

The Needlet CMB Trispectrum

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Abstract. We propose a computationally feasible estimator for the needlet trispectrum, which develops earlier work on the bispectrum by Donzelli *et al.* (2012). Our proposal seems to enjoy a number of useful properties, in particular a) the construction exploits the localization properties of the needlet system, and hence it automatically handles masked regions; b) the procedure incorporates a quadratic correction term to correct for the presence of instrumental noise and sky-cuts; c) it is possible to provide analytic results on its statistical properties, which can serve as a guidance for simulations. The needlet trispectrum we present here provides the natural building blocks for the efficient estimation of nonlinearity parameters on CMB data, and in particular for the third order constants g_{NL} and τ_{NL} .

Keywords. cosmic microwave background, early universe

1. Introduction

The Inflationary models describe the dynamics of the first instants of the Universe. Each model introduces a characteristic signature of non-Gaussianity into the anisotropy distribution of the Cosmic Microwave Background (CMB) radiation. The level of non-Gaussianity is described by the so-called *Bardeen's Potential* (Bardeen (1980)):

$$\Phi(\mathbf{x}) = \Phi_L(\mathbf{x}) + f_{NL}[\Phi_L^2(\mathbf{x}) - \langle \Phi_L^2(\mathbf{x}) \rangle] + g_{NL}[\Phi_L^3(\mathbf{x})] \quad (1.1)$$

where $\Phi(\mathbf{x})$ is the gravitational potential field of the Universe and $\Phi_L(\mathbf{x})$ is its Gaussian part. The constants f_{NL} and g_{NL} , which parametrize the non-Gaussian part, can be measured from the amplitude of the bispectrum and trispectrum of the field, i.e. the harmonic counterpart of the 3-point and 4-point correlation function of the field, respectively.

There exist several optimal bispectrum estimators that allow to evaluate f_{NL} with high confidence (see for instance Komatsu *et al.* (2005) or PLANCK Coll. (2013)). Optimal trispectrum estimators has been already applied to CMB data at WMAP resolution (e.g. Sekiguchi & Sugiyama (2013) in real space and Regan *et al.* (2013) in needlet space), but the lack of an optimal trispectrum estimator applied to high-resolution sky simulations (e.g. PLANCK resolution) has prevented the evaluation of strong constraints on g_{NL} so far.

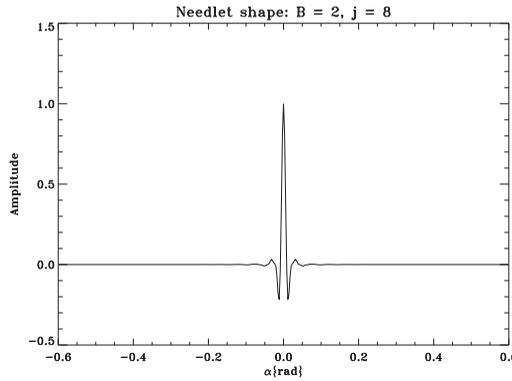


Figure 1. 1D-Spherical needlelet. The figure shows the characteristic needle shape of the function, from which take its name (from Marinucci *et al.* (2008))

2. Spherical Needlelet System

2.1. Definition

Spherical needlelets (Narcowich *et al.* (2006), Baldi *et al.* (2009)) are a wavelet system on the sphere defined by setting:

$$\psi_{jk}(x) := \sqrt{\lambda_{jk}} \sum_l b\left(\frac{l}{B^j}\right) \sum_{m=-l}^l Y_{lm}(\xi_{jk}) \overline{Y_{lm}(x)}; \tag{2.1}$$

where $x \in S^2$, $\{\lambda_{jk}, \xi_{jk}\}$ are a set of cubature points and weights on the sphere, $B > 1$ is a constant related to the width of the needlelet and $b(\cdot)$ is a weight function satisfying the three following conditions: a) Compact Support, $b(\xi) > 0$ if $\xi \in (B^{-1}, B)$, 0 otherwise; b) Partition of Unity, for all $\zeta \geq 1$, $\sum_{j=0}^\infty b^2\left(\frac{\zeta}{B^j}\right) = 1$; c) Smoothness, $b(\cdot) \in C^M$, i.e., $b(\cdot)$ is M times continuously differentiable, for some $M = 1, 2, \dots$ or $M = \infty$.

2.2. Reconstruction Formula

The spherical needlelet coefficients are provided by the analytical formula:

$$\beta_{jk} = \int_{S^2} T(x) \psi_{jk}(x) d\sigma(x) = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{l}{B^j}\right) \sum_{m=-l}^l a_{lm} Y_{lm}(\xi_{jk}). \tag{2.2}$$

As a consequence of the partition of unity property, the following well-known reconstruction formula holds:

$$f(x) = \sum_{jk} \beta_{jk} \psi_{jk}(x) \equiv \sum_{lm} a_{lm} Y_{lm}(x). \tag{2.3}$$

2.3. Localization and Uncorrelation Properties

As argued earlier in Baldi *et al.* (2009), spherical needlelets enjoy an excellent localization property in real domain (fig. 1):

$$|\psi_{jk}(x)| \leq \frac{c_M B^j}{(1 + B^j \arccos\langle \xi_{jk}, x \rangle)^M}. \tag{2.4}$$

More explicitly, spherical needlelets are then quasi-exponentially localized around any cubature point ξ_{jk} . Moreover, as extensively argued in the literature, the needlelet coefficients

evaluated on isotropic random fields are asymptotically uncorrelated, under mild regularity conditions. It is then possible to derive analytically their statistical properties, and to understand the role and expressions for correction terms under realistic experimental conditions (noise and masks, see Donzelli *et al.* for the bispectrum case)

3. Optimal Trispectrum Estimator

The sample trispectrum estimator is usually written using the spherical harmonics coefficients as:

$$\sum_{m_1 m_2 m_3 m_4} \mathcal{G}(l_1, m_1, l_2, m_2, l_3, m_3, l_4, m_4) \times a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4}, \quad (3.1)$$

where $\mathcal{G}(l_1, m_1, l_2, m_2, l_3, m_3, l_4, m_4)$ is the Gaunt integral defined for instance in Marinucci & Peccati (2011). This estimator is clearly unfeasible under realistic experimental conditions, due to the presence of missing data and anisotropic noise. As a consequence of the previous discussion, it is possible to exploit the needlet coefficients to derive an alternative, computationally feasible and statistically sound estimators of the trispectrum. More precisely, exploiting Wick theorem on higher order moments of Gaussian variables we propose the following needlet trispectrum:

$$J_{j_3 j_4}^{j_1 j_2} = \frac{1}{\sigma_{j_1} \sigma_{j_2} \sigma_{j_3} \sigma_{j_4}} \sum_k [\beta_{j_1 k} \beta_{j_2 k} \beta_{j_3 k} \beta_{j_4 k} - \{\langle \beta_{j_1 k} \beta_{j_2 k} \rangle \beta_{j_3 k} \beta_{j_4 k} + 5 \text{ perms.} \} + \{\langle \beta_{j_1 k} \beta_{j_2 k} \rangle \langle \beta_{j_3 k} \beta_{j_4 k} \rangle + 2 \text{ perms.} \}] \quad (3.2)$$

Heuristically, the needlet trispectrum is constructed combining quadruples of coefficients, evaluated at the scales of interest, and subtracting linear and quadratic components in order to cancel bias and minimize the variance. Further details, and a software which exploits the needlet trispectrum to evaluate g_{NL} on CMB maps, will be provided in the forthcoming paper Troja *et al.* (2014) .

References

- Baldi, P., Kerkycharian, G. Marinucci, D., & Picard, D. 2009, *Annals of Statistics*, 37, 1150
 Bardeen, J. M. 1980, *Phys. Rev. D*, 22, 1882
 Donzelli, S., Hansen, F. K., Liguori, M., Marinucci, D., & Matarrese, S. *ApJ*
 Komatsu, E., Spergel, D. N., & Wandelt, B. D. 2005, *ApJ*, 634, 14
 Marinucci, D. & Peccati, G. 2011, *Random Fields on the Sphere*, Cambridge University Press
 Marinucci, D., Pietrobon, D., Balbi, A., Baldi, P., Cabella, P., Kerkycharian, G., Natoli, P., Picard, D., & Vittorio, N. 2008, *MNRAS*, 383, 539
 Narcowich, F. J., Petrushev, P., & Ward, J. D. 2006, *SIAM J. Math. Anal.*, 38, 574
 PLANCK Collaboration 2013, *ArXiv e-prints* arxiv:1303.5084
 Regan, D., Gosenca M., Seery D. 2013, *ArXiv e-prints* arxiv:1310.8617
 Sekiguchi, T., Sugiyama, N. 2013, *JCAP*, 09, 2
 Troja, A. *et al.* 2014, Manuscript in preparation
 Wick, G. C. 1950, *Physical Review*, 80, 268