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Regular metabelian groups of prime-power order

R. J. Faudree

Let H be a finite metabelian p-group which is nilpotent of class c. In this paper we will prove that for any prime $p > 2$ there exists a finite metacyclic p-group G which is nilpotent of class c such that H is isomorphic to a section of a finite direct product of G.

Introduction

Subgroups and factor groups of regular groups are regular but the direct product of two regular groups is not necessarily regular (see [5]). Moreover the variety generated by all finite regular p-groups for $p > 3$ is the variety of all groups (see [2]) and the variety generated by all finite metabelian regular p-groups for $p > 2$ is the variety of all metabelian groups (see [4]). The main result of this paper extends some of the results mentioned above by showing for $p > 2$ that not only do you get irregular p-groups by taking finite direct products of regular p-groups but every finite metabelian p-group can be obtained by taking factor groups of subgroups of finite direct products of finite metabelian regular p-groups.

Proof of theorem

In the remaining discussion we will assume $p > 2$ and H is a fixed finite metabelian p-group of exponent p^e which is nilpotent of class c. For any positive integer r let

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$$G_r = \left\langle a, b : a^{p^{rc}} = 1 , b^{p^c} = 1 , \text{ and } a^{-1}b^{-1}ab = a^{p^r} \right\rangle .$$

G_r is a metacyclic group which is nilpotent of class c and of exponent p^{rc}. G_r is also regular (see [1]).

We will prove

THEOREM. *H is in the variety generated by G_r for r sufficiently large.*

A well-known property of finite groups in the variety generated by a finite group (see [3]) gives the following:

COROLLARY. *H is isomorphic to a section of a finite direct product of G_r for sufficiently large r.*

Let X_∞ denote the free group of countable rank on generators $\{x_1, x_2, \ldots\}$. Let g be an element of X_∞. g is a *simple commutator in normal form of weight* v, weight $g = v$, *and sign* u, sign $g = u$ and involving precisely $\{x_1, \ldots, x_t\}$ if

$$g(x_1, x_2, \ldots, x_t) = \left(x_{i(1)}, x_{i(2)}, \ldots, x_{i(v)}\right)$$

where $i(1) = 1$, $i(2) = u$, $i(j) \leq i(k)$ if $2 < j \leq k$ and $\{i(1), \ldots, i(v)\} = \{1, \ldots, t\}$. $d_j(g)$ will denote the number of occurrences of x_j in g.

An element f of X_∞ is in *normal form* of weight $\leq c$ if

$$f = \prod_{i=1}^{l} f_i^{\gamma_i}$$

where the f_i are distinct simple commutators in normal form of weight $\leq c$ involving precisely $\{x_1, \ldots, x_t\}$, l is an arbitrary positive integer, t is a positive integer $\leq c$, and the γ_i are non-zero integers. Let L_r denote the words of X_∞ which are in normal form of weight $\leq c$ and are laws of G_r. A basis for the laws of G_r is

$L_r \cup \left\{\left((x_1, x_2), (x_3, x_4)\right), \left(x_1, \ldots, x_{c+1}\right)\right\}$, (see [6]). Therefore to prove the theorem it is sufficient to prove:

PROPOSITION. *For sufficiently large* r, f *in* L_r *implies* p^e *divides* γ_i $(1 \le i \le l)$.

Before we can complete the proof of the proposition it will be necessary to state and prove some elementary lemmas.

For m a positive integer let $\theta(m)$ be the highest power of p dividing $m!$.

LEMMA 1.

i) If $m = \sum_{i=0}^{t} k_i p^i$ *with* $0 \le k_i \le p-1$ *then*

$$\theta(m) = \left(m - \sum_{i=0}^{t} k_i\right)/(p-1) ;$$

ii) for positive integers n *and* m, $p^{\theta(m)}$ *divides* $(m+n)!/n!$.

Proof. The number of positive multiples of p^i, $(1 \le i \le t)$, less than or equal to m is $\sum_{j=i}^{t} k_j p^{j-i}$. Thus

$$\theta(m) = \sum_{i=1}^{t} \sum_{j=i}^{t} k_j p^{j-i} = \left(m - \sum_{i=0}^{t} k_i\right)/(p-1) .$$

ii) is a consequence of the fact that $m!$ divides $(m+n)!/n!$.

Let Z denote the integers, $R = Z\left[y_1, \ldots, y_t\right]$ the polynomial ring over Z in indeterminates $y_1, \ldots, y_t$, and J_n the ideal of R generated by p^n for n a non-negative integer. Denote $Z/\left(Z \cap J_n\right)$ by Z_n.

LEMMA 2. *Let* $h = h\left(y_1, \ldots, y_t\right)$ *be an element of* R *such that the degree in each variable is* $\le c$ *and* $tc < n$. *If* h, *considered as*

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a function of Z *into* Z *, has only values in* $J_n \cap Z$ *then* h *is in* J_m *for* $m = n - t\theta(c)$.

Proof. Let $t = 1$. We can assume that $y_1 = y$ and h is a polynomial of degree c . Since $h(1)$ is in $J_n \cap Z$

$$h(y) = (y-1)h_1(y) + h_1'(y)$$

where h_1' is a polynomial in J_n . Assume for $j \le c-1$ that

$$* \qquad h(y) = \left(\prod_{i=1}^{j} (y-i)\right)h_j(y) + h_j'(y)$$

where h_j' is a polynomial in J_m which as a function has only values in J_n . Thus $(j!)h_j(j+1)$ is in $J_n \cap Z$ and

$$h_j(y) = (y-j-1)\cdot h_{j+1}(y) + k(y)$$

for $k(y)$ in $J_{n-\theta(j)}$.

$$h_{j+1}'(y) = \prod_{i=1}^{j} (y-i)\cdot k(y) + h_j'(y)$$

has the same properties as $h_j'(y)$. Therefore $*$ is true for $1 \le j \le c$.

Since $h(y)$ has degree c

$$h(y) = \prod_{i=1}^{c} (y-i)\cdot b + h_c(y)$$

where b is in Z . Hence $(c!)b$ is in $J_n \cap Z$, b is in $J_m \cap Z$ and h is in J_m . Induction on t will complete the proof.

For non-negative integers $i \le j$ let $\binom{j}{i} = j!/\big((j-i)!i!\big)$. For any positive integer r let σ_r be the function on the non-negative integers defined by

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$$\sigma_r(j) = \sum_{i=1}^{j} \binom{j}{i} p^{r(i-1)} .$$

For any positive integer n, σ_r induces a map σ'_r of Z_n into Z_n.

LEMMA 3. *σ'_r is onto.*

Proof. It suffices to show that σ'_r is injective. Assume $\sigma_r(k) - \sigma_r(j)$ is in $Z \cap J_n$ for $0 \le j < k < p^n$. Clearly $k - j$ is in $Z \cap J_s$ for $s = \min\{n, r\}$. $k - j$ in $Z \cap J_{lr}$ $(l \ge 1)$ implies $p^{r(i-1)}\left[\binom{k}{i}-\binom{j}{i}\right]$ is in $Z \cap J_{r(l+1)}$ if $2 \le i \le j$ and $\binom{k}{i}p^{r(i-1)}$ is in $J_{r(l+1)}$ if $i > j$. Hence $k - j$ is in $Z \cap J_s$ for $s = \min\{(l+1)r, n\}$. Induction on l gives that $k - j$ is in $Z \cap J_n$.

Proof of the proposition. Let $f = \prod_{i=1}^{l} f_i^{\gamma_i}$ be an element of L_r. We can assume f involves precisely $\{x_1, \ldots, x_t\}$ and $\gamma_i = \beta_i p^{\alpha_i}$ $(1 \le i \le l)$ with β_i relatively prime to p. Let u be a fixed integer between 2 and t, $n_{ij} = d_j(f_i) - \delta_{ju}$ $(1 \le i \le l,\ 1 \le j \le t)$ where δ_{ju} is the Kronecker delta, and $w_i = \sum_{j=1}^{t} n_{ij}$.

For any function j of Z into Z define the homomorphism τ_j of X_∞ into G_r by $\tau_j(x_k) = b^{j(k)}$ if $k \ne u$ and $\tau_j(x_k) = a^{-1}b^{j(k)}$ if $k = u$. By assumption $\tau_j(f) = 1$ for all j. Also $\tau_j(f_i) = 1$ for all j if sign $f_i \ne u$. If sign $f_i = u$ then $\tau_j(f) = a^{\phi(i)}$ where $\phi(i) = p^{rw_i} \prod_{k=1}^{t} \left(\sigma_r(j(k))\right)^{n_{ik}}$. If only the first s of the f_i have sign u then

$$\sum_{i=1}^{s} \beta_i p^{rw_i+\alpha_i} \left(\prod_{k=1}^{t} \sigma_r(j(k))^{n_{ik}}\right)$$

is in $J_{rc} \cap Z$. By Lemma 3 the only values of the polynomial

$$h(y_1, \ldots, y_t) = \sum_{i=1}^{s} \beta_i p^{rw_i+\alpha_i}\left(\prod_{k=1}^{t} y_k^{n_{ik}}\right)$$

considered as a function are in $J_{rc} \cap Z$. Lemma 2 implies $rw_i+\alpha_i \geq rc-t\theta(c)$ $(1 \leq i \leq s)$. Thus

$$\alpha_i \geq r(c-w_i) - t\theta(c) ;$$

$w_i < c$ and $t \leq c$ since each f_i involves precisely $\{x_1, \ldots, x_t\}$. The above argument is true for any u . Therefore $\alpha_i \geq e$ $(1 \leq i \leq l)$ if r is sufficiently large.

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University of Illinois at Urbana-Champaign,
Urbana, Illinois, USA.