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L^p BEHAVIOR OF THE EIGENFUNCTIONS OF THE INVARIANT LAPLACIAN

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ABSTRACT. Let $\tilde{\Delta}$ be the invariant Laplacian on the open unit ball B of C^n and let X_λ denote the set of those $f \in C^2(B)$ such that $\tilde{\Delta}f = \lambda f$. X_λ counterparts of some known results on X_0, *i.e.* on M-harmonic functions, are investigated here. We distinguish those complex numbers λ for which the real parts of functions in X_λ belongs to X_λ. We distinguish those λ for which the Maximum Modulus Priniple remains true. A kind of weighted Maximum Modulus Principle is presented. As an application, setting $\alpha \geq \frac{1}{2}$ and $\lambda = 4n^2\alpha(\alpha-1)$, we obtain a necessary and sufficient condition for a function f in X_λ to be represented as

$$f(z) = \int_{\partial B}\Big(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\Big)^{n\alpha} F(\zeta)\,d\sigma(\zeta)$$

for some $F \in L^p(\partial B)$.

1. **Introduction.** Let $\mathbf{C}^n$ be the n-dimensional complex Euclidean space with the norm $|z| = \sqrt{\sum_j |z_j|^2}$ and the Hermitian inner product $\langle z,w\rangle = \sum_j^n z_j\bar{w}_j$, $z = (z_1,\ldots,z_n)$, $w = (w_1,\ldots,w_n)$. Let B denote the open unit ball of $\mathbf{C}^n$ and let S be its boundary. Let Aut(B) denote the Möbius group, *i.e.* the group of those bijective holomorphic maps of B onto itself. Let ψ_z denote one such map with $\psi_z(0) = z$. For $f \in C^2(B)$, $\tilde{\Delta}f$ is defined by

$$(\tilde{\Delta}f)(z) = 4(1-|z|^2)\sum_{i,j=1}^n(\delta_{ij}-z_i\bar{z}_j)\Big(\frac{\partial^2}{\partial z_j\partial\bar{z}_j}f\Big)(z) \tag{1.1}$$

[R, 4.1.3] and is called the *invariant Laplacian* because $\tilde{\Delta}(f\circ\psi) = (\tilde{\Delta}f)\circ\psi$ for $\psi \in$ Aut(B) [R, 4.1.2]. If $f \in C^2(B)$ satisfies $(\tilde{\Delta}f)(z) = 0$, $z \in B$, then f is said to be *M-harmonic*. Here M refers to the Möbius group. For a complex number λ, X_λ denotes the set of those $f \in C^2(B)$ such that $\tilde{\Delta}f = \lambda f$. X_λ is an M-invariant closed subspace of $C^2(B)$ in the topology of uniform convergence on compact sets. If $\lambda \neq \lambda'$ then $X_\lambda \cap X_{\lambda'}$ is trivial. *i.e.* $X_\lambda \cap X_{\lambda'} = \{0\}$. An outstanding feature of X_λ we need is that if $f \in X_\lambda$ and $\lambda = 4n^2\alpha(\alpha-1)$ then f satisfies the weighted mean value property (and conversely) [R, 4.2.4]:

$$\int_S f\big(\psi_z(r\zeta)\big)\,d\sigma(\zeta) = f(z)\int_S P^\alpha(r\eta,\zeta)\,d\sigma(\zeta), \quad 0 \leq r < 1,\ \eta \in S. \tag{1.2}$$

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Here index α refers to the principal branch, σ denotes the rotation invariant probability measure on S, and $P(z,\zeta)$ denotes the invariant Poisson kernel:

(1.3) $$P(z,\zeta) = \left(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\right)^n, \quad z \in B,\ \zeta \in S.$$

See [K], [KK], and [R] for X_λ theory.

Throughout, two complex numbers α and λ are related to be

(1.4) $$\lambda = 4n^2\alpha(\alpha-1),$$

and the radial function $\int_S P^\alpha(z,\zeta)\,d\sigma(\zeta)$ is denoted by $g_\alpha(z)$. The function g_α is used both as a radial function on the ball and as a function on $\mathbf{R}^+$.

If $f \in X_0$, *i.e.* if f is M-harmonic, then the real part of f, $\operatorname{Re} f$, is also M-harmonic. Our question in Section 2 is whether this remains true for functions of X_λ. Theorem 1 and Theorem 2 distinguish those complex numbers λ for which the real parts of functions in X_λ also belongs to X_λ. If $f \in X_0$ then f satisfies the Maximum Modulus Principle, *i.e.* $|f|$ can't obtain a local maximum unless f is a constant. In Section 3, we distinguish those λ for which every function of X_λ satisfies the the Maximum Modulus Principle. Also, it is observed that functions of X_λ, α real, satisfy a weighted type Maximum Modulus Principle with the weight function g_α (Theorem 4). As an application to this, in Section 4, we obtain a necessary and sufficient growth condition for a function f of X_λ, $\alpha \geq \frac{1}{2}$, to be represented as

$$f(z) = \int_{\partial B}\big(P(z,\zeta)\big)^\alpha F(\zeta)\,d\sigma(\zeta),$$

for some $F \in L^p(S)$ (Theorem 6).

2. **Real parts of X_λ.**

THEOREM 1. *If* $\operatorname{Re}\alpha \neq \frac{1}{2}$ *then the following are equivalent.*
(1) X_λ *has a nontrivial real function;*
(2) λ *is real;*
(3) α *is real;*
(4) $g_\alpha(z)$ *is a real function;*
(5) $f \in X_\lambda$ *if and only if* $\operatorname{Re} f \in X_\lambda$ *and* $\operatorname{Im} f \in X_\lambda$.

THEOREM 2. *If* $\operatorname{Re}\alpha = \frac{1}{2}$, *then we have*
(1) λ *is real;*
(2) $g_\alpha(z)$ *is a real function;*
(3) $f \in X_\lambda$ *if and only if* $\operatorname{Re} f \in X_\lambda$ *and* $\operatorname{Im} f \in X_\lambda$.

PROOF OF THEOREM 1. (1) $\Rightarrow$ (2): From (1.1), we have $\overline{\tilde{\Delta}\bar{f}} = \tilde{\Delta}f$. Let f be a nontrivial real function of X_λ. Then

$$\lambda f = \tilde{\Delta}f = \overline{\tilde{\Delta}\bar{f}} = \overline{\tilde{\Delta}f} = \overline{\lambda f} = \bar{\lambda}f.$$

Thus $\lambda = \bar{\lambda}$. *i.e.* λ is real.

(2) $\Rightarrow$ (3): Let λ be real and let $\alpha = a + ib$, a, b real. Then $0 = \operatorname{Im}\lambda = 4n^2b(2a-1)$. Since $a = \operatorname{Re}\alpha \neq \frac{1}{2}$, $b = 0$. *i.e.* α is real.

(3) $\Rightarrow$ (4): Since $P(z,\zeta)$ is real, $g_\alpha(z)$ is real if α is real.

(4) $\Rightarrow$ (5): Let $f \in X_\lambda$. Supposing g_α real, from (1.2), we have

$$\int_S (\operatorname{Re} f \circ \psi_z)(r\zeta)\, d\sigma(\zeta) = \operatorname{Re} f(z) g_\alpha(r), \qquad 0 < r < 1,\ z \in B.$$

Hence it follows from [R, 4.2.4] that $\operatorname{Re} f \in X_\lambda$. Similar arguments give us that $\operatorname{Im} f \in X_\lambda$ also. Conversely, if $\operatorname{Re} f \in X_\lambda$ and $\operatorname{Im} f \in X_\lambda$ then it obviously follows that $f \in X_\lambda$.

(5) $\Rightarrow$ (1): Suppose (5). Since $g_\alpha(z) \in X_\lambda$ [R, 4.2.2], $\operatorname{Re} g_\alpha \in X_\lambda$. Since $g_\alpha(0) = 1$, real part of g_α is a non-trivial real function of X_λ.

PROOF OF THEOREM 2. (1) Let $\alpha = \frac{1}{2} + ib$, b real. Then $\lambda = 4n^2\alpha(\alpha-1) = 4n^2(\frac{1}{4} + b^2)$, so that λ is real.

(2) Since $\bar{\alpha} = 1 - \alpha$, from [R, 4.2.3 Corollary] it follows that

$$g_\alpha = g_{1-\alpha} = g_{\bar{\alpha}} = \overline{g_\alpha}.$$

Hence g_α is real.

(3) Let $f \in X_\lambda$, then (1.2) holds. Taking real parts, we conclude that $\operatorname{Re} f \in X_\lambda$ as in the proof (4) $\Rightarrow$ (5) of Theorem 1. Similarly, $\operatorname{Im} f \in X_\lambda$.

3. On maximum modulus principle.

We will say that f defined on B satisfies *Maximum Modulus Principle* (abbreviated as MMP) if $|f|$ cannot have a local maximum in B unless f is a constant function. M-harmonic functions satisfy MMP. But MMP is no longer true for functions of X_λ in general even when λ is real.

THEOREM 3. *Let $\alpha = a + ib$, a, b real. Then the following (1) and (2) are equivalent.*

(1) Every function of X_λ satisfy MMP.

(2) $a(a-1) > b^2$ or $\lambda = 0$.

PROOF. (1) $\Rightarrow$ (2): Consider the radial function $g_\alpha(z)$. Note that

$$g_\alpha(r) = (1-r^2)^{n\alpha} F(n\alpha, n\alpha, n; r^2) \tag{3.1}$$

[KK, Corollary 2.4], where F is the Gaussian hypergeometric function:

$$F(a,b,c;t) = \sum_0^\infty \frac{(a)_k(b)_k}{(c)_k} \frac{t^k}{k!}$$

[S]. Let

$$y_\alpha(t) = (1-t)^{n\alpha} F(n\alpha, n\alpha, n; t), \qquad -1 < t < 1. \tag{3.2}$$

Then it follows from differentiating (3.2) that

$$\Big(\frac{d}{dt}|y_\alpha|^2\Big)(0) = 2n\big(a(a-1) - b^2\big) \tag{3.3}$$

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and

$$\text{if } a(a-1)=b^2 \text{ then } \frac{d^2}{dt^2}|y_\alpha|^2(0) = -2a\big(4n^2a(a-1)^2+n^2a-n^2\big). \tag{3.4}$$

Now if $a(a-1)-b^2<0$ then by (3.3) we know $\frac{d}{dt}|y_\alpha|^2<0$ near $t=0$. That is, the radial function $|y_\alpha|$ is decreasing near the origin, so that $|g_\alpha(0)|=1$ is a local maximum of $|g_\alpha|$. Hence $g_\alpha(z)=y_\alpha(|z|^2)$ is a function of X_λ for which MMP fails. If $a(a-1)=b^2$ and $\lambda\neq 0$, then by (3.3) and (3.4) we have

$$\frac{d}{dt}|y_\alpha|^2(0)=0 \text{ and } \frac{d^2}{dt^2}|y_\alpha|^2(0)<0,$$

so that $|y_\alpha|$ has a local maximum at 0. Hence MMP fails for g_α.

(2) $\Rightarrow$ (1): Let $f\in X_\lambda$. Suppose $|f|$ has a local maximum, say at a. Take r_0 sufficiently small so that $|f(a)|\geq|f(z)|$, $z\in\phi_a\big(D(0,r_0)\big)$. Here $D(0,r_0)$ denotes the open ball of radius r centered at 0. Then by the maximality of $|f|$ and (1.2), we have

$$\begin{aligned}|f(a)| &\geq \int_S |f\circ\phi_a(r\zeta)|\,d\sigma(\zeta)\\ &\geq \left|\int_S f\circ\phi_a(r\zeta)\,d\sigma(\zeta)\right| = |f(a)|\,|y_\alpha(r^2)|, \quad 0<r<r_0.\end{aligned} \tag{3.5}$$

Now if $a(a-1)-b^2>0$ then, by (3.3), $\frac{d}{dt}|y_\alpha|^2>0$ in a neighborhood of 0, so that $|y_\alpha(r^2)|>|y_\alpha(0)|=1$ for sufficiently small r. Thus, from (3.5), $f(a)=0$. Since any local maximum of $|f|$ is zero, we have $f\equiv 0$. If $\lambda=0$ then $|y_\alpha(r^2)|=1$, so that equality holds in (3.5), which implies that $|f|=\gamma f$ for some constant γ, on $D(a,r_0)$. Thus, γf is a nonnegative function of X_0 having local maximum in $D(a,r_0)$. This is impossible by the Maximum Principle of nonnegative M-harmonic functions [R, 4.3.2] unless f is a constant function.

Though MMP failed for some real λ, there is a MMP of weighted type in case α is real. Note that if α is real then g_α is nonzero and positive.

THEOREM 4. *Let α be real. Then $g_\alpha^{-1}u$ has* MMP *for every $u\in X_\lambda$.*

PROOF. Let $u\in X_\lambda$ and $f=g_\alpha^{-1}u$. From (1.2) we have

$$g_\alpha(r)u(z)=\int_S u\circ\phi_z(r\zeta)\,d\sigma(\zeta), \quad z\in B,\ 0\leq r<1. \tag{3.6}$$

Hence

$$\begin{aligned}|f(z)| &= \frac{1}{g_\alpha(r)g_\alpha(z)}\left|\int_S u\circ\phi_z(r\zeta)\,d\sigma(\zeta)\right|\\ &= \frac{1}{g_\alpha(r)g_\alpha(z)}\left|\int_S g_\alpha\big(\phi_z(r\zeta)\big)f\circ\phi_z(r\zeta)\,d\sigma(\zeta)\right|\\ &\leq \frac{1}{g_\alpha(r)g_\alpha(z)}\int_S g_\alpha\big(\phi_z(r\zeta)\big)|f\circ\phi_z(r\zeta)|\,d\sigma(\zeta)\\ &\leq \sup_{\zeta\in S}|f\circ\phi_z(r\zeta)|, \quad z\in B,\ 0\leq r<1,\end{aligned} \tag{3.7}$$

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where we used (3.6) once more with g_α instead of u in the last inequality. We conclude from (3.7) that $|f|$ can't have a local maximum unless $f = \gamma|f| =$ constant for some constant γ.

COROLLARY 5. *Let α be real and let Ω be an open subset of B. Let $u \in X_\lambda$ and $f = g_\alpha^{-1}u \in C(\bar{B})$. If $|f| \leq M$ on $\partial\Omega$ then $|f| \leq M$ on Ω.*

PROOF. The proof is typically routine. Suppose $|f(z)| \leq M$ on $\partial\Omega$ but $|f(z)| > M$ for some $z \in \Omega$. Then the set E of the points in $\bar{\Omega}$ on which $|f|$ takes its maximum is nonempty closed. Since $f \in C(\bar{\Omega})$, we can take $z_0 \in \Omega$ such that $\operatorname{dist}(z_0, \Omega) = \operatorname{dist}(E, \Omega)$. But for z_0 in place of z we have the strict inequality in (3.7):

$$|f(z_0)| < \sup_{\zeta \in S} |f \circ \phi_{z_0}(r\zeta)|, \quad 0 < r < 1.$$

This contradicts the maximality of $|f(z_0)|$, and so completes the proof.

4. L^p **behavior of functions in** X_λ. Throughout this section, we let α be real and $\beta = \alpha - 1$. For $1 \leq p \leq \infty$, $L^p(\sigma)$ norm of an $F \in L^p(\sigma)$ is denoted by $\|F\|_p$. For f continuous on B and $0 \leq r < 1$, we denote

$$M_p(r,f) = \left(\int_S |f(r\zeta)|^p \, d\sigma(\zeta)\right)^{1/p}$$

if $p < \infty$, and

$$M_\infty(r,f) = \sup_{\zeta \in S} |f(r\zeta)|.$$

For a complex Borel measure μ, $P^\alpha[\mu]$ is defined by

$$P^\alpha[\mu](z) = \int_S \left(P(z,\zeta)\right)^\alpha d\mu(\zeta),$$

where $P(z,\zeta)$ is the invariant Poisson kernel defined in (1.3). Note that $P^\alpha[\sigma] = g_\alpha$.

We define the function spaces $h^{p,t}$, $1 \leq p \leq \infty$, $-\infty < t < \infty$, by

$$h^{p,t} = \left\{f : \sup_{0 \leq r < 1} (1 - r^2)^t M_p(r,f) < \infty\right\}.$$

and

$$h^{p,t-} = \left\{f : \sup_{0 \leq r < 1} r^2 (1 - r^2)^t \log(1 - r^2) M_p(r,f) < \infty\right\}.$$

It is well-known that if $f \in X_0 \cap h^{p,0}$, $1 < p \leq \infty$, then there is a function $F \in L^p(\sigma)$ such that $f = P[F]$, and conversely [R, 4.3.3]. The goal of this section is in a generalization of this fact. Since, as a function of α, λ defined by (1.4) satisfies $\lambda(\alpha) = \lambda(1 - \alpha)$, we confine ourselves to $\alpha \geq \frac{1}{2}$.

THEOREM 6. *Let $\alpha \geq \frac{1}{2}$. If $F \in C(S)$ and if we define*

$$f(z) = \begin{cases} g_\alpha^{-1}(z)P^\alpha[F](z), & z \in B \\ F(z), & z \in S \end{cases} \tag{4.2}$$

then $f(z) \in C(\bar{B})$. Conversely, if $u(z) \in X_\lambda$ and $g(z) = g_\alpha^{-1}(z)u(z)$ is continuous up to S so that $g(z) \in C(\bar{B})$ then $u(z) = P^\alpha[G](z)$, where $G(\zeta) = \lim_{r\to 1} g(r\zeta)$, $\zeta \in S$.

PROOF. Let $F \in C(S)$ and let $f(z)$ be defined as (4.2). Consider

$$k(r) = \frac{(1-r^2)^{n(\alpha+\beta)}}{F(-n\beta, -n\beta, n; r^2)}, \quad 0 \leq r < 1.$$

If $\alpha > \frac{1}{2}$, then $k(r)$ is dominated by $(1-r^2)^{n(\alpha+\beta)}$, so that it tends to 0 as $r \longrightarrow 1$. If $\alpha = \frac{1}{2}$, then $\alpha + \beta = 0$; but since $F(n/2, n/2, n; r^2) \sim -\log(1-r^2)$ as $r \longrightarrow 1$, $k(r)$ also tends to 0 as $r \longrightarrow 1$. Now set

$$K(z,\eta) = \frac{P^\alpha(z,\eta)}{g_\alpha(z)}, \quad z \in B,\ \eta \in S,$$

and

$$Q = Q(\zeta, \delta) = \{\eta \in S : |1 - \langle \zeta, \eta\rangle| < \delta\}, \quad \delta > 0.$$

Then $|1 - \langle r\zeta, \eta\rangle| \geq \delta - (1-r)$ on $S - Q$, so that by (3.1) and above argument on $k(r)$,

$$\int_{S-Q} K(r\zeta,\eta)\, d\sigma(\eta) = k(r) \int_{S-Q} \frac{d\sigma(\eta)}{|1 - \langle r\zeta, \eta\rangle|^{2n\alpha}} \longrightarrow 0 (r \longrightarrow 1). \tag{4.3}$$

From (4.3) and the fact $\int_S K(z,\eta)\, d\eta = 1$, we conclude that

$$f(r\zeta) - f(\zeta) = \int_S K(r\zeta,\eta)\big(F(\eta) - F(\zeta)\big)\, d\sigma(\zeta)$$

tends to 0 uniformly on $\zeta \in S$ as $r \longrightarrow 1$. Therefore $f \in C(\bar{B})$.

Conversely, suppose $u(z) \in X_\lambda$ and $g(z) = g_\alpha^{-1}(z)u(z)$ is continuous up to S so that $g(z) \in C(\bar{B})$. Let $v(z) = P^\alpha[G](z)$, where $G(\zeta) = \lim_{r\to 1} g(r\zeta)$, $\zeta \in S$. We will show that $u(z) = v(z)$, $z \in B$. Define

$$f(z) = \begin{cases} g_\alpha^{-1}(z)v(z), & z \in B \\ G(z), & z \in S. \end{cases}$$

Then $f(z)$ and $g(z)$ have the same boundary function $G(z)$, and by what we have just proven (the first part of this theorem), $f(z) - g(z) \in C(\bar{B})$. Therefore we can conclude $u \equiv v$ by Corollary 5.

THEOREM 7. *Let $1 \leq p \leq \infty$ and let $F \in L^p(\sigma)$. If $\alpha > \frac{1}{2}$ then $P^\alpha[F] \in X_\lambda \cap h^{p,n\beta}$ and if $\alpha = \frac{1}{2}$ then $P^{1/2}[F] \in X_{-n^2} \cap h^{p,-n/2-}$.*

Conversely, suppose either $f \in X_\lambda \cap h^{p,n\beta}$, $\alpha > \frac{1}{2}$, or $f \in X_{-n^2} \cap h^{p,-n/2-}$. If $1 < p \leq \infty$ then there is an $F \in L^p(\sigma)$ such that $f = P^\alpha[F]$. If $p = 1$ then there is a measure μ such that $f = P^\alpha[\mu]$.

$p = 2$ case of Theorem 7 appeared at [KK] by an approach using orthogonality in $L^2(\sigma)$. In proving Theorem 7 all we need now are, as in the proof of X_0 case [R, 4.2.4], an equicontinuity argument of D. Ullrich [R, 4.2.4], MMP (Corollary 5), and duality. We include here the equicontinuity as a lemma, and give a proof of Theorem 7 for the completeness.

Let $\mathcal{U}$ denote the unitary group on S, and let dU denote the Haar measure on $\mathcal{U}$. $\mathcal{U}$ is compact subgroup of $O(2n)$ (See [R 1.4.6]). For $G(z)$ defined on B and for $0 < r < 1$, let us denote the dilation by $G_r(z) = G(rz)$, $z \in B$.

LEMMA 8 [R, PP. 56–57]. *Let $1 \le p \le \infty$. Let $\nu\colon \mathcal{U} \longrightarrow [0,\infty)$ be continuous such that $\int_{\mathcal{U}} \nu(U)\,dU = 1$. If $G(z), z \in B$, is defined by*

$$G(z) = \int_{\mathcal{U}} u(Uz)\nu(U)\,dU$$

for some $u \in h^{p,0}$ then we have

(4.4) $\{G_r : 0 < r < 1\}$ is equicontinuous subset of $C(S)$,

(4.5) $G(z)$ is uniformly bounded by $\|u\|_{p,0}\big(\int_{\mathcal{U}} \nu^q(U)\,dU\big)^{1/q}$, where q is the conjugate exponent of p, and

(4.6) $M_p(r,G) \le \|u\|_{p,0}$.

PROOF OF THEOREM 7. Note first that if $\alpha > \frac{1}{2}$ then $(1-r)^{n\beta} g_\alpha(r) = O(1)$ and that $-\log(1-r) g_{\frac{1}{2}}(r) = O(1)$ as $r \uparrow 1$.

If $F \in L^p(\sigma)$ and $f = P^\alpha[F]$ then it follows from Hölder's inequality that

$$\|f\|_{p,n\beta} \le C\Big(\int_S |g_\alpha^{-1} f|^p\,d\sigma\Big)^{1/p} \le \|F\|_p < \infty,$$

so that $f \in h^{p,n\beta}$. On the other hand, $\tilde{\Delta} f(z) = \lambda f(z)$. This proves the first half of Theorem 7.

For the converse, let $1 < p \le \infty$ and suppose either $f \in X_\lambda \cap h^{p,n\beta}$, $\alpha > \frac{1}{2}$, or $f \in X_{-n^2} \cap h^{p,\frac{n}{2}-}$. We assume $\|g_\alpha^{-1} f\|_{p,0} = 1$ without loss of generality. Let $\nu_j\colon \mathcal{U} \longrightarrow [0,\infty)$, $j = 1,2,\dots$ be continuous such that $\int_{\mathcal{U}} \nu_j(U)\,dU = 1$ and the support of ν_j shrink to the identity of $\mathcal{U}$ as $j \longrightarrow \infty$. Apply Lemma 8 with $g_\alpha^{-1}(z) f(z)$ and $\nu_j(z)$ in places of $u(z)$ and $\nu(z)$. Let G_j be the corresponding G. We fix j for a moment. Then by (4.4) and (4.5) there is a sequence $r_i = r(j,i)$ tending to 1 (as $i \longrightarrow \infty$) such that $(G_j)_{r_i}$ converges to a function $g_j \in C(S)$ uniformly.

Let

$$\epsilon_{j,i} = \sup_{\zeta\in S}\left| G_j(r_i\zeta) - \frac{P^\alpha[g_j](r_i\zeta)}{g_\alpha(r_i\zeta)}\right|. \tag{4.7}$$

By Theorem 6, $g_\alpha^{-1}(z) P^\alpha[g_j](r_i\zeta)$ tends to $g_j(\zeta)$, uniformly as $i \longrightarrow \infty$. Thus $\epsilon_{j,i} \longrightarrow 0$ as $i \longrightarrow \infty$. By Corollary 5 and (4.7),

$$\left| G_j(z) - \frac{P^\alpha[g_j](z)}{g_\alpha(z)}\right| \le \epsilon_{j,i}, \quad |z| \le r_i,$$

for every i. Hence $G_j(z) = g_\alpha^{-1}(z) P^\alpha[g_j](z)$, $z \in B$. Now, letting $j \longrightarrow \infty$, $G_j(z) \longrightarrow g_\alpha^{-1}(z) f(z)$ pointwise. On the other hand, since $\|g_j\|_p \le 1$ by (4.6), there is a subsequence of $\{g_j\}$ that converges to some $F \in L^p(\sigma)$ in the weak*-topology of $L^P(\sigma)$. In particular,

$$P^\alpha[g_j](z) \longrightarrow P^\alpha[F](z), \quad z \in B.$$

Therefore we have $f(z) = P^{\alpha}[F](z)$.

When $p = 1$, the proof is same except using the dual of $C(S)$.

COROLLARY 9. *Let $\alpha \geq \frac{1}{2}$. If f is positive and $f \in X_\lambda$, then there is a positive measure μ on S such that $f = P^{\alpha}[\mu]$.*

PROOF. If f is positive and $f \in X_\lambda$, then by (1.2), $g_\alpha^{-1}(r)\int_S f(r\zeta)\,d\sigma(\zeta) = f(0)$. Thus, by Theorem 7, $f = P^{\alpha}[\mu]$ for some μ. This μ is positive being weak*-limit of the positive function $h_r = g_\alpha^{-1}(r)f_r$. In fact, for $g \in C(S)$,

$$\begin{aligned}\int_S h_r g\,d\sigma &= \frac{1}{g_\alpha(r)}\int_S g(\eta)\,d\sigma(\eta)\int_S P^{\alpha}(r\eta,\zeta)\,d\mu(\zeta)\\ &= \frac{1}{g_\alpha(r)}\int_S P^{\alpha}[g](r\zeta)\,d\mu(\zeta),\end{aligned}$$

and this last integral tends to $\int_S g(\zeta)\,d\mu(\zeta)$ by Theorem 6.

REFERENCES

[K] E. G. Kwon, *One radius theorem for the eigenfunctions of the invariant Laplacian,* Proc. AMS. **116**(1992), 27–34.

[KK] H. O. Kim and E. G. Kwon, *M-invariant subspaces of X_λ,* Illinois J. Math, to appear.

[R] Walter Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

[S] Lucy John Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966.

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