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L. G. KOVÁCS' WORK ON LIE POWERS

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In memoriam Laci Kovács

Abstract

From the mid-1990s onwards, the main focus of L. G. Kovács' research was on Lie powers. This brief survey presents some of the key results on Lie powers obtained by Kovács and his collaborators, and discusses some subsequent developments and applications of this work.

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L. G. Kovács (or Laci as he was known to those around him) made numerous significant contributions to the study of Lie powers, and from the mid-1990s onwards this was the main focus of his research. Much of this work [6–10, 18–22] was carried out in collaboration with Manchester-based researchers Roger Bryant and Ralph Stöhr (who together are responsible for much of my own mathematical education). Such collaborations necessarily involve a number of research visits, and I had the pleasure of meeting Laci on just one such occasion when he visited Karin Erdmann in Oxford to complete some joint work [12]. In this brief survey I shall present some of the key results on Lie powers obtained by Laci and his collaborators. I thank Ralph Stöhr for his many insightful remarks and comments whilst preparing this article.

1. A brief introduction to Lie powers

Let G be a group, K a field and V a KG-module. Let $L(V)$ denote the free Lie algebra on V over K (that is, the Lie algebra which is freely generated by any K-basis of V). This is a graded algebra $L(V) = \bigoplus_{n\geq 1} L^n(V)$, where $L^n(V)$ is the K-subspace containing all homogeneous elements of degree n. The universal enveloping algebra of $L(V)$ is the tensor algebra $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, and it is clear that each homogeneous component $L^n(V) = V^{\otimes n} \cap L(V)$ can be considered as a subspace of the corresponding tensor power $V^{\otimes n}$. The action of G on V extends uniquely to $L(V)$ and $T(V)$, with the elements of G acting as algebra automorphisms, hence turning these algebras into KG-modules. Moreover, since the action respects the degree of each homogeneous element, it is easy to see that each of the subspaces $L^n(V)$ is actually a KG-submodule of $V^{\otimes n}$. These are the so-called Lie powers of V.

If V is finite-dimensional, then so is each Lie power $L^n(V)$ and in this case the dimension of the nth Lie power is given by Witt's dimension formula,

$$\dim(L^n(V)) = \frac{1}{n}\sum_{d|n}\mu(d)(\dim(V))^{n/d},$$

where μ denotes the Möbius function [34]. Observe that these dimensions increase very rapidly as n is increases. Now fix a group G and a field K and choose your favourite finite-dimensional KG-module V. The above considerations show that these initial conditions can be used to generate an interesting infinite family of finite-dimensional KG-modules, the Lie powers. Now suppose that you have some nice detailed information about the structure of V as a KG-module, such as a decomposition into indecomposable modules (V was your favourite, after all). Can you give a similar description of the Lie powers (up to isomorphism)? For the classical case of Lie powers in characteristic zero, this 'decomposition problem' is well understood via the early works of Thrall [32], Brandt [1] and Wever [33], with further qualitative information and combinatorial insights later given by Klyachko [17] and Kraśkiewicz and Weyman [23]. To set the scene, let us begin with a brief overview of these works.

In his 1942 paper, Thrall considered the case where G is the general linear group $\mathrm{GL}_n(K)$, K a field of characteristic zero, and V is the natural KG-module. In this case it is well known [28] that the simple submodules $L(\lambda)$ of $V^{\otimes n}$ are indexed by the partitions λ of n into at most $\dim V$ parts. Moreover, each tensor power $V^{\otimes n}$ is semisimple and the simple module $L(\lambda)$ occurs as a direct summand of $V^{\otimes n}$ with multiplicity t_λ equal to the number of standard tableaux of shape λ. Thrall asked for the decomposition of the nth Lie power $L^n(V)$. Since $L^n(V)$ can be regarded as a submodule of $V^{\otimes n}$, it follows that the nth Lie power will be isomorphic to a KG-module direct sum of those same simple modules $L(\lambda)$ occurring with multiplicities l_λ, where $0 \leq l_\lambda \leq t_\lambda$. Thrall was able to calculate the multiplicities l_λ for all partitions λ of $n \leq 10$, with a correction in the case $n = 10$ later given by Brandt [1], who also gave a formula for the character of the nth Lie representation. In 1949 Wever [33] gave a formula for the multiplicities l_λ in terms of characters of the symmetric group of degree n. Although these character theoretic formulae are useful and indeed beautiful, it is not easy to see from this information alone which of the multiplicities l_λ are positive, or in other words, which simple modules actually do occur in the Lie powers. Moreover, those who enjoy counting things may have hoped for a combinatorial formula for l_λ, similar to the interpretation of the multiplicities t_λ in the tensor power. In 1974 Klyachko [17] was able to answer the question of which simple modules occur in the Lie power. He proved that for $n > 6$ almost every simple module $L(\lambda)$ occurs in $L^n(V)$, the exceptions being the simple modules corresponding to the partitions $\lambda = (n)$ and $\lambda = (1^n)$. In 1987 Kraśkiewicz and Weyman [23] provided a combinatorial formula for the multiplicities, proving that the multiplicity l_λ is equal to the number of standard tableaux of shape λ with major index congruent to a modulo n, for any fixed positive integer a which is coprime to n. Using this result, one obtains the following rather intriguing combinatorial version of Klyachko's theorem: for each partition $\lambda \neq (n), (1^n)$ of $n > 6$ there is a standard tableau of shape λ whose major index is coprime to n. A completely combinatorial proof of this result was given in [15], which when combined with the result of Kraśkiewicz and Weyman provides an alternative proof of Klyachko's theorem.

We conclude our discussion of the characteristic-zero case by mentioning some of Laci's joint work with Stöhr [21], where another combinatorial proof of Klyachko's above-mentioned result was given. The key observation is [21, Lemma 1], which says that whenever $n = k + l$ with $k > l > (k/2)$, the subspace $[L^k(V), L^l(V)]$ of $L^n(V)$ is a *submodule* isomorphic to the tensor product $L^k(V) \otimes L^l(V)$. Together with the Littlewood–Richardson rule and the preliminary calculations of Thrall, this neat result was used to give a short inductive proof of Klyacho's theorem. Their method gives an improved lower bound $l_\lambda \geq (n/6) - 1$ for all multiplicities $l_\lambda > 1$.

In fact, Laci's interest in Lie powers focused at first on positive characteristic. His collaboration with Bryant and Stöhr stemmed from the following problem, which Laci had entered into the 1990 edition of the Kourovka Notebook [26, Problem 11.47].

> Let L^n, be the homogeneous component of degree n in a free Lie algebra L of rank two over the field of order 2. What is the dimension of the fixed point space in L^n, for the automorphism of L which interchanges two elements of a free generating set of L?

This problem was subsequently solved by Bryant and Stöhr [3], and marked the beginning of a long-term collaboration on Lie powers between these three. A major topic of this collaboration was the solution of the decomposition problem for modular Lie representations of groups with cyclic Sylow p-subgroup in defining characteristic, with the first two cases of interest being symmetric group of degree p [6] and $\mathrm{GL}(2, p)$ [9]. In situations where the Sylow p-subgroups of G are cyclic, there are only finitely many isomorphism classes of indecomposable KG-modules. Thus, once armed with an understanding of the finitely many indecomposable modules which could occur, the decomposition problem for Lie powers of a given module V asks for the Krull–Schmidt multiplicities of those indecomposables in $L^n(V)$ for all n.

Work on $\mathrm{GL}(2, p)$ began in Laci's joint work with Stöhr [19], in which the Lie powers $L^n(E)$ of the natural module E for $\mathrm{GL}(2, p)$ over the field of p elements were considered, with results obtained for $p = 2$ and $p = 3$. This work was brought to a conclusion in joint work of the three authors [9]. The main ingredient of the conclusive results on $\mathrm{GL}(2, p)$ is the three-authored paper [8], which is of interest in its own right. The paper [8] concerns Lie powers for cyclic groups of prime order and gives a complete solution of the decomposition problem in this case. The result is highly complex, requiring several clever ideas, many of which laid the groundwork for later progress on modular Lie powers. This article gives a roughly chronological account of the research leading up to the influential paper [8] and some of the developments that followed.

2. Early experiments in prime characteristic

The smallest case one can think to consider is that of Lie powers of the regular module for the cyclic group of order two in characteristic two. In this case there are just two indecomposable modules: the one-dimensional trivial module and the regular module. Thus the decomposition problem for Lie powers asks for the Krull–Schmidt multiplicities of these two indecomposables in the nth Lie power. This was the topic of Bryant and Stöhr's paper [3] (which set out to answer the above-mentioned question of Laci). This work was subsequently extended by those two authors in [4], where the decomposition problem for Lie powers of a free module for the cyclic group of order p in characteristic p was solved. Of course, the cyclic group of order two can also be considered as the symmetric group of degree two, and it is therefore natural to enquire more generally about the structure of the Lie powers for the natural module for the symmetric group S_r.

2.1. Modular Lie powers for the symmetric group. In characteristic zero, the ordinary characters of Lie representations of the symmetric group can be obtained from the character formula of Brandt [1], and from this the multiplicities of the simple summands of the nth Lie power can be obtained using orthogonality relations. In prime characteristic, there is a completely analogous formula for the *Brauer character* of the Lie representation of the symmetric group, and again one can use orthogonality relations to identify the relevant composition factors and their multiplicities. When $r < p$ the Lie powers are semisimple, and so in this case the analysis above solves the decomposition problem. For $r \geq p$, things are not so simple (if you will pardon the pun).

In [6], Bryant *et al.* considered the cases where $p \leq r < 2p$. For $p = 2$, these cases had essentially already been dealt with by these three authors in [3] (for $r = p = 2$) and in [18] (for $p = 2$, $r = 3$). The paper [6] therefore concentrates on the case of odd characteristic. Their main results identify the nonprojective indecomposable summands of the nth Lie power as Specht or dual Specht modules corresponding to certain partitions. Their most detailed results apply in the case where $r = p$. Here they show that each nonprojective indecomposable summand of the Lie module for S_p in characteristic p is isomorphic to one of the Specht modules S^λ or its dual, where the modules S^λ occurring are indexed by the hook partitions $\lambda = (p-k, 1^k)$ with k odd and $1 \leq k \leq p-2$, and furthermore, they give a formula for the multiplicity of each of these modules inside the nth Lie power.

2.2. Modular Lie powers of the natural module for GL(2, *p*). Let K be a field of prime characteristic p, $G = \mathrm{GL}_r(K)$ and let E denote the natural KG-module. This situation is rather more delicate than its characteristic-zero counterpart, since (unlike the characteristic-zero case) the Lie power is unlikely to be semisimple, and

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moreover, may not turn out to be isomorphic to a module direct summand of the tensor power (which is not semisimple in general). In fact, it transpires that the nth Lie power *is* a direct summand of $V^{\otimes n}$ whenever $p \nmid n$. This fact was exploited by Donkin and Erdmann [11] to obtain a formula for the Krull–Schmidt multiplicities in the case where K is infinite. The first case of interest to Laci was the simplest case at the other extreme, namely when $r = 2$ and K is the field with p elements, so that G is the finite group $\mathrm{GL}(2, p)$. In this case the Sylow p-subgroups of G are cyclic and hence there are finitely many indecomposables up to isomorphism. For instance, when $p = 2$ this group has three isomorphism types of indecomposable modules. These are the one-dimensional trivial module, the two-dimensional natural module, and a nonsimple indecomposable of dimension 2. The decomposition problem therefore asks for the Krull–Schmidt multiplicities ($a(n)$, $b(n)$, $c(n)$, say) of these three indecomposables in $L^n(E)$. It is clear that $a(n) + 2b(n) + 2c(n)$ is equal to the dimension of $L^n(E)$, which is given by Witt's dimension formula. It is also straightforward to verify that $a(n) + b(n) + c(n)$ is the dimension of the fixed point space in $L^n(E)$ under the automorphism which interchanges the two standard basis vectors of E. (In other words, $a(n) + b(n) + c(n)$ is equal to the dimension of the fixed point space elucidated by Bryant and Stöhr [3].) To complete the analysis in this case, one can use the Brauer character of the Lie representation to obtain a third linear equation in the multiplicities $a(n), b(n)$ and $c(n)$, and hence obtain closed formulae for these Krull–Schmidt multiplicities [19, Theorem 3.1]. The Brauer character formula for the Lie representation of the general linear group had been known for some time (it is the exact analogue of the ordinary character formula of [1], and the methods of Wever [33] can be adapted to give an analogous proof of this), so the final ingredient of the theorem was the result about the dimension of the fixed point space; this is precisely what prompted Laci to pose his problem in the Kourovka Notebook.

The bulk of the paper [19] consists of a thorough examination of Lie powers of the natural $\mathrm{GL}(2, p)$-module in the case $p = 3$, resulting in the complete solution for the decomposition problem in this case [19, Theorem 6.1]. The central argument built upon previous work of Bryant and Stöhr [4], which dealt with Lie powers of free modules for the group of order p. However, the methods employed in [19] did not immediately carry over to larger primes; the theory needed to be extended and completely new ideas were required. This was accomplished in [8], which conquered the decomposition problem conclusively for Lie powers for groups of prime order.

3. Lie powers for groups of prime order

By the end of the 1990s it is fair to say that progress on the decomposition problem in prime characteristic had been limited to an understanding of Lie powers of degree not divisible by p, together with a study of the Lie powers of some rather special modules (as discussed above). In stark contrast, the paper [8] provides a *complete* solution decomposition problem for Lie powers for groups of prime order, where it is explained how to compute the Krull–Schmidt multiplicities for Lie powers of *any* finite-dimensional module for a group of prime order p over *any* field of

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characteristic p. The first key idea is to use *Lazard elimination* to show that if $V \cong V_1 \oplus \cdots \oplus V_k$ is a decomposition of V into indecomposable modules, then the Lie power $L^n(V)$ can be expressed in terms of Lie powers of the form $L^n(V_i)$ for $i = 1, \ldots, k$ together with modules of the form $L^j(W)$, where $j < n$ and W is an indecomposable summand of a tensor product formed from the V_i. If U and V are vector spaces, Lazard elimination gives a vector space decomposition of the free Lie algebra $L(U \oplus V)$ as the direct sum of $L(U)$ and $L(V \wr U)$, where $V \wr U$ is the subspace of $L(U \oplus V)$ spanned by all products of the form $[v, u_1, \ldots, u_m]$ with $m \geq 0$, $v \in V$ and $u_1, \ldots, u_m \in U$. The first important observation made in [8, Lemma 2.2] is that if U and V are KG-modules, then elimination yields *module* decompositions, not just vector space decompositions, where $V \wr U$ is a KG-module isomorphic to $V \otimes T(U)$. By induction, this essentially reduces the decomposition problem for $L^n(V)$ to first determining the Krull–Schmidt multiplicities of all Lie powers of indecomposable modules up to degree n, and then keeping track of these multiplicities when elimination is (repeatedly) applied. Notice that for this scheme to work, we must have a good understanding of the indecomposable KG-modules and be able to decompose tensor products of these. In the case considered in [8] where G is a group of prime order p and K is a field of characteristic p, there are (up to isomorphism) only p indecomposable modules $J_1, \ldots, J_p$, where $J_r \cong KG/(g-1)^r KG$, and detailed information about how to decompose tensor products of indecomposables is readily available. Thus the most technically challenging part of [8] is in decomposing the Lie powers of indecomposables.

A dimension shifting argument is used to find an expression for $L^n(J_r)$ for each $r \geq 2$ in terms of modules of the form $L^i(V)$, with $i < n$ and V finite-dimensional. Since the dimension of J_r is r, it is clear that $L^1(J_1) = J_1$ and $L^n(J_1) = 0$ for $n > 1$. For $r \geq 2$, the authors construct a particular graded submodule U of the enveloping algebra $T(J_r)$ of $L(J_r)$ such that $U = U_2 \oplus U_3 \oplus \cdots$ and U_k is a direct summand of $T^k(J_r)$. The main technical results of the paper concern the Lie subalgebra of $T(J_r)$ generated by U, termed the *shifted Lie algebra* $L(U)$. It is shown that this Lie subalgebra is free on U, and moreover, that

$$L(U) \cap T^k(J_r) = L^k(J_r) \quad \text{for } k \notin \{1, p\},$$
$$(L(U) \cap T^p(J_r)) \oplus J_{p-1} \cong L^p(J_r) \oplus J_p.$$

Since the elements of U_k are homogeneous of degree k, in order to calculate $L^n(J_r)$ up to isomorphism it is therefore sufficient to find

$$L(U_2 \oplus \cdots \oplus U_n) \cap T^n(J_r) = \bigoplus_{i=1}^{n-1} L^i(U_2 \oplus \cdots \oplus U_n) \cap T^n(J_r),$$

which can be calculated by using Lazard elimination.

The remainder of the paper explains how one can keep track of the indecomposable summands obtained at each degree from this delicate elimination procedure, by using graded modules and formal power series with coefficients in the Green ring. In the case

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where V is a free KG-module (that is, V is a direct sum of copies of J_p), it is shown that every indecomposable module occurring in the Lie power $L^n(V)$ is isomorphic to either J_p or J_{p-1} and explicit formulae are obtained for the multiplicities of these indecomposables (these results were already proved in [4] by a different method). For $p \geq 3$, a similar result is shown to hold for the Lie powers $L^n(J_{p-1})$, which also consist only of copies of J_p and J_{p-1}, and again explicit formulae for the corresponding multiplicities are obtained. For the Lie powers $L^n(J_r)$ with $2 \leq r \leq p-2$, some qualitative information is given about which indecomposables can occur and it is shown that for every sufficiently large n, every indecomposable occurs as a summand of the Lie power $L^n(V)$.

4. Integral Lie representations

Another topic of interest to Laci was the much more difficult case of integral Lie representations [7, 18, 19]. Consider the free Lie ring $\mathcal{L}(V)$, whose elements take coefficients from $\mathbb{Z}$, where V is a $\mathbb{Z}$-free $\mathbb{Z}G$-module. It is easy to see that in this case $\mathcal{L}(V)$ is itself a $\mathbb{Z}$-free $\mathbb{Z}G$-module which decomposes as the direct sum of integral Lie powers $\mathcal{L}^n(V)$. Of course, the 'decomposition problem' for integral representations is no longer well posed, since there is no Krull–Schmidt theorem in this case (two direct sum decompositions of a given $\mathbb{Z}G$-module can contain different collections of indecomposable summands up to isomorphism). Nevertheless, Laci and his co-authors were able to give detailed information about the integral Lie powers in a few special situations.

In joint work with Stöhr [19], Laci investigated the $\mathbb{Z}\mathrm{GL}(2,\mathbb{Z})$-module structure of $\mathcal{L}(V)$, where V is the natural $\mathrm{GL}(2,\mathbb{Z})$-module. The main result of [3] can be used to deduce the structure of the Lie powers $\mathcal{L}^n(V)$ considered as modules for any indecomposable subgroup of order 2 in $\mathrm{GL}(2,\mathbb{Z})$. In [19] this result is substantially extended, essentially giving full information about the Lie powers $\mathcal{L}^n(V)$ considered as modules for any finite subgroup of $\mathrm{GL}(2,\mathbb{Z})$. This is achieved by first considering the integral Lie powers as modules for a maximal finite subgroup G of $\mathrm{GL}(2,\mathbb{Z})$. Such maximal subgroups fall into two conjugacy classes and it therefore turns out to be enough to consider the integral Lie powers as modules with respect to two particular subgroups of $\mathrm{GL}(2,\mathbb{Z})$, namely $C \times H$ and D, where C is the centre of $\mathrm{GL}(2,\mathbb{Z})$, H is a dihedral subgroup of order six and D is a dihedral subgroup of order eight. The general strategy is then to use knowledge of the integral representation theory of the dihedral groups (in particular, the work of Lee [25]) together with (amongst other things) an understanding of the restriction to a subgroup of order two provided by [3] and an understanding of the integral Lie powers upon reduction modulo 3; this latter having been described in detail in an earlier section of their paper where Lie powers of the natural $\mathrm{GL}(2,3)$-module are considered. In both cases ($G = C \times H$, $G = D$) it turns out that there are finitely many indecomposable G-modules which can arise as summands of the integral Lie power, and that the multiplicity with which a particular indecomposable module occurs does not depend upon the decomposition (a rather surprising result for these integral representations). Recursive formulae for the multiplicities are given, hence completely determining the structure of these Lie powers. A very similar approach was also adopted by these two authors in [18] to completely describe the integral Lie powers of the natural $\mathbb{Z}S_3$-module.

Finally, in joint work with Bryant and Stöhr [7], Laci studied the free Lie ring $\mathcal{L}(V)$ where V is a $\mathbb{Z}$-free module of rank $2r$ under the action of the subgroup of S_{2r} generated by a fixed-point-free permutation τ of order two. They were able to provide a highly symmetric $\mathbb{Z}$-basis $B = B_0 \cup B_{-1}$ for $\mathcal{L}^n(V)$, with the property that for each $b \in B_0$ there is a unique b' such that τ interchanges b and b', whilst τ acts by multiplication by -1 on each $b \in B_{-1}$. It is then easy to see that each $b \in B_{-1}$ generates a one-dimensional indecomposable $\mathbb{Z}\langle\tau\rangle$-module, whilst each of the pairs $b, b' \in B_0$ generates a two-dimensional indecomposable $\mathbb{Z}\langle\tau\rangle$-module and it follows that their basis construction yields a decomposition of $\mathcal{L}(V)$ into indecomposable $\mathbb{Z}\langle\tau\rangle$-modules.

5. Further developments and applications

In recent years modular Lie representations have developed into an important research topic, and there is no doubt that Laci played a major role in this, both as an eminent author and as a tireless promoter of the subject. Since the publication of [8], substantial progress has been made on modular Lie powers by a number of authors including Bryant, Stöhr, Erdmann and the late Manfred Schocker. A highlight in this development was the Bryant–Schocker decomposition theorem. As we have seen, Lie powers in degrees not divisible by the characteristic are direct summands of the corresponding tensor power, and in this case it is possible to exploit knowledge of the tensor power to obtain information about the Lie power. The Bryant–Schocker decomposition theorem concerns Lie powers in degrees divisible by the characteristic in the most general case (K a field of characteristic p, G a group, V a finite-dimensional KG-module).

The first case of difficulty, namely $n = p$, was dealt with in [5] by Bryant and Stöhr via an analysis of the structure of $V^{\otimes p}$. They proved that $L^p(V) \cong B^p(V) \oplus M^p(V)$, where $B^p(V) = L''(V) \cap L^p(V)$ and $M^p(V)$ is the pth metabelian Lie power of V. In the case where K is an infinite field, G is the general linear group and V is the natural KG-module, they also calculated the indecomposable summands of $B^p(V)$, along with their multiplicities. Since $M^p(V)$ is also known to be indecomposable, this gives full information for $L^p(V)$ in this case. We note that the metabelian Lie powers $M^d(V)$ in this setting were later studied in more detail by Laci in joint work with Erdmann [12]. These modules are in fact the dual Weyl modules of weight $(d-1, 1)$ and, as such, are of interest to representation theorists. The paper [12] completely describes the submodule structure of these metabelian powers, giving a composition series for each and identifying the composition factors and their dimensions. Moreover, it is shown that the composition factors are pairwise nonisomorphic, from which it follows that the submodule lattice is finite and distributive.

The next step forward in the decomposition problem for Lie powers was achieved by Erdmann and Schocker [13] who considered Lie powers of degree $n = pk$, where

$p \nmid k$, and were able to prove that the study of $L^{pk}(V)$ can be reduced to the study of $L^p(L^k(V))$. Their methods made use of the Solomon descent algebra – a certain subring of the symmetric group ring (see [27] for further reference). Bryant and Schocker [2] were able to develop this idea much further, proving that the study of arbitrary Lie powers can, in a sense, be reduced to the study of Lie powers of p-power degree. Their decomposition theorem states that for all $k > 0$, with $p \nmid k$, and for all $m \geq 0$, there exists a KG-module direct sum decomposition

$$L^{p^m k}(V) = L^{p^m}(B_k) \oplus L^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L^1(B_{p^m k}),$$

for some submodules $B_{p^i k}$ of $L^{p^i k}(V)$, which are in turn isomorphic to KG-module direct summands of $V^{\otimes p^i k}$.

Apart from the intrinsic interest in this fascinating subject, there is also considerable interest from researchers in both group theory and representation theory. We note that an additional and somewhat unexpected impetus came from algebraic topology, where the relevance of modular representations became apparent in the work of Selick and Wu [29], prompting several topologists to undertake research in this area. We shall restrict ourselves here to a discussion of one particular application in which Laci had direct involvement.

It turns out that modular Lie powers can be used to give insight into an interesting problem, concerning torsion in free central extensions of groups. In 1973 Gupta [14] discovered elements of finite order in free centre-by-metabelian groups. More precisely, she proved that the relatively free group $F/[F'', F]$ (where F is a free group of rank $d \geq 4$) contains an elementary abelian 2-group of rank $\binom{d}{4}$ in its centre. This was the first result to remark upon an, at the time, surprising phenomenon: the appearance of torsion in free central extensions of certain torsion-free groups. Gupta's proof was purely group-theoretical, consisting of several pages of intricate commutator calculations. Kuz'min [24] introduced homological methods into the study of Gupta's torsion elements, and subsequently was able to identify the torsion subgroup of $F/[F'', F]$ with the fourth homology group of the free abelian group F/F' reduced modulo 2. The unexpected presence of torsion and the connection with group homology created considerable interest in such groups.

More generally, for G a group given by free presentation $G = F/R$, the quotient $F/[\gamma_c R, F]$, where $c \geq 2$ and $\gamma_c R$ denotes the cth term of the lower central series of R, is a free central extension of $F/\gamma_c R$. The quotient $F/\gamma_c R$ is in turn an extension of $G = F/R$ with free nilpotent kernel $R/\gamma_c R$. Whilst $F/\gamma_c R$ is always torsion-free [30], Gupta's result shows that elements of finite order can occur in the centre of $F/[\gamma_c R, F]$, that is, in the quotient $\gamma_c R/[\gamma_c R, F]$. The problem is then to determine the torsion subgroup of $\gamma_c R/[\gamma_c R, F]$. In a similar spirit to Kuz'min's result, Stöhr [31] was able to identify the torsion subgroup with a certain (nontrivial) homology group in the cases where $c = 4$ and $c = p$ a prime, provided that the group G does not contain any c-torsion.

Lie powers of the relation module can be brought to bear via the isomorphism

$$\gamma_c R/[\gamma_c R, F] \cong \mathcal{L}^c(R_{ab}) \otimes_G \mathbb{Z},$$

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where $\mathcal{L}^c(R_{ab})$ denotes the cth homogeneous component of the free Lie ring $\mathcal{L}(R_{ab})$. It is this connection which motivated the paper [22]. Reduction modulo p, where p is a prime, turns the relation module into the $(\mathbb{Z}/p\mathbb{Z})G$-module $M_p = R_{ab} \otimes (\mathbb{Z}/p\mathbb{Z})$. The main result of [22] is that the Lie power $L^c(M_p)$ is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module, provided that $c > 1$ and c is not divisible by p. In joint work with Stöhr [16] we extended this result, showing that if G does not contain any elements of order p, then each of the Lie powers $L^c(M_p)$ with c not equal to a power of p is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module. This result, together with (an infinite-dimensional version of) the Bryant–Schocker decomposition theorem, was then applied in [16] to show that whenever c is divisible by at least two primes and G has no c-torsion the group $\gamma_c R/[\gamma_c R, F]$ (and hence $F/[\gamma_c R, F]$) is torsion-free.

References

[1] A. J. Brandt, 'The free Lie ring and Lie representations of the full linear group', *Trans. Amer. Math. Soc.* **56** (1944), 528–536.

[2] R. M. Bryant and M. Schocker, 'The decomposition of Lie powers', *Proc. Lond. Math. Soc.* (3) **93**(1) (2006), 175–196.

[3] R. M. Bryant and R. Stöhr, 'Fixed points of automorphisms of free Lie algebras', *Arch. Math. (Basel)* **67**(4) (1996), 281–289.

[4] R. M. Bryant and R. Stöhr, 'On the module structure of free Lie algebras', *Trans. Amer. Math. Soc.* **352**(2) (2000), 901–934.

[5] R. M Bryant and R. Stöhr, 'Lie powers in prime degree', *Q. J. Math.* **56**(4) (2005), 473–489.

[6] R. M. Bryant, L. G. Kovács and R. Stöhr, 'Free Lie algebras as modules for symmetric groups', *J. Aust. Math. Soc. Ser. A* **67**(2) (1999), 143–156.

[7] R. M. Bryant, L. G. Kovács and R. Stöhr, 'Invariant bases for free Lie rings', *Q. J. Math.* **53**(1) (2002), 1–17.

[8] R. M. Bryant, L. G. Kovács and R. Stöhr, 'Lie powers of modules for groups of prime order', *Proc. Lond. Math. Soc.* (3) **84**(2) (2002), 343–374.

[9] R. M. Bryant, L. G. Kovács and R. Stöhr, 'Lie powers of modules for GL(2, p)', *J. Algebra* **260**(2) (2003), 617–630.

[10] R. M. Bryant, L. G. Kovács and R. Stöhr, 'Subalgebras of free restricted Lie algebras', *Bull. Aust. Math. Soc.* **72**(1) (2005), 147–156.

[11] S. Donkin and K. Erdmann, 'Tilting modules, symmetric functions, and the module structure of the free Lie algebra', *J. Algebra* **203**(1) (1998), 69–90.

[12] K. Erdmann and L. G. Kovács, 'Metabelian Lie powers of the natural module for a general linear group', *J. Algebra* **352** (2012), 232–267.

[13] K. Erdmann and M. Schocker, 'Modular Lie powers and the Solomon descent algebra', *Math. Z.* **253**(2) (2006), 295–313.

[14] C. K. Gupta, 'The free centre-by-metabelian groups', *J. Aust. Math. Soc.* **16** (1973), 294–299; Collection of articles dedicated to the memory of Hanna Neumann, III.

[15] M. Johnson, 'Standard tableaux and Klyachko's theorem on Lie representations', *J. Combin. Theory Ser. A* **114**(1) (2007), 151–158.

[16] M. Johnson and R. Stöhr, 'Free central extensions of groups and modular Lie powers of relation modules', *Proc. Amer. Math. Soc.* **138**(11) (2010), 3807–3814.

[17] A. A. Klyachko, 'Lie elements in the tensor algebra', *Siberian Math. J.* **15**(6) (1975), 914–920; translated from *Sibirsk. Mat. Zh.* **15** (6) (1974), 1296–1304.

[18] L. G. Kovács and R. Stöhr, 'Module structure of the free Lie ring on three generators', *Arch. Math. (Basel)* **73**(3) (1999), 182–185.

https://doi.org/10.1017/S1446788715000117 Published online by Cambridge University Press

[19] L. G. Kovács and R. Stöhr, 'Lie powers of the natural module for GL(2)', *J. Algebra* **229**(2) (2000), 435–462.
[20] L. G. Kovács and R. Stöhr, 'On Lie powers of regular modules in characteristic 2', *Rend. Semin. Mat. Univ. Padova* **112** (2004), 41–69.
[21] L. G. Kovács and R. Stöhr, 'A combinatorial proof of Klyachko's theorem on Lie representations', *J. Algebraic Combin.* **23**(3) (2006), 225–230.
[22] L. G. Kovács and R. Stöhr, 'Lie powers of relation modules for groups', *J. Algebra* **326** (2011), 192–200.
[23] W. Kraśkiewicz and J. Weyman, 'Algebra of coinvariants and the action of a Coxeter element', *Bayreuth. Math. Schr.* **63** (2001), 265–284; Preprint, 1987.
[24] Yu. V. Kuz'min, 'Free center-by-metabelian groups, Lie algebras and $\mathcal{D}$-groups', *Izv. Akad. Nauk SSSR Ser. Mat.* **41**(1) (1977), 3–33 (in Russian).
[25] M. P. Lee, 'Integral representations of dihedral groups of order $2p$', *Trans. Amer. Math. Soc.* **110** (1964), 213–231.
[26] V. D. Mazurov (ed.), *Kourovka Notebook: Unsolved Problems in Group Theory*, 11th edn, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1990.
[27] M. Schocker, 'The descent algebra of the symmetric group', in: *Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry*, Fields Institute Communications, 40 (American Mathematical Society, Providence, RI, 2004), 145–161.
[28] I. Schur, 'Über die rationalen Darstellungen der allgemeinen linearen Gruppe, (1927)', in: *Gesammelte Abhandlungen, III* (eds. A. Brauer and H. Rohrbach) (Springer, Berlin, 1973), 68–85.
[29] P. Selick and J. Wu, 'Some calculations of $\mathrm{Lie}(n)_{\max}$ for low n', *J. Pure Appl. Algebra* **212**(11) (2008), 2570–2580.
[30] A. L. Shmelkin, 'Wreath products and varieties of groups', *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 149–170 (in Russian).
[31] R. Stöhr, 'On torsion in free central extensions of some torsion-free groups', *J. Pure Appl. Algebra* **46** (1987), 249–289.
[32] R. M. Thrall, 'On symmetrized Kronecker powers and the structure of the free Lie ring', *Amer. J. Math.* **64** (1942), 371–388.
[33] F. Wever, 'Über Invarianten in Lie'schen Ringen', *Math. Ann.* **120** (1949), 563–580.
[34] E. Witt, 'Die Unterringe der freien Lieschen Ringe', *Math. Z.* **64** (1956), 195–216.

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