.

Define six more elements of $\mathscr{A}$.

(4.4a) $\quad H_1 = -1 - a_2\partial_{a_2} - a_3\partial_{a_3} - a_{23}\partial_{a_{23}}$

(4.4b) $\quad H_2 = -1 - a_1\partial_{a_1} - a_3\partial_{a_3} - a_{31}\partial_{a_{31}}$

(4.4c) $H_3 = -1 - a_1\partial_{a_1} - a_2\partial_{a_2} - a_{12}\partial_{a_{12}}$

(4.4d) $H_4 = -1 - a_{12}\partial_{a_{12}} - a_{23}\partial_{a_{23}} - a_{31}\partial_{a_{31}}$

(4.4e) $X = 1 + a_1\partial_{a_1} + a_2\partial_{a_2} + a_3\partial_{a_3}$

(4.4f) $Y = -H_4$.

Notice that X and Y commute with $\mathfrak{g}$. On the subspace $V_{(ji0)}$ of V, X acts as scalar multiplication by $j - i + 1$ and Y as scalar multiplication by $i + 1$.

The following important theorem summarizes many commutation calculations.

THEOREM 4.5. *The eighteen generators of $\mathscr{A}$, $\mathfrak{g}$, X, and Y span a twenty-eight dimensional lie algebra isomorphic to so_8.*

COROLLARY 4.6. *$\mathscr{A}$ is isomorphic to a quotient of the universal enveloping algebra of so_8.*

COROLLARY 4.7. *V may be viewed as an irreducible representation of so_8.*

We want to give explicitly the isomorphism with so_8.

Let $J = (\delta_{i,9-i})$ be the 8×8 matrix all of whose entries are zero except those on the second diagonal which are equal to one. We will take for so_8 the lie algebra of 8×8 complex matrices A such that

$$^tAJ + JA = 0.$$

These are precisely the 8×8 matrices which are antisymmetric with respect to the second diagonal.

The identification of matrices in so_8 with elements of $\mathscr{A}$ is given in Table 1, where F_{ij} is the 8×8 matrix of all of whose entries are zero except the ij^{th} which is one.

One can now ask about subalgebras of so_8. Here is an easy result.

PROPOSITION 4.8. *The three $\begin{pmatrix} 010 \\ \dots \end{pmatrix}$, the three $\begin{pmatrix} 101 \\ \dots \end{pmatrix}$, $\mathfrak{g}$, and $X - Y$ span a fifteen dimensional lie algebra isomorphic to sl_4.*

Each of the subspaces H^j of V is irreducible as a representation of this sl_4.

We want next to show that the τ-action of G on $\mathrm{End}_{\mathbf{C}}(V)$ restricts to an action on the algebra $\mathscr{A}$.

Let $\mathfrak{k}$ denote the subspace of diagonal matrices of so_8. We continue to identify so_8 and its isomorphic lie algebra in $\mathscr{A}$, so that $\mathfrak{k}$ is spanned by the four H_i.

PROPOSITION 4.9. *For each $\sigma \in G$, $\tau(\sigma)$ preserves $\mathfrak{k}$, sl_4, and so_8. G acts Through τ as a group of automorphisms of $\mathscr{A}$.*

Table 1

$\mathscr{A}$ and so_8

$E_{12} = F_{12} - F_{78}$	$E_{21} = F_{21} - F_{87}$
$E_{13} = F_{13} - F_{68}$	$E_{31} = F_{31} - F_{86}$
$E_{23} = F_{23} - F_{67}$	$E_{32} = F_{32} - F_{76}$
$\begin{pmatrix}010\\100\end{pmatrix} = F_{14} - F_{58}$	$\begin{pmatrix}101\\011\end{pmatrix} = -F_{41} + F_{85}$
$\begin{pmatrix}010\\010\end{pmatrix} = F_{24} - F_{57}$	$\begin{pmatrix}101\\101\end{pmatrix} = -F_{42} + F_{75}$
$\begin{pmatrix}010\\001\end{pmatrix} = F_{34} - F_{56}$	$\begin{pmatrix}101\\110\end{pmatrix} = -F_{43} + F_{65}$
$\begin{pmatrix}001\\100\end{pmatrix} = F_{15} - F_{48}$	$\begin{pmatrix}110\\011\end{pmatrix} = -F_{51} + F_{84}$
$\begin{pmatrix}001\\010\end{pmatrix} = F_{25} - F_{47}$	$\begin{pmatrix}110\\101\end{pmatrix} = -F_{52} + F_{74}$
$\begin{pmatrix}001\\001\end{pmatrix} = F_{35} - F_{46}$	$\begin{pmatrix}110\\110\end{pmatrix} = -F_{53} + F_{64}$
$\begin{pmatrix}100\\100\end{pmatrix} = F_{62} - F_{73}$	$\begin{pmatrix}011\\011\end{pmatrix} = -F_{26} + F_{37}$
$\begin{pmatrix}100\\010\end{pmatrix} = F_{83} - F_{61}$	$\begin{pmatrix}011\\101\end{pmatrix} = -F_{38} + F_{16}$
$\begin{pmatrix}100\\001\end{pmatrix} = F_{71} - F_{82}$	$\begin{pmatrix}011\\110\end{pmatrix} = -F_{17} + F_{28}$
$H_i = F_{ii} - F_{9-i,9-i}$	$i = 1, 2, 3, 4.$

Proof. One must check the first assertion explicitly for generators σ of G. The last assertion follows because so_8 generates $\mathscr{A}$.

The actions of G on $\mathfrak{k}$ and on $\mathfrak{k} \cap sl_4$ are faithful. Indeed, the subgroup G' acts as the full permutation group of the set of H_i, and the element $(1\ 23)(2\ 31)(3\ 12) \in G$ acts as scalar multiplication by -1 on $\mathfrak{k} \cap sl_4$.

Denote by R the root system of sl_4 associated to the cartan subalgebra $\mathfrak{k} \cap sl_4$.

Denote by $\mathrm{Aut}(R)$ the automorphism group of R, a finite subgroup of linear automorphisms of $(\mathfrak{k} \cap sl_4)^*$. Let $W(R)$ be the Weyl group of R, a subgroup of index 2 in $\mathrm{Aut}(R)$.

For $\sigma \in G$, let $\epsilon(\sigma)$ be the contragredient of the restriction of $\tau(\sigma)$ to $\mathfrak{k} \cap sl_4$. The previous proposition shows that $\epsilon(\sigma) \in \mathrm{Aut}(R)$.

PROPOSITION 4.10. i) *The map* $\epsilon: G \rightarrow \mathrm{Aut}(R)$ *is an isomorphism.*
ii) $\epsilon(G') = W(R)$.

Proof. See the explicit description of $W(R)$ in **[2]**.

5. The commutant $\mathscr{B}$ of $\{E_{12}, E_{23}\}$ in $\mathscr{A}$. We want to decompose the representation ρ of $\mathfrak{g}$ on $\mathscr{A}$. Because $\mathscr{A}$ is a sum of finite dimensional representations, this amounts to the determination of the space of a in $\mathscr{A}$ such that

$$\rho(E_{12})a = \rho(E_{23})a = 0.$$

This is precisely the commutant of E_{12}, E_{23} in $\mathscr{A}$.

It is easily verified that the commutant of E_{12}, E_{23} in so_8 is the nine dimensional lie subalgebra spanned by the following:

$$(5.1) \qquad X,\ Y,\ E_{13},\ \begin{pmatrix}110\\110\end{pmatrix},\ \begin{pmatrix}101\\110\end{pmatrix},\ \begin{pmatrix}011\\110\end{pmatrix},\ \begin{pmatrix}100\\100\end{pmatrix},\ \begin{pmatrix}010\\100\end{pmatrix},\ \begin{pmatrix}001\\100\end{pmatrix}.$$

Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by the nine operators above.

The nine generators of $\mathscr{B}$ are not independent. We note two relations in addition to the commutation rules.

$$(5.2a) \qquad \begin{pmatrix}011\\110\end{pmatrix}\begin{pmatrix}100\\100\end{pmatrix} - \begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix} - XE_{13} = 0$$

$$(5.2b) \qquad \begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix} - \begin{pmatrix}110\\110\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix} - YE_{13} = 0.$$

LEMMA 5.3. *The vector space $\mathscr{B}$ is spanned by elements of the form SX^eY^f where*

$$(5.4) \qquad S = E_{13}{}^{a}\begin{pmatrix}101\\110\end{pmatrix}^{b_1}\begin{pmatrix}010\\100\end{pmatrix}^{b_2}\begin{pmatrix}110\\110\end{pmatrix}^{c_1}\begin{pmatrix}001\\100\end{pmatrix}^{c_2}\begin{pmatrix}011\\110\end{pmatrix}^{d_1}\begin{pmatrix}100\\100\end{pmatrix}^{d_2},$$

with $c_1c_2 = d_1d_2 = 0.$

Proof. Use the relations.

THEOREM 5.5. *$\mathscr{B}$ is the commutant of $\{E_{12}, E_{23}\}$ in $\mathscr{A}$.*

Proof. Let U be the $\mathfrak{g}$-module generated by $\mathscr{B}$. The theorem is equivalent to the equality: $U = \mathscr{A}$. Because X and Y commute with $\mathfrak{g}$, we have UX, $UY \subset U$.

Let $\sigma = (1\ 12)(2\ 31)(3\ 23) \in G$. Because $\sigma(\mathscr{B}) = \mathscr{B}$ and $\sigma(\mathfrak{g}) = \mathfrak{g}$, we have that $\sigma(U) = U$.

LEMMA 5.6. $\mathscr{B}\cdot\begin{pmatrix}001\\001\end{pmatrix},\ \mathscr{B}\cdot\begin{pmatrix}100\\001\end{pmatrix} \subset U.$

Proof. The proof consists of tedious calculations, mainly consisting of finding enough relations in $\mathscr{A}$ amongst the elements of so_8. Only an outline will be given.

We list three equalities in $\mathscr{A}$.

$$(5.7a) \qquad E_{23}\begin{pmatrix}001\\100\end{pmatrix} = E_{13}\begin{pmatrix}001\\010\end{pmatrix} + \begin{pmatrix}011\\110\end{pmatrix}\begin{pmatrix}101\\110\end{pmatrix}$$

$$(5.7b) \qquad \begin{pmatrix}010\\010\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix} = \begin{pmatrix}010\\100\end{pmatrix}\begin{pmatrix}001\\010\end{pmatrix} + (Y-1)\begin{pmatrix}011\\110\end{pmatrix}$$

(5.7c) $\begin{pmatrix}100\\010\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix} = \begin{pmatrix}100\\100\end{pmatrix}\begin{pmatrix}001\\010\end{pmatrix} + (X + Y - 1)\begin{pmatrix}101\\110\end{pmatrix}.$

Using these relations one shows that

$$\left[E_{21}, S\begin{pmatrix}001\\100\end{pmatrix}\right] \in (a + b_2 + c_2 + d_2 + 1)S\begin{pmatrix}001\\010\end{pmatrix} + \mathscr{B},$$

whence

$$\mathscr{B} \cdot \begin{pmatrix}001\\010\end{pmatrix} \subset U.$$

Quite similarly, one proves that

$$\mathscr{B} \cdot \begin{pmatrix}010\\010\end{pmatrix}, \mathscr{B} \cdot \begin{pmatrix}100\\010\end{pmatrix}, \mathscr{B} \cdot E_{23} \subset U.$$

By applying σ, one deduces that also

$$\mathscr{B} \cdot \begin{pmatrix}011\\101\end{pmatrix}, \mathscr{B} \cdot \begin{pmatrix}101\\101\end{pmatrix}, \mathscr{B} \cdot \begin{pmatrix}110\\101\end{pmatrix}, \mathscr{B} \cdot E_{12} \subset U.$$

Next by considering both $[E_{21}, SE_{12}]$ and $[E_{21}, SE_{12}] + [E_{31}, SE_{13}] + [E_{32}, SE_{23}]$ one shows that

$$\mathscr{B} \cdot H_1, \mathscr{B} \cdot H_2, \mathscr{B} \cdot H_3 \subset U.$$

Finally, consideration of $\left[E_{31}, S\begin{pmatrix}001\\100\end{pmatrix}\right] + \left[E_{32}, S\begin{pmatrix}001\\010\end{pmatrix}\right]$ establishes the inclusion

$$\mathscr{B} \cdot \begin{pmatrix}001\\001\end{pmatrix} \subset U,$$

and consideration of $\left[E_{31}, S\begin{pmatrix}100\\100\end{pmatrix}\right] + \left[E_{32}, S\begin{pmatrix}100\\010\end{pmatrix}\right]$ establishes

$$\mathscr{B} \cdot \begin{pmatrix}100\\001\end{pmatrix} \subset U.$$

The lemma is proved.

We now quickly prove the theorem.
By applying σ,

$$\mathscr{B} \cdot \begin{pmatrix}011\\011\end{pmatrix}, \mathscr{B} \cdot \begin{pmatrix}110\\011\end{pmatrix} \subset U.$$

Because E_{21}, E_{32} commute with $\begin{pmatrix}\cdots\\001\end{pmatrix}$ and $\begin{pmatrix}\cdots\\011\end{pmatrix}$, and since

$$U = \rho(\mathscr{E}) \cdot B$$

where $\mathscr{E}$ is the enveloping algebra of span$\{E_{21}, E_{32}, E_{31}\}$, we conclude that

$$U \cdot \begin{pmatrix}001\\001\end{pmatrix}, U \cdot \begin{pmatrix}100\\001\end{pmatrix}, U \cdot \begin{pmatrix}011\\011\end{pmatrix}, U \cdot \begin{pmatrix}110\\011\end{pmatrix} \subset U.$$

Next, apply E_{13}, E_{12}, E_{23} to these last inclusions to show that $U \cdot \mathscr{C} \subset U$ where $\mathscr{C}$ is the subalgebra of $\mathscr{A}$ generated by the twelve operators $\begin{pmatrix}100\\\cdots\end{pmatrix}$, $\begin{pmatrix}001\\\cdots\end{pmatrix}$, $\begin{pmatrix}110\\\cdots\end{pmatrix}$, $\begin{pmatrix}011\\\cdots\end{pmatrix}$.

It remains but to observe that $\mathscr{C} = \mathscr{A}$.

Define $\mathscr{A}^{\circ}$ to be the algebra of all T in $\mathscr{A}$ such that $T(V_\lambda) \subset V_\lambda$ for all dominant weights λ.

LEMMA 5.8. *$\mathscr{A}^{\circ} \cap \mathscr{B}$ is generated as an algebra by X, Y, E_{13}, $\begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix}$, $\begin{pmatrix}010\\100\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix}\begin{pmatrix}100\\100\end{pmatrix}$, and $\begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}110\\110\end{pmatrix}\begin{pmatrix}011\\110\end{pmatrix}$.*

Proof. The condition on a member of the spanning set (5.4) of $\mathscr{B}$ to be in $\mathscr{A}^{\circ}$ is that

$$b_1 + c_1 + d_2 = b_2 + c_1 + d_1 = b_1 + c_2 + d_1.$$

Consideration of the four cases arising from the condition $c_1c_2 = d_1d_2 = 0$ shows that the elements meeting this condition can be written in terms of the six operators given in the lemma and the elements

$$T_n = \begin{pmatrix}101\\110\end{pmatrix}^n\begin{pmatrix}010\\100\end{pmatrix}^n.$$

That the T_n are unnecessary is shown by the calculation:

$$T_n = T_{n-1}\left((n-1)E_{13} + \begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix}\right).$$

PROPOSITION 5.9. *$\mathscr{A}^{\circ}$ is the subalgebra of $\mathscr{A}$ generated by $\mathfrak{g}$, X, and Y.*

Proof. $\mathscr{A}^{\circ}$ is the $\mathfrak{g}$-module generated by $\mathscr{A}^{\circ} \cap \mathscr{B}$. To show that $\mathscr{A}^{\circ}$ is contained within the algebra generated by $\mathfrak{g}$, X, and Y we need only show that $\mathscr{A}^{\circ} \cap \mathscr{B}$ is so contained. Combine Lemma (5.8) and the following identities.

$$\text{(5.10a)}\quad \begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix} = E_{12}E_{23} + \frac{1}{2}(H_1 - H_2 + H_3 - H_4)E_{13}$$

$$\text{(5.10b)}\quad \begin{pmatrix}010\\100\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix}\begin{pmatrix}100\\100\end{pmatrix} = E_{23}E_{12}^2 - E_{32}E_{13}^2 - (H_2 - H_3)E_{12}E_{13}$$

$$\text{(5.10c)}\quad \begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}110\\110\end{pmatrix}\begin{pmatrix}011\\110\end{pmatrix} = E_{12}E_{23}^2 - E_{21}E_{13}^2 + (H_1 - H_2)E_{23}E_{13}.$$

6. Structure of $\mathscr{B}$. For weights λ, μ of $\mathfrak{g}$, define $\mathscr{B}\binom{\mu}{\lambda}$ to be the set of $T \in \mathscr{B}$ such that the following two conditions are satisfied:

6.1a) T is a λ vector of the $\mathfrak{g}$-representation ρ on $\mathscr{A}$.

6.1b) $T(V_\alpha) \subset V_{\alpha+\mu}$ for all dominant weights α of $\mathfrak{g}$.

Because the generators of $\mathscr{B}$ are all dominant weight vectors, unless λ is dominant, $\mathscr{B}\binom{\mu}{\lambda} = (0)$.

One has a grading of $\mathscr{B}$:

$$\mathscr{B} = \oplus\, \mathscr{B}\binom{\mu}{\lambda} \quad \text{and} \quad \mathscr{B}\binom{\mu}{\lambda} \cdot \mathscr{B}\binom{\mu'}{\lambda'} \subset \mathscr{B}\binom{\mu+\mu'}{\lambda+\lambda'}.$$

PROPOSITION 6.2. $\mathscr{B}\binom{0}{0} = \mathbf{C}[X, Y]$.

Proof. The algebra $\mathscr{B}\binom{0}{0}$ is spanned by those monomials in the six generators from Lemma 5.8 of $\mathscr{A}^{\circ} \cap \mathscr{B}$ which actually lie in $\mathscr{B}\binom{0}{0}$. Thus it is spanned by monomials in X and Y.

For weights μ and λ and dominant weight α of $\mathfrak{g}$ denote by $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$ the space of all $T \in \mathrm{Hom}_{\mathbf{C}}(V_\alpha, V_{\alpha+\mu})$ which are restrictions of elements of $\mathscr{B}\binom{\mu}{\lambda}$.

LEMMA 6.3. (i) *For μ, λ, α weights of* $\mathfrak{g}$ *with λ and α dominant,*

$$\dim \mathscr{B}\binom{\mu}{\lambda}(\alpha) = \dim \mathrm{Hom}_{\mathfrak{g}}(\pi_\lambda, \mathrm{Hom}_{\mathbf{C}}(\pi_\alpha, \pi_{\alpha+\mu})).$$

(ii) *The space $\mathscr{B}\binom{\mu}{\lambda}$ is nonzero if and only if λ is dominant and μ is a weight of π_λ.*

Proof. (i) The elements of $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$ are λ vectors of the $\mathfrak{g}$-representation $\mathrm{Hom}_{\mathbf{C}}(V_\alpha, V_{\alpha+\mu})$ which are highest weight vectors. By Proposition 3.4 we find all such in $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$.

(ii) This is a trivial consequence of (i) and Lemma 1.1.

Let Φ be the set of S in $\mathscr{B}$ as in (5.4).

Let $\Phi\binom{\mu}{\lambda}$ equal $\Phi \cap \mathscr{B}\binom{\mu}{\lambda}$.

For dominant weights α, denote by $\Phi\binom{\mu}{\lambda}(\alpha)$ the set of restrictions to

V_α of the elements of $\Phi\binom{\mu}{\lambda}$.

LEMMA 6.4. *The set* $\Phi\binom{\mu}{\lambda}(\alpha)$ *is a basis of* $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$ *for each dominant weight* α *such that* $\alpha + (210) + \sigma\lambda$ *is dominant for every* $\sigma \in \mathscr{S}_3$.

Proof. By Lemma 5.3 it is seen that the set $\Phi\binom{\mu}{\lambda}(\alpha)$ spans $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$ for all α.

To establish linear independence we must show that the cardinality of $\Phi\binom{\mu}{\lambda}$ equals dim $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$ for α as in the lemma.

Let

$$\binom{\mu}{\lambda} = \begin{pmatrix} a\ b\ c \\ p\ q\ 0 \end{pmatrix}$$

with $(p\ q\ 0)$ dominant, $(a\ b\ c)$ a weight of $\pi_{(pq0)}$, and $a + b + c = p + q$.

An easy calculation enumerates the elements of $\Phi\binom{\mu}{\lambda}$:

$$(6.5)\qquad \Phi\binom{\mu}{\lambda} = \left\{ E_{13}^{d}\binom{101}{110}^{\delta-d}\binom{010}{100}^{\delta+b-q-d}\binom{001}{100}^{c-q} \times \binom{100}{100}^{a-q} \right\}_{0 \leqq d \leqq \inf\{\delta, \delta+b-q\}}$$

where: i) For $n \geqq 0$ we have written $\binom{100}{100}^{-n}$ for $\binom{011}{110}^{n}$ and $\binom{001}{100}^{-n}$ for $\binom{110}{110}^{n}$.

ii) We compute δ from the table below:

δ	$q \geqq a$	$q \leqq a$
$q \geqq c$	$p - b$	c
$q \leqq c$	a	q

On the other hand, the dimension of $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$, which equals $\mathrm{mult}_\mu(\pi_\lambda)$ by Lemmas 1.1 and 6.3 for α as above, can also be computed explicitly. Choose $\sigma \in \mathscr{S}_3$ such that $\sigma\mu$ is dominant. Then $\mathrm{mult}_\mu(\pi_\lambda)$ equals $\mathrm{mult}_{\sigma\mu}(\pi_\lambda)$, and the latter is given by Lemma 1.3.

It is now a simple matter to conclude the proof by showing the two numbers card $\Phi\binom{\mu}{\lambda}$ and dim $\mathscr{B}\binom{\mu}{\lambda}(\alpha)$ to be equal.

THEOREM 6.6. *$\mathscr{B}\binom{\mu}{\lambda}$ is a free* $\mathbf{C}[X, Y]$*-module of rank equal to* $\text{mult}_\mu(\pi_\lambda)$. *The set $\Phi\binom{\mu}{\lambda}$ is a basis.*

Proof. By Lemma 5.3, the set $\Phi\binom{\mu}{\lambda}$ generates $\mathscr{B}\binom{\mu}{\lambda}$ as a $\mathbf{C}[X, Y]$-module.

Let the elements of $\Phi\binom{\mu}{\lambda}$ be denoted S_i.

Suppose given polynomials $f_i(X, Y)$ in $\mathbf{C}[X, Y]$ such that

$$\sum S_i f_i(X, Y) = 0.$$

Recall that $f_i(X, Y)$ acts as scalar multiplication by $f_i(r - s + 1, s + 1)$ on $V_{(rs0)}$.

A dominant weight $\alpha = (r\, s\, 0)$ satisfies the condition of Lemma 6.4 with $\lambda = (p\, q\, 0)$ if $s + 1 \geqq p$ and $r - s + 1 \geqq p$. The restriction of $S_i f_i(X, Y)$ to V_α for such α must be zero, and hence also each $f_i(r - s + 1, s + 1)$ must equal zero. This implies that each f_i is zero.

COROLLARY 6.7. *Let $\mathscr{U}$ be the universal enveloping algebra of the nine dimensional lie algebra spanned by the nine generators of $\mathscr{B}$. Let $\phi:\mathscr{U} \rightarrow \mathscr{B}$ be the canonical surjection.*

The kernel of ϕ is the ideal I of $\mathscr{U}$ generated by the two elements below:

$$\binom{011}{110}\binom{100}{100} - \binom{101}{110}\binom{010}{100} - XE_{13}$$

$$\binom{101}{110}\binom{010}{100} - \binom{110}{110}\binom{001}{100} - YE_{13}.$$

Proof. By Theorem 6.6 the elements SX^eY^f of Lemma 5.3 which span $\mathscr{U}/I$ are linearly independent in $\mathscr{B}$.

As an illustration of what can be done with Theorem 6.6, we find explicitly a basis for the space of (210) vectors in the $\pi_{(210)}$-isotypic subrepresentation of each $\mathfrak{g}$-module $\text{Hom}_{\mathbf{C}}(V_\alpha, V_\alpha)$.

Observe that

$$\Phi\binom{111}{210} = \left\{E_{13}, \binom{101}{110}\binom{010}{100}\right\}.$$

The conditions of Lemma 6.4 are met for $\alpha = (r\, s\, 0)$ if $r > s > 0$. For such α, $\Phi\binom{111}{210}$ is the sought for basis.

Next notice that $V_{(r00)}$ is the space of homogeneous polynomials of degree r in the variables a_1, a_2, a_3 and that $V_{(rr0)}$ is the space of homogeneous polynomials of degree r in a_{12}, a_{23}, a_{31}.

On $V_{(000)}$, both E_{13} and $\begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix}$ vanish and so $\mathrm{Hom}_{\mathbf{C}}(V_{(000)}, V_{(000)})$ contains no subrepresentation isomorphic to $\pi_{(210)}$.

Calculations show that E_{13} is nonzero on $V_{(r00)}$ and on $V_{(rr0)}$ if $r > 0$, and that on each of these spaces $\begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix}$ is linearly dependent upon E_{13}. Thus for $r > 0$, E_{13} is a highest weight vector in the unique irreducible subrepresentation of $\mathrm{Hom}_{\mathbf{C}}(V_{(r00)}, V_{(r00)})$ or of $\mathrm{Hom}_{\mathbf{C}}(V_{(rr0)}, V_{(rr0)})$ which is isomorphic to $\pi_{(210)}$.

7. The so_8-representation $\mathscr{A}$. The action ρ of $\mathfrak{g}$ on $\mathscr{A}$ extends to an action, also denoted ρ, of so_8 on $\mathscr{A}$:

$$\rho(x)a = [x, a] \quad \text{for } x \in so_8,\ a \in \mathscr{A}.$$

We want to decompose explicitly the representation ρ of so_8.

The group of weights of so_8 will be identified with $\mathbf{Z}^4 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbf{Z}$ as follows: For $\eta = (p_1 p_2 p_3 p_4)$ a weight and $H = \sum b_i H_i \in \mathfrak{k}$, define

$$\eta(H) = \sum p_i b_i.$$

A weight η is *dominant* if

$$p_1 \geqq p_2 \geqq p_3 \geqq |p_4|.$$

An element w of an so_8-module is an η *vector* if

$$Hw = \eta(H)w \quad \text{for all } H \in \mathfrak{k}.$$

We say that η is a *weight* of a representation if there is a nonzero η vector and refer to the dimension of the space of η vectors as the *multiplicity* of η.

Every finite dimensional irreducible representation of so_8 has a unique weight η, called its *highest weight*, for which there is a nonzero η vector annihilated by E_{12}, E_{23}, $\begin{pmatrix}010\\001\end{pmatrix}$, and $\begin{pmatrix}001\\001\end{pmatrix}$. It is a dominant weight and of multiplicity one; it determines the isomorphism class of the representation. We shall write π_η to denote an irreducible representation of highest weight η.

THEOREM 7.1. i) $\mathbf{C}\left[\begin{pmatrix}011\\110\end{pmatrix}\right]$ *is the commutant of* $\left\{E_{12}, E_{23}, \begin{pmatrix}010\\001\end{pmatrix}, \begin{pmatrix}001\\001\end{pmatrix}\right\}$ *in* $\mathscr{A}$.

ii) *There is an isomorphism of so_8-representations*:

$$\rho \simeq \bigoplus_{p=0}^{\infty} \pi_{(pp00)}.$$

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Proof. The commutant is surely contained within $\mathcal{B}$, the commutant of E_{12} and E_{23} in $\mathcal{A}$.

We list the nine generators of $\mathcal{B}$ and their so_8 weights.

X	$(0, 0, 0, 0)$	$\begin{pmatrix}110\\110\end{pmatrix}$	$(0, 0, -1, -1)$
Y	$(0, 0, 0, 0)$	$\begin{pmatrix}001\\100\end{pmatrix}$	$(1, 0, 0, 1)$
E_{13}	$(1, 0, -1, 0)$	$\begin{pmatrix}011\\110\end{pmatrix}$	$(1, 1, 0, 0)$
$\begin{pmatrix}101\\110\end{pmatrix}$	$(0, 0, -1, 1)$	$\begin{pmatrix}100\\100\end{pmatrix}$	$(0, -1, -1, 0)$
$\begin{pmatrix}010\\100\end{pmatrix}$	$(1, 0, 0, -1)$		

An eigenvector of $\mathfrak{k}$ in the commutant of E_{12}, E_{23}, $\begin{pmatrix}010\\001\end{pmatrix}$ and $\begin{pmatrix}001\\001\end{pmatrix}$ must be a dominant weight vector. The list above shows that it can be written in the form

$$\begin{pmatrix}010\\100\end{pmatrix}^a\begin{pmatrix}001\\100\end{pmatrix}^a\begin{pmatrix}011\\110\end{pmatrix}^b f(X, Y),$$

where a, b, and the polynomial f are uniquely determined.

To facilitate computations we will change variables. Let $W = X + Y - 2$, and let $Z = Y - 1$. A dominant weight vector is uniquely expressible in the form:

$$T = \begin{pmatrix}010\\100\end{pmatrix}^a\begin{pmatrix}001\\100\end{pmatrix}^a\begin{pmatrix}011\\110\end{pmatrix}^b g(W, Z). \tag{7.3}$$

We first show that a must be zero. This follows from explicit calculation, for all $\alpha, \gamma \geqq a$, of both sides of the equality (7.4). The right hand side is always zero.

$$\begin{pmatrix}001\\001\end{pmatrix}T \cdot a_2{}^{\alpha}a_3{}^{b}a_{23}{}^{\gamma} = T\begin{pmatrix}001\\001\end{pmatrix} \cdot a_2{}^{\alpha}a_3{}^{b}a_{23}{}^{\gamma}. \tag{7.4}$$

We next show that the polynomial $g(W, Z)$ must be independent of Z. This can be done by calculating explicitly, for all $\alpha \geqq b$ and $\beta \geqq 0$, both sides of the equality (7.5).

$$\begin{pmatrix}010\\001\end{pmatrix}T \cdot a_1a_3{}^{\alpha}a_{31}{}^{\beta} = T\begin{pmatrix}010\\001\end{pmatrix} \cdot a_1a_3{}^{\alpha}a_{31}{}^{\beta}. \tag{7.5}$$

At last, calculations for all $\alpha \geqq b$ of (7.6) shows that $g(W)$ is constant.

$$\begin{pmatrix}001\\001\end{pmatrix}T \cdot (a_{12}a_3{}^{\alpha} - \alpha a_3{}^{\alpha-1}a_2a_{31}) \tag{7.6}$$

$$= T\begin{pmatrix} 001 \\ 001 \end{pmatrix} \cdot (a_{12}a_3{}^{\alpha} - \alpha a_3{}^{\alpha-1}a_2a_{31}).$$

8. Simplicity of $\mathscr{A}$.

THEOREM 8.1. *The algebra $\mathscr{A}$ contains no nonzero proper two-sided ideal.*

Proof. Let $\mathscr{A}(p)$ denote the irreducible so_8-submodule of $\mathscr{A}$ with highest weight $(pp00)$ and highest weight vector $\begin{pmatrix} 011 \\ 110 \end{pmatrix}^p$.

A two-sided ideal J is an so_8-submodule of $\mathscr{A}$, hence must be a sum of $\mathscr{A}(p)$. If $\mathscr{A}(p)$ is contained in J, then $\begin{pmatrix} 011 \\ 110 \end{pmatrix}^n$ is contained in J for all $n \geqq p$. Thus

$$J = \bigoplus_{p \geqq N} \mathscr{A}(p),$$

where N is the smallest integer for which $\mathscr{A}(N) \subset J$. We see thus that the nontrivial two-sided ideals, if any, form a chain and that each is of finite codimension in $\mathscr{A}$.

Let J be a nontrivial ideal of $\mathscr{A}$.

The quotient algebra $\mathscr{A}/J$, being a finite dimensional quotient of $\mathscr{U}(so_8)$, the universal enveloping algebra of so_8, is semisimple; that is, it is isomorphic to a finite product of full matrix algebras. Since the ideals in $\mathscr{A}/J$ form a chain, there can be at most one factor in the product. We deduce that J is maximal.

Let I be the inverse image of J in $\mathscr{U}(so_8)$. There is a finite dimensional irreducible representation π_η of so_8 such that I equals the kernel of π_η in $\mathscr{U}(so_8)$.

Let Z be the center of $\mathscr{U}(so_8)$, and let $\chi_\eta : Z \to \mathbf{C}$ be the central character of π_η. Let $\chi : Z \to \mathbf{C}$ be the central character of the representation of so_8 on V.

It is clear that $\chi = \chi_\eta$. We will show that this equality leads to a contradiction.

The representation of so_8 on V is irreducible with highest weight $(-1, -1, -1, -1)$. Indeed the element $1 \in V$ is a $(-1, -1, -1, -1)$-vector which is annihilated by E_{12}, E_{23}, $\begin{pmatrix} 010 \\ 001 \end{pmatrix}$, and $\begin{pmatrix} 001 \\ 001 \end{pmatrix}$.

The equality $\chi = \chi_\eta$ implies the existence of an element w in the Weyl group of $\mathfrak{k}$ such that

$$\eta + (3, 2, 1, 0) = w((-1, -1, -1, -1) + (3, 2, 1, 0)).$$

But this is impossible for a dominant weight η.

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Duke University,
Durham, North Carolina