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A NOTE ON OHM'S RATIONALITY CRITERION FOR CONICS

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An irreducible quadratic polynomial $P(X, Y)$ in two variables over a field k is called a conic over k. It is called rational if its function field is simple transcendental over k (equivalently if P is parameterisable by rational functions). Ohm's rationality criterion states that P is rational if and only if (i) the locus of P is non-empty and (ii) k is algebraically closed in the function field of P. To show the irredundancy of (ii), Ohm gives an example of a non-rational conic with a non-empty locus. That locus, however, consists of a single point.

In this note, we show that a better example cannot exist by showing that if the locus of a conic contains more than one point then it is rational. We also show that the only rational conic whose locus consists of one point is the conic $XY + 1$ over the field of two elements.

Let k be any field. An irreducible polynomial $P = P(X, Y) \in k[X, Y]$ of total degree 2 is called a conic over k (or simply a conic). A point $(\alpha, \beta) \in k^2$ with $P(\alpha, \beta) = 0$ is called a k-zero of P and the set of all k-zeros of P is referred to as the locus P. The quotient field of $k[X, Y]/(P)$ is called the function field of P. If one denotes the images of X, Y under the canonical map

$$k[X, Y] \to k[X, Y]/(P)$$

by x, y, then the function field of P is nothing but the field extension $k(x, y)$ of k. It is easy to see that the generators x, y are characterised by the properties that (i) the transcendence degree $dt_k k(x, y)$ of $k(x, y)$ over k is 1 and (ii) the ideal of polynomials in $k[X, Y]$ that vanish on (x, y) is the principal ideal generated by P. Also, it is easy to see that a non-singular affine change of variables

$$X \to aX + bY + c, \quad Y \to \alpha X + \beta Y + \gamma$$

transforms P into a conic Q having a k-isomorphic function field. Such conics P, Q are called equivalent.

The function field $k(x, y)$ of P is said to be rational if it is simple transcendental over k, that is if it is generated by a single element t. In this case, P itself is called

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a rational conic. It is easy to see that the rationality of a conic is equivalent to its parameterisability by rational functions (in the variable t).

In [1, Theorem 1.4, (ii) $\Longleftrightarrow$ (iii)] Ohm gives necessary and sufficient conditions for a conic to be rational. His criterion asserts that the function field $k(x, y)$ of the conic P is rational if and only if the following two conditions are satisfied:

(A) P has a k-zero (that is, P has a non-empty locus)
(B) k is algebraically closed in $k(x, y)$.

At the end of the proof, he remarks that (B) is not redundant by considering the function field of the conic $P(X, Y) = X^2 + Y^2$ over the field of real numbers.

The example above does indeed satisfy (A) (since $P(0, 0) = 0$) and does not satisfy (B) (since y/x is an element in $k(x, y) - k$ that is algebraic over k). However, it satisfies (A) only very weakly: true, the locus of P is not empty, but it is almost empty since it consists of the single point $(0, 0)$. This observation leads naturally to the search for a non-rational conic whose locus contains more than one point.

In this note, we prove that if the locus of a conic P has more than one point, then P is rational. Further, we prove that the only rational conic whose locus is a singleton is the conic

$$P(X, Y) = XY + 1, \quad k = \mathbb{Z}_2.$$

THEOREM 1. *Let $k(x, y)$ be the function field of the conic P and suppose that the locus of P is nonempty. If P is not rational then the locus of P consists of a single point. In this case, P is equivalent to*

$$aX^2 + bXY + cY^2.$$

Conversely, if P is irreducible of the form above, then the locus of P consists of a single point and P is not rational.

PROOF: Since the locus of P is not empty, then by a suitable change of variables, one may assume that $(0, 0)$ lies on the locus of P. Thus $P(0, 0) = 0$ and hence

$$P(X, Y) = aX^2 + bXY + cY^2 + dX + eY.$$

Suppose that P is not rational. If $x = 0$, then $cy^2 + ey = 0$ and hence $y \in k$. Hence $k(x, y) = k$, contradicting the assumption that $dt_k k(x, y) = 1$. Hence $x \neq 0$. Let $t = y/x$. Then

$$(a + bt + ct^2)x + (d + dt) = 0.$$

If $(a + bt + ct^2) \neq 0$, then x would belong to $k(t)$. Hence

$$k(x, y) = k(x, xt) = k(t),$$

contradicting the assumption that $k(x, y)$ is not rational. Thus $a + bt + ct^2 = 0$ and therefore $d + et = 0$. But $t \notin k$ (since $k(x, y) \neq k(x)$). So then $e = d = 0$. Therefore

$$P(X, Y) = aX^2 + bXY + cY^2.$$

Let (r, s) be another k-zero of P. Since P is irreducible, then neither a nor c is 0. Therefore neither r nor s is 0. Hence

$$sX - rY$$

is a factor of P, contradicting the irreducibility of P. This shows that $(0, 0)$ is the only k-zero of P (and that P is equivalent to the given form).

Conversely, if P is irreducible and of the given form, then the same argument above shows that the locus of P contains no point other than $(0, 0)$ and that y/s is an element in $k(x, y) - k$ that is algebraic over k. Thus k is not algebraically closed in $k(x, y)$ and hence P is not rational. □

Corollary 2. *If the locus of the conic P has more than one point, then P is rational.*

Can the locus of a rational conic consist of a single point? The following example shows the existence of such a conic, while the theorem that follows shows its uniqueness.

Example 3. *Let $k = \mathbb{Z}_2$ and let $P(X, Y) = XY + 1$. To see that the locus of P has exactly one point, we try all possibilities*

$$(0, 0), (0, 1), (1, 0), (1, 1)$$

and easily see that $(1, 1)$ is the only zero of P. To see that it is rational, we note that $y = 1/x$ and hence $k(x, y) = k(x)$.

Theorem 4. *If the locus of the k-conic P consists of a single point, and if P is rational then $k = \mathbb{Z}_2$ and P is equivalent to $XY + 1$.*

Proof: Let $K = k(x, y)$ be the function field of P (with $P(x, y) = 0$). Suppose that P is rational and that the locus of P consists of a single point. Then by a suitable change of variables, one may assume that point to be $(0, 0)$. Then

$$(*) \qquad P(X, Y) = aX^2 + bXY + cY^2 + dX + eY.$$

If $x = 0$, then $cy^2 + ey = 0$ and hence $y \in k$. This contradicts the fact that $dt_k k(x, y) = 1$. Therefore $x \neq 0$. Let $t = y/x$. Then

$$x[a + bt + ct^2] + [d + et] = 0.$$

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If $a + bt + ct^2 = 0$, then

$$ax^2 + bxy + cy^2 = 0,$$

and hence

$$P(X, Y) = aX^2 + bXY + cY^2.$$

In view of Theorem 1, this contradicts the rationality of P. Therefore

$$a + bt + ct^2 \neq 0$$

and hence $d + et \neq 0$. Hence $(a, b, c) \neq (0, 0, 0)$ and

$$(d, e) \neq (0, 0). \tag{1}$$

Also,

$$x = -[d + et]/[a + bt + ct^2], \quad y = -t[d + et]/[a + bt + ct^2].$$

If k has more than 3 elements, then one can find α in k such that

$$(d + e\alpha)(a + b\alpha + c\alpha^2) \neq 0.$$

Then

$$\left(-(d + e\alpha)/(a + b\alpha + c\alpha^2), -\alpha(d + e\alpha)/(a + b\alpha + c\alpha^2)\right)$$

would be another point on the locus of P, contrary to the hypothesis. Thus k cannot have more than 3 elements and consequently k must be either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

CASE 1. $k = \mathbb{Z}_2$. Plugging $Y = 0$ in $(*)$, we see that if $da \neq 0$ then $(-d/a, 0)$ is another k-zero of P, contrary to the hypothesis. Therefore $da = 0$. Similarly $ec = 0$. Also plugging $X = Y$ in $(*)$, we see that if $(a + b + c)(d + e) \neq 0$, then $((d + e)/(a + b + c), (d + e)/(a + b + c))$ is another k-zero of P. Hence we conclude that

$$da = ec = (d + e)(a + b + c) = 0.$$

If $(d, e) = (1, 1)$, then $a = c = 0$ and

$$P(X, Y) = XY + X + Y = (X + 1)(Y + 1) + 1.$$

Making the change of variables

$$X \to X + 1, \quad Y \to Y + 1,$$

we see that P is equivalent to $XY + 1$.

If $(d, e) = (1, 0)$, then $a = b + c = 0$. Hence $b = c$ and

$$P(X, Y) = XY + Y^2 + X = (X + Y + 1)(Y + 1) + 1.$$

Making the change of variables

$$X \to X + Y + 1, \quad Y \to Y + 1,$$

we see that P is equivalent to $XY + 1$.

The case $(d, e) = (0, 1)$ is similar while the case $(d, e) = (0, 0)$ is not feasible by (1).

Hence in all cases, P is equivalent to $XY + 1$ as desired.

CASE 2. $k = \mathbb{Z}_3$. Plugging $Y = 0$ in $(*)$, we see that if $da \neq 0$ then $(-d/a, 0)$ is another k-zero of P, contrary to the hypothesis. Therefore $da = 0$. Similarly by plugging $X = 0$, $X = Y$, $X = -Y$, we conclude that

$$da = ec = (d + e)(a + b + c) = (d - e)(a - b + c) = 0.$$

Case by case computations reveal that every case leads to a contradiction.

Thus the only rational conic whose locus consists of a single point is the conic $XY + 1$ over $\mathbb{Z}_2$. □

In conclusion, we summarise our results for a conic P over a field k as follows:

(i) If the locus of P is empty, then P is not rational.

(ii) If the locus of P contains more than one point, then P is rational.

(iii) If the locus of P consists of one point, then (a) P is not rational if and only if P is equivalent to an irreducible polynomial of the form $aX^2 + bXY + cY^2$ and (b) P is rational if and only if $k = \mathbb{Z}_2$ and P is equivalent to $XY + 1$.

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