Canad. Math. Bull. Vol. **18** (4), 1975

ON A RESULT OF FAITH

BY
I. N. HERSTEIN

In a paper several years ago, Faith [2] proved an extension of a well-known theorem of Kaplansky [4]. His proof, even for the division ring case, was somewhat complicated. Using an old trick of Brauer [1] we show how Faith's theorem follows from Kaplansky's immediately.

THEOREM (*Faith*). *Let D be a division ring and $A \neq D$ a sub-ring of D. Suppose that for every $x \in D$, $x^{n(x)} \in A$ where $n(x) \geq 1$ depends on x. Then D is commutative.*

Proof. A must be a subdivision ring; for, if $a \neq 0 \in A$ then $a^{-n} = (a^{-1})^n \in A$ for some $n \geq 1$, hence $a^{-n}a^{n-1} = a^{-1}$ must be in A.

Let $x \in D$, $x \notin A$ and suppose that $a \in A$. Then, for a suitable $m \geq 1$ both $(xax^{-1})^m$ and $((1+x)a(1+x)^{-1})^m$ are in A. These give us

$$(1) \qquad \begin{aligned} xa^m &= a_1x \\ (1+x)a^m &= a_2(1+x) \end{aligned}$$

where $a_1, a_2 \in A$. Subtracting we get $a^m - a_2 = (a_2 - a_1)x$; since $x \notin A$ and A is a subdivision ring of D, we must have $a_1 = a_2$ and so $a^m = a_2$. Thus $a^m = a_1$; hence (1) gives us $xa^m = a^mx$. If $b \in A$ then $x+b \notin A$, hence by the above $(x+b)a^n = a^n(x+b)$ for some $n \geq 1$. Thus we have a^{mn} commutes with b. In other words, if $a \in A$ then some power of a commutes with y, for $y \in D$.

If $x, y \in D$ then $x^m \in A$ for some $m \geq 1$ hence x^{mn} commutes with y, by the above. By a trivial extension of Kaplansky's theorem [3], D must be commutative.

BIBLIOGRAPHY

1. Richard Brauer, *On a theorem of H. Cartan*, BAMS **55** (1949), 6. 9-620.
2. Carl Faith, *Algebraic division ring extensions*, PAMS **11** (1960), 43–53.
3. I. N. Herstein, *Two remarks on the commutativity of rings*, Canadian Journal Math. **7** (1955), 411–412.
4. Irving Kaplansky, *A theorem on division rings*, Canadian Journal Math. **3** (1951), 290–292.