Canad. Math. Bull. Vol. **42** (3), 1999 pp. 263–273

Mellin Transforms of Mixed Cusp Forms

Youngju Choie and Min Ho Lee

Abstract. We define generalized Mellin transforms of mixed cusp forms, show their convergence, and prove that the function obtained by such a Mellin transform of a mixed cusp form satisfies a certain functional equation. We also prove that a mixed cusp form can be identified with a holomorphic form of the highest degree on an elliptic variety.

1 Introduction

Given nonnegative integers l and m, mixed cusp forms of type (l, m) for a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ are defined using automorphy factors of the form $J(\gamma, z)^l J\big(\chi(\gamma), \omega(z)\big)^m$, where $J(\gamma, z) = cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and ω is a map of the the Poincaré upper half plane that is equivariant with respect to a homomorphism $\chi \colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$. Families of abelian varieties parametrized by an arithmetic variety play an important role in the theory of automorphic forms (see *e.g.* [11]). An elliptic variety is one of such families of abelian varieties, and it is a fiber variety over a compact Riemann surface whose generic fiber is isomorphic to the product of a finite number of elliptic curves. It is known that mixed cusp forms of type $(2, m)$ can be interpreted as holomorphic forms of the highest degree on a certain type of elliptic variety (*cf.* [2], [6]). Various aspects of such cusp forms were studied in a number of papers (see *e.g.* [7], [8]), and mixed automorphic forms of several variables have also been investigated (*cf.* [9], [11], [10]). The goal of this paper is to describe geometric interpretations of mixed cusp forms of one variable of type (l, m) for an arbitrary integer $l \geq 2$ and discuss the L-functions attached to such mixed cusp forms.

A classical automorphic form for a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ has a Fourier expansion of the form

$$\sum_{n=0}^{\infty} a_n e^{2\pi i n z / h}$$

at each cusp for some positive real number h, and is called a *cusp form* if $a_0 = 0$ for all cusps. To each cusp form $f(z)$ we associate the Dirichlet series

$$D(s, f) = \sum_{n=1}^{\infty} a_n n^{-s},$$

where a_n's are the Fourier coefficients of $f(z)$ at the cusp infinity. If $f(z)$ is a cusp form for the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; c \equiv 0 (\mathrm{mod}\, N) \right\},$$

Received by the editors September 24, 1997; revised May 6, 1998.
The research of the first author was partially supported by BSRI 97-1431 and KOSEF 941-0100-001-2.
AMS subject classification: Primary: 11F12, 11F66, 11M06; secondary: 14K05.

then it is well-known that its Mellin transform

$$L(s, f) = \int_0^\infty f(iy) y^{s-1}\, dy \tag{1.1}$$

is equal to $(2\pi)^{-s}\Gamma(s)D(s, f)$, where $\Gamma(s)$ is the Γ-function. It is also known that $L(s, f)$ converges in some half plane, can be analytically continued to the whole complex plane as an entire function, and satisfies a certain functional equation (see *e.g.* [12], [13] for details).

In this paper we construct a certain type of elliptic varieties and prove that the space of holomorphic forms of the highest degree on such an elliptic variety is isomorphic to the space of mixed cusp forms of type (l, m) for some $l \geq 2$ (Theorem 3.1). We then introduce a generalized Mellin transform

$$L^{2k,m}_{\omega,\chi}(s, f) = \int_0^\infty f(iy)\omega(iy)^{s-k} y^{k-1}\, dy$$

of a mixed cusp form $f(z)$ of type $(2k, m)$ associated to a discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$, a holomorphic map ω of the Poincaré upper half plane and a group homomorphism $\chi\colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$, and prove that it converges (Theorem 4.2) and that $L^{2k,m}_{\omega,\chi}(s, f)$ satisfies a functional equation (Theorem 4.4) which generalizes the usual functional equation for $L(s, f)$. For the proofs of these theorems we use the fact that a mixed cusp form $f(z)$ of type (l, m) associated to Γ, ω and χ can be written as a linear combination of functions of the form $f_1(z) f_2\big(\omega(z)\big)$, where f_1 is a cusp form of weight l for Γ and f_2 is a cusp form of weight m for $\chi(\Gamma)$ (Proposition 2.4).

2 Mixed Automorphic Forms

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian group of the first kind, and let $\chi\colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$ a homomorphism of groups. Thus both Γ and the image $\chi(\Gamma)$ of Γ under χ operate on the Poincaré upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by linear fractional transformations. Let $\omega\colon \mathcal{H} \to \mathcal{H}$ be a Γ-equivariant holomorphic map, *i.e.*, a holomorphic map that satisfies $\omega(gz) = \chi(g)\omega(z)$ for all $g \in \Gamma$ and $z \in \mathcal{H}$. We assume that the image of a parabolic element of Γ under χ is a parabolic element in $\chi(\Gamma)$. Given a cusp s of Γ we set

$$\Delta_s = \{\sigma \in \mathrm{SL}(2, \mathbb{R}) \mid \sigma(\infty) = s\},$$

and let Δ be the union $\bigcup \Delta_s$ of Δ_s for all cusps s of Γ. We assume that the homomorphism χ can be extended to a mapping $\chi\colon \Gamma \cup \Delta \to \mathrm{SL}(2, \mathbb{R})$.

Given a pair of nonnegative even integers l and m, we set

$$J^{l,m}_{\omega,\chi}(g, z) = (cz + d)^l \big(c_\chi \omega(z) + d_\chi\big)^m$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \cup \Delta, \quad \chi(g) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Then $J^{l,m}_{\omega,\chi}\colon (\Gamma \cup \Delta) \times \mathcal{H} \to \mathbb{C}$ is a factor of automorphy, *i.e.*, it satisfies the condition

$$J^{l,m}_{\omega,\chi}(gh, z) = J^{l,m}_{\omega,\chi}(g, hz) \cdot J^{l,m}_{\omega,\chi}(h, z)$$

for all $g, h \in \Gamma$ and $z \in \mathcal{H}$.

Let s be a cusp of Γ, and let σ be an element of $\mathrm{SL}(2,\mathbb{R})$ such that $\sigma(\infty) = s$. We set

$$\big(f \mid [\sigma]\big)(z) = J^{l,m}_{\omega,\chi}(\sigma, z)^{-1} f(\sigma z).$$

Then ∞ is a cusp of $\Gamma^\sigma = \sigma^{-1}\Gamma\sigma$, and we have the Fourier expansion of $f \mid [\sigma]$ at ∞ of the form

$$\big(f \mid [\sigma]\big)(z) = \sum_{n \geq n_0} a_n e^{2\pi i n z/h},$$

which is called the *Fourier expansion of f at s*.

Definition 2.1 Let Γ, ω, and χ as above. A *mixed automorphic form of type* (l,m) associated to Γ, ω and χ is a holomorphic function $f \colon \mathcal{H} \to \mathbb{C}$ satisfying the following conditions:

(i) $f(\gamma z) = J^{l,m}_{\omega,\chi}(\gamma, z) f(z)$ for all $\gamma \in \Gamma$.

(ii) The Fourier coefficients a_n of f at each cusp s satisfy the condition that $n \geq 0$ whenever $a_n \neq 0$.

The holomorphic function f is a *mixed cusp form* if (ii) is replaced with the following condition:

(ii)$'$ The Fourier coefficients a_n of f at each cusp s satisfy the condition that $n > 0$ whenever $a_n \neq 0$.

We shall denote by $S_{l,m}(\Gamma, \omega, \chi)$ the space of mixed cusp forms of type (l, m) associated to Γ, ω and χ.

Remark 2.2 If $S_k(\Gamma)$ denotes the space of cusp forms of weight k for Γ, then we have

$$S_{l,0}(\Gamma, \omega, \chi) = S_l(\Gamma), \quad S_{l,m}(\Gamma, \mathrm{id}, \mathrm{id}) = S_{l+m}(\Gamma).$$

On the other hand for $l = 0$ the elements of $S_{0,m}(\Gamma, \omega, \chi)$ are generalized automorphic forms of weight m in the sense of W. Hoyt and P. Stiller (see *e.g.* [15, p. 31]).

Now we discuss the interpretation of mixed cusp forms in terms of line bundles which will be used in Section 4 to prove the convergence of Mellin transforms. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$, $\omega \colon \mathcal{H} \to \mathcal{H}$ and $\chi \colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ be as above, and let Σ be the the set of cusps of Γ. Let $\mathcal{H}^* = \mathcal{H} \cup \Sigma$, and assume that Γ does not contain elements of finite order. Then the quotient $\Gamma \setminus \mathcal{H}^*$ has the structure of a compact Riemann surface.

Let $\mathcal{O}_{\mathcal{H}}$ be the sheaf of holomorphic functions on $\mathcal{H}$, and we extend $\mathcal{O}_{\mathcal{H}}$ to the sheaf $\mathcal{O}_{\mathcal{H}^*}$ on $\mathcal{H}^*$ by defining the stalk at each cusp $s \in \Sigma$ by

$$\mathcal{O}_{\mathcal{H}^*,s} = \{f \in (j_*\mathcal{O}_{\mathcal{H}})_s \mid f(\sigma z) = O(|z|^n) \text{ for some } n \in \mathbb{Z}\},$$

where σ is an element of $\mathrm{SL}(2,\mathbb{R})$ with $s = \sigma\infty$ and $j \colon \mathcal{H} \to \mathcal{H}^*$ is the inclusion map. Let $\mathcal{F}_{\mathcal{H}^*}$ be the subsheaf of the tensor product $\mathcal{O}_{\mathcal{H}^*} \otimes \mathbb{C}^2$ of $\mathcal{O}_{\mathcal{H}^*}$ and the constant sheaf $\mathbb{C}^2$ generated by the global section $\binom{z}{1}$, *i.e.*,

$$\mathcal{F}_{\mathcal{H}^*} = \left\{ f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \;\middle|\; f \in \mathcal{O}_{\mathcal{H}^*} \right\}.$$

Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$\gamma \cdot \left(f(z) \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = f(\gamma^{-1}z) \begin{pmatrix} a(\gamma^{-1}z) + b \\ c(\gamma^{-1}z) + d \end{pmatrix}$$
$$= \big(c(\gamma^{-1}z) + d \big) f(\gamma^{-1}z) \begin{pmatrix} z \\ 1 \end{pmatrix} = (-cz + a) f(\gamma^{-1}z) \begin{pmatrix} z \\ 1 \end{pmatrix};$$

hence $\mathcal{F}_{\mathcal{H}^*}$ is Γ-invariant. Let $\mathcal{O}_{\mathcal{H}^*}(-\Sigma)$ be the sheaf of functions on $\mathcal{H}^*$ which are holomorphic on $\mathcal{H}$ and zero on Σ. For each nonnegative integer m we set

$$\mathcal{F}^m_{\mathcal{H}^*}(-\Sigma) = \mathcal{F}^m_{\mathcal{H}^*} \otimes \mathcal{O}_{\mathcal{H}^*}(-\Sigma),$$

where $\mathcal{F}^m_{\mathcal{H}^*}$ is the m-th tensor power of $\mathcal{F}_{\mathcal{H}^*}$, and denote by

$$\mathcal{F}^m_{\Gamma} = \big(\mathcal{F}^m_{\mathcal{H}^*}(-\Sigma) \big)^{\Gamma}$$

the Γ-fixed sheaf of $\mathcal{F}^m_{\mathcal{H}^*}(-\Sigma)$ on $X_\Gamma = \Gamma \setminus \mathcal{H}^*$.

Proposition 2.3 *The space $S_m(\Gamma)$ of cusp forms of weight m for Γ is canonically isomorphic to the space $H^0(X_\Gamma, \mathcal{F}^m_\Gamma)$ of sections of $\mathcal{F}^m_\Gamma$.*

Proof See [2, Proposition 1.4] (see also [1]). ■

Let $\Gamma' = \chi(\Gamma)$ be the image of Γ under χ, and let Σ' be the set of cusps of Γ'. Then for each nonnegative integer n we can define the sheaf $\mathcal{F}^n_{\Gamma'}$ over the Riemann surface $X_{\Gamma'} = \Gamma' \setminus (\mathcal{H} \cup \Sigma')$ whose sections are cusp forms of weight n for Γ'. Let $\omega_X \colon X_\Gamma \to X_{\Gamma'}$ be the morphism of complex algebraic curves induced by the holomorphic map $\omega \colon \mathcal{H} \to \mathcal{H}$.

Proposition 2.4 *The space $S_{l,m}(\Gamma, \omega, \chi)$ of mixed cusp forms associated to Γ, ω and χ is canonically isomorphic to the space $H^0(X_\Gamma, \mathcal{F}^l_\Gamma \otimes \omega_X^* \mathcal{F}^m_{\Gamma'})$ of sections of the sheaf $\mathcal{F}^l_\Gamma \otimes \omega_X^* \mathcal{F}^m_{\Gamma'}$ over X_Γ.*

Proof This proposition is an extension of Theorem 1.6 in [2]. By Proposition 2.3 each section of the sheaf $\mathcal{F}^l_\Gamma \otimes \omega_X^* \mathcal{F}^m_{\Gamma'}$ is a linear combination of sections of the form $z \mapsto \big(f_1 \otimes (f_2 \circ \omega)\big)(z) = f_1(z) f_2\big(\omega(z)\big)$ with $f_1 \in S_l(\Gamma)$ and $f_2 \in S_m(\Gamma')$, and therefore has the same transformation property as the one for an element in $S_{l,m}(\Gamma, \omega, \chi)$. Now the proposition follows from the fact that the Γ-cusps and Γ'-cusps correspond via ω and χ, since χ maps parabolic elements to parabolic elements. ■

3 Elliptic Varieties

In this section we describe the interpretation of mixed cusp forms as holomorphic forms on certain families of abelian varieties. Let E be an elliptic surface in the sense of Kodaira [3]. Thus E is a compact smooth surface over $\mathbb{C}$, and it is the total space of an elliptic fibration $\pi \colon E \to X$ over a Riemann surface X whose generic fiber is an elliptic curve. Let E_0 be the union of the regular fibers of π, and let $\Gamma \subset PSL(2, \mathbb{R})$ be the fundamental group

https://doi.org/10.4153/CMB-1999-032-3 Published online by Cambridge University Press

of $X_0 = \pi(E_0)$. Then Γ acts on the universal covering space $\mathcal{H}$ of X_0 by linear fractional transformations, and we have

$$X = \Gamma \setminus \mathcal{H} \cup \{\Gamma\text{-cusps}\}.$$

For $z \in X_0$, let Φ be a holomorphic 1-form on the fiber $E_z = \pi^{-1}(z)$, and choose an ordered basis $\{\gamma_1(z), \gamma_2(z)\}$ for $H_1(E_z, \mathbb{Z})$ that depends on the parameter z in a continuous manner. Consider the periods ω_1 and ω_2 of E given by

$$\omega_1(z) = \int_{\gamma_1(z)} \Phi, \quad \omega_2(z) = \int_{\gamma_2(z)} \Phi.$$

Then ω_1/ω_2 is a many-valued function from X_0 to $\mathcal{H}$ which can be lifted to a single-valued function $\omega\colon \mathcal{H} \to \mathcal{H}$ on the universal cover of X_0 such that

$$\omega(\gamma z) = \chi(\gamma)\omega(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, where $\chi\colon \Gamma \to \mathrm{SL}(2, \mathbb{Z})$ is the monodromy representation of the elliptic fibration $\pi\colon E \to X$.

In order to discuss connections of elliptic varieties with mixed cusp forms we shall regard Γ as a subgroup of $\mathrm{SL}(2, \mathbb{R})$. As in Section 2, we denote by $S_{j+2,m}(\Gamma, \chi, \omega)$ the space of mixed cusp forms of type $(j + 2, m)$ associated to Γ, ω and χ. Let $E(\chi)$ (resp. $E(1)$) be an elliptic surface over X whose monodromy representation is χ (resp. the inclusion map), and let $\pi(\chi)\colon E(\chi) \to X$ (resp. $\pi(1)\colon E(1) \to X$) be the associated elliptic fibration. We set

$$E(\chi)_0 = \pi(\chi)^{-1}(\Gamma \setminus \mathcal{H}), \quad E(1)_0 = \pi(1)^{-1}(\Gamma \setminus \mathcal{H}),$$

and denote by $(E^{j,m}_{\chi,1})_0$ the fiber product of j-copies of $E(1)_0$ and m copies of $E(\chi)_0$ over X corresponding to the maps $\pi(1)$ and $\pi(\chi)$, respectively. The space $(E^{j,m}_{\chi,1})_0$ can also be constructed as below. Consider the semidirect product $\Gamma \ltimes_{1,\chi} \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$ consisting of the triples (γ, μ, ν) in $\Gamma \times \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$ whose multiplication law is defined as follows. Let

$$(\gamma, \mu, \nu), (\gamma', \mu', \nu') \in \Gamma \times \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$$

with

$$\mu' = (\mu'_1, \mu'_2) = (\mu'_{11}, \dots, \mu'_{j1}, \mu'_{12}, \dots, \mu'_{j2}) \in \mathbb{Z}^{2j},$$
$$\nu' = (\nu'_1, \nu'_2) = (\nu'_{11}, \dots, \nu'_{m1}, \nu'_{12}, \dots, \nu'_{m2}) \in \mathbb{Z}^{2m},$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \chi(\gamma) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Then we have

$$(\gamma, \mu, \nu) \cdot (\gamma', \mu', \nu') = \bigl(\gamma\gamma', \gamma \cdot (\mu', \nu') + (\mu, \nu)\bigr),$$

https://doi.org/10.4153/CMB-1999-032-3 Published online by Cambridge University Press

where $\gamma \cdot (\mu', \nu') = (\mu'', \nu'')$ with

$$\mu'' = (a\mu'_{11} + b\mu'_{12}, \dots, a\mu'_{j1} + b\mu'_{j2}, c\mu'_{11} + d\mu'_{12}, \dots, c\mu'_{j1} + d\mu'_{j2}) \in \mathbb{Z}^{2j},$$

$$\nu'' = (a_\chi\nu'_{11} + b_\chi\nu'_{12}, \dots, a_\chi\nu'_{m1} + b_\chi\nu'_{m2}, c_\chi\nu'_{11} + d_\chi\nu'_{12}, \dots, c_\chi\nu'_{m1} + d_\chi\nu'_{m2}) \in \mathbb{Z}^{2m}.$$

The group $\Gamma_{1,\chi}\, \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$ acts on the space $\mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m$ by

$$\begin{aligned}(3.1)\qquad &(\gamma, \mu, \nu) \cdot (z, \xi, \zeta)\\ &= \Big(\gamma z, (cz+d)^{-1}(\xi + z\mu_1 + \mu_2), \big(c_\chi\omega(z) + d_\chi\big)^{-1}\big(\zeta + \omega(z)\nu_1 + \nu_2\big)\Big)\end{aligned}$$

for $\gamma \in \Gamma$, $z \in \mathcal{H}$, $\xi \in \mathbb{C}^j$, $\zeta \in \mathbb{C}^m$, $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^{2j}$, and $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^{2m}$. Then we have

$$(3.2)\qquad (E^{j,m}_{1,\chi})_0 = \Gamma_{1,\chi}\, \mathbb{Z}^{2j} \times \mathbb{Z}^{2m} \setminus \mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m.$$

Now we obtain the *elliptic variety* $E^{j,m}_{1,\chi}$ by resolving the singularities of the compactification of $(E^{j,m}_{1,\chi})_0$ (*cf.* [14]). The elliptic fibration π induces a fibration $\pi^{j,m}_{1,\chi}\colon E^{j,m}_{1,\chi} \to X$ whose generic fiber is the product of $(j+m)$ elliptic curves.

Theorem 3.1 *Let $E^{j,m}_{1,\chi}$ be an elliptic variety described above. Then there is a canonical isomorphism*

$$H^0(E^{j,m}_{1,\chi}, \Omega^{j+m+1}) \cong S_{j+2,m}(\Gamma, \omega, \chi)$$

between the space of holomorphic $(j+m+1)$-forms on $E^{j,m}_{\chi,1}$ and the space of mixed cusp forms of type $(j+2, m)$.

Proof Let $(E^{j,m}_{1,\chi})_0$ be as in (3.2). Then a holomorphic $(j+m+1)$-form on $E^{j,m}_{1,\chi}$ can be regarded as a holomorphic $(j+m+1)$-form on $\mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m$ that is invariant under the operation of $\Gamma_{1,\chi}\, \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$. Since the complex dimension of the space $\mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m$ is $j+m+1$, a holomorphic $(j+m+1)$-form on $\mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m$ is of the form

$$\Theta = \tilde{f}(z, \xi, \zeta)dz \wedge d\xi \wedge d\zeta,$$

where $\tilde{f}$ is holomorphic. For $z_0 \in \mathcal{H}$, the holomorphic form Θ descends to a holomorphic $(j+m)$-form on the fiber $(\pi^{j,m}_{1,\chi})^{-1}(z_0)$. However, the dimension of the fiber is $j+m$, and therefore the space of holomorphic $(j+m)$-forms on $(\pi^{j,m}_{1,\chi})^{-1}(z_0)$ is one. Hence the map $(\xi, \zeta) \mapsto \tilde{f}(z, \xi, \zeta)$ is a holomorphic function on a compact complex manifold, and consequently is a constant function. Thus we have $\tilde{f}(z, \xi, \zeta) = f(z)$, where f is a holomorphic function on $\mathcal{H}$. From (3.1) the action of $(\gamma, \mu, \nu) \in \Gamma_{1,\chi}\, \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$ on the form $\Theta = f(z)dz \wedge d\xi \wedge d\zeta$ is given by

$$\begin{aligned}\Theta \cdot (\gamma, \mu, \nu) &= f(gz)d(gz) \wedge d\big((cz+d)^{-1}(\xi + z\mu_1 + \mu_2)\big)\\ &\qquad \wedge d\Big(\big(c_\chi\omega(z) + d_\chi\big)^{-1}\big(\zeta + \omega(z)\nu_1 + \nu_2\big)\Big)\\ &= f(gz)(cz+d)^{-2}(cz+d)^{-j}\big(c_\chi\omega(z) + d_\chi\big)^{-m} dz \wedge d\xi \wedge d\zeta.\end{aligned}$$

Thus it follows that $f(z)$ satisfies the condition (i) of Definition 2.1, and it remains to show that f satisfies the cusp condition. Using Theorem 3.1 in [5], we see that the differential form Θ can be extended to $E^{j,m}_{1,\chi}$ if and only if

$$\int_{(E^{j,m}_{1,\chi})_0} \Theta \wedge \bar{\Theta} < \infty.$$

From (3.1) it follows that a fundamental domain F in $\mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m$ for the action of $\Gamma_{1,\chi} \mathbb{Z}^{2j} \times \mathbb{Z}^{2m}$ can be chosen in the form

$$F = \{(z, \xi, \zeta) \in \mathcal{H} \times \mathbb{C}^j \times \mathbb{C}^m \mid z \in F_0,\ \xi = \mathbf{s} + \mathbf{t}z,\ \zeta = \mathbf{u} + \mathbf{v}\omega(z), \mathbf{s}, \mathbf{t} \in I^j,\ \mathbf{u}, \mathbf{v} \in I^m\},$$

where $F_0 \subset \mathcal{H}$ is a fundamental domain of Γ and I is the closed interval $[0, 1] \subset \mathbb{R}$. Thus we have

$$\begin{aligned}\int_{(E^{j,m}_{1,\chi})_0} \Theta \wedge \bar{\Theta} = \int_F \Theta \wedge \bar{\Theta} &= \int_F |f(z)|^2\, dz \wedge d\xi \wedge d\zeta \wedge d\bar{z} \wedge d\bar{\xi} \wedge d\bar{\zeta} \\ &= K \int_F |f(z)|^2 (\operatorname{Im} z)^j \big(\operatorname{Im} \omega(z)\big)^m\, dz \wedge d\bar{z},\end{aligned}$$

where K is a nonzero constant. Thus the integral $\int_F \Theta \wedge \bar{\Theta}$ is a nonzero constant multiple of the Petersson inner product $\langle f, f\rangle$ described in Proposition 2.1 in [8]; hence it is finite if and only if f satisfies the cusp condition, and the proof of the theorem is complete. ∎

Remark 3.2 Theorem 3.1 is an extension of the results of [2, Theorem 1.2] and [6, Theorem 3.2], where mixed cusp forms of types $(2, 1)$ and $(2, m)$, respectively, were considered (see also [2, Theorem 1.6]).

4 Generalized Mellin Transforms

In this section, we define an L-function given by a generalized Mellin transform of a mixed cusp form and prove that it satisfies a certain functional equation. Given any positive integer N, we set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; c \equiv 0 (\operatorname{mod} N) \right\},$$

$$\nu_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Then we have $\nu_N^2 = -1$ and $\nu_N^{-1}\Gamma_0(N)\nu_N = \Gamma_0(N)$; hence we can form a discrete subgroup

$$\Gamma_0^*(N) = \Gamma_0(N) \cup \Gamma_0(N)\nu_N$$

of $\mathrm{SL}(2, \mathbb{R})$ (see [13, p. 27]).

https://doi.org/10.4153/CMB-1999-032-3 Published online by Cambridge University Press

Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that contains $\Gamma_0^*(N)$, $\chi\colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ a homomorphism, and $\omega\colon \mathcal{H} \to \mathcal{H}$ a holomorphic map satisfying $\omega(gz) = \chi(g)\omega(z)$ for all $g \in \Gamma$ and $z \in \mathcal{H}$ as in Section 2. We assume that $\chi(\nu_N) = \nu_{N_\chi}$ for some positive integer N_χ. Thus, in particular, we can choose ω such that it satisfies

$$\omega\left(\frac{-1}{Nz}\right) = \omega(\nu_N z) = \chi(\nu_N)\omega(z) = \nu_{N_\chi}\omega(z) = \frac{-1}{N_\chi \omega(z)} \tag{4.1}$$

for all $z \in \mathcal{H}$. Given any holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$, we set

$$f \mid [\nu_N]_{2k,m}(z) = f(\nu_N z) j(\nu_N, z)^{-2k} j\big(\chi(\nu_N), \omega(z)\big)^{-m}$$

for $z \in \mathcal{H}$. Thus, using (4.3), we obtain

$$f \mid [\nu_N]_{2k,m}(z) = f\big(-1/(Nz)\big)(\sqrt{N}z)^{-2k}\big(\sqrt{N_\chi}\omega(z)\big)^{-m}.$$

In particular, we have

$$f \mid [\nu_N]_{2k,m}(iy) = N^{-k} N_\chi^{-m/2} f\big(i/(Ny)\big)(iy)^{-2k}\big(\omega(iy)\big)^{-m} \tag{4.2}$$

for positive real numbers y.

Now we define the L-function $L_{\omega,\chi}^{2k,m}(s, f)$ of a mixed cusp form f of type $(2k, m)$ associated to Γ, ω and χ by

$$L_{\omega,\chi}^{2k,m}(s, f) = \int_0^\infty f(iy)\omega(iy)^{s-k} y^{k-1}\, dy. \tag{4.3}$$

Remark 4.1 If $k = 1$ in (4.3), the integral defining $L_{\omega,\chi}^{2,m}$ becomes

$$\int_0^\infty f(iy)\omega(iy)^{s-1}\, dy,$$

where f is a mixed cusp form of type $(2, m)$. Certain properties of such an integral for $s = 1, \dots, m+1$ were investigated in [7].

In order to discuss the convergence of the integral in (4.3), we assume that the image curve $\omega(\{iy \mid y \geq 0\})$ of the vertical line $\{iy \mid y \geq 0\}$ under ω joining $\omega(0)$ and $\omega(\infty)$ is analytic at the end points in the sense of [4, p. 58]. Since the image of a translation of the form $z \mapsto z+b$ under χ is a translation $w \mapsto w+b_\chi$ for some $b_\chi \in \mathbb{R}$, we have $\omega(\infty) = \infty$, and the analyticity at the end points implies that the curve $\omega(\{iy \mid y \geq 0\})$ is contained in a vertical strip of finite width.

Theorem 4.2 *Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that contains $\Gamma_0^*(N)$, and assume that $\chi(\nu_N) = \nu_{N_\chi}$ for some positive integer N_χ. If f is a mixed cusp form of type $(2k, m)$ associated to Γ, ω and χ, then the integral in (4.3) defining the L-function $L_{\omega,\chi}^{2k,m}(s, f)$ converges for all $s \in \mathbb{C}$.*

Proof Let $a = N^{-1/2}$, and set

$$I_1 = \int_0^a f(iy)\omega(iy)^{s-k}y^{k-1}\,dy, \quad I_2 = \int_a^\infty f(iy)\omega(iy)^{s-k}y^{k-1}\,dy, \tag{4.4}$$

so that the integral defining $L^{2k,m}_{\omega,\chi}(s, f)$ can be written in the form

$$\int_0^\infty f(iy)\omega(iy)^{s-k}y^{k-1}\,dy = I_1 + I_2.$$

By Proposition 2.4 a mixed cusp form $f(z)$ in $S_{2k,m}(\Gamma,\omega,\chi)$ can be written as a linear combination of functions of the form

$$f_1(z) f_2\big(\omega(z)\big)$$

for all $z \in \mathcal{H}$, where $f_1 \in S_{2k}(\Gamma)$ and $f_2 \in S_m(\Gamma')$. Thus we have the Fourier expansions

$$f_1(z) = \sum_{n\geq 1} a_n e^{2\pi inz/h}, \quad f_2\big(\omega(z)\big) = \sum_{n\geq 1} a'_n e^{2\pi in\omega(z)/h'}$$

for some positive real numbers h and h', and hence we obtain the estimations

$$|f_1(iy)| = O(e^{-2\pi y/h}), \quad |f_2\big(\omega(iy)\big)| = O(e^{-2\pi\,\mathrm{Im}(\omega(iy))/h'})$$

for $y \to \infty$. On the other hand, since $\omega(iy)$ is contained in a vertical strip of finite width, there is a real number $B > 0$ such that

$$|\mathrm{Re}\big(\omega(iy)\big)| \leq B$$

for all $y \geq 0$. Thus it follows that the integral I_2 converges for each $s \in \mathbb{C}$. As for the integral I_1, using the change of variable $y \mapsto 1/(Ny)$, we obtain

$$I_1 = \int_\infty^a f\left(\frac{i}{Ny}\right)\omega\left(\frac{i}{Ny}\right)^{s-k}\left(\frac{1}{Ny}\right)^{k-1}\left(\frac{-1}{Ny^2}\right)dy.$$

On the other hand, using (4.1) and (4.2), we have

$$\omega\left(\frac{i}{Ny}\right) = \omega\left(\frac{-1}{N(iy)}\right) = \frac{-1}{N_\chi\omega(iy)}$$

and

$$f\left(\frac{i}{Ny}\right) = N^k N_\chi^{m/2}(iy)^{2k}\omega(iy)^m\big(f \mid [\nu_N]_{2k,m}(iy)\big).$$

Hence we see that

(4.5)

$$\begin{aligned} I_1 &= \int_\infty^a N^k N_\chi^{m/2}(iy)^{2k}\omega(iy)^m\big(f \mid [\nu_N]_{2k,m}(iy)\big)\big(-N_\chi\omega(iy)\big)^{-s+k}(Ny)^{1-k}(-N^{-1}y^{-2})\,dy \\ &= (-1)^s N_\chi^{m/2+k-s}\int_a^\infty \big(f \mid [\nu_N]_{2k,m}(iy)\big)\omega(iy)^{(m+2k-s)-k}y^{k-1}\,dy. \end{aligned}$$

Now, using the argument similar to the one for the case of I_2 above, we see that I_1 also converges. ■

https://doi.org/10.4153/CMB-1999-032-3 Published online by Cambridge University Press

Remark 4.3 If ω is the identity map, then the integral in (4.3) is the usual Mellin transform given in (1.1) of the function f up to a multiple depending on s. Thus it can be considered as a generalized Mellin transform of f, and the function $L^{2k,m}_{\omega,\chi}(s,f)$ is essentially a generalization of the usual Dirichlet series of a classical cusp form described in Section 1 (see [12], [13]).

Theorem 4.4 *Let Γ and N_χ be as in Theorem 4.2, and let f be a mixed cusp form of type $(2k,m)$ associated to Γ, ω and χ. Then the L-function $L^{2k,m}_{\omega,\chi}(s,f)$ of f satisfies the functional equation*

$$L^{2k,m}_{\omega,\chi}(s,f) = (-1)^s N_\chi^{m/2+k-s} L^{2k,m}_{\omega,\chi}\big(2k+m-s, f \mid [\nu_N]_{2k,m}\big). \tag{4.6}$$

Proof Let $a = N^{-1/2}$, and let I_1, I_2 be as in (4.4). Then we have

$$L^{2k,m}_{\omega,\chi}(s,f) = I_1 + I_2,$$

and by (4.5) we obtain

$$I_1 = (-1)^s N_\chi^{m/2+k-s} \int_a^\infty \big(f \mid [\nu_N]_{2k,m}(iy)\big)\omega(iy)^{(m+2k-s)-k} y^{k-1}\, dy.$$

On the other hand, using the change of variable $y \mapsto 1/(Ny)$ for $a \le y < \infty$ and computing in a way similar to the case of I_1 in the proof of Theorem 4.2, we have

$$I_2 = (-1)^s N_\chi^{m/2+k-s} \int_0^a \big(f \mid [\nu_N]_{2k,m}(iy)\big)\omega(iy)^{(m+2k-s)-k} y^{k-1}\, dy.$$

Hence it follows that

$$\begin{aligned} L^{2k,m}_{\omega,\chi}(s,f) &= (-1)^s N_\chi^{m/2+k-s} \int_0^\infty \big(f \mid [\nu_N]_{2k,m}(iy)\big)\omega(iy)^{(m+2k-s)-k} y^{k-1}\, dy \\ &= (-1)^s N_\chi^{m/2+k-s} L^{2k,m}_{\omega,\chi}\big(2k+m-s, f \mid [\nu_N]_{2k,m}\big), \end{aligned}$$

and the proof of the relation (4.6) is complete. ■

References

[1] P. Bayer and J. Neukirch, *On automorphic forms and Hodge theory*. Math. Ann. **257**(1981), 135–155.
[2] B. Hunt and W. Meyer, *Mixed automorphic forms and invariants of elliptic surfaces*. Math. Ann. **271**(1985), 53–80.
[3] K. Kodaira, *On compact analytic surfaces II*. Ann. of Math. **77**(1963), 563–626.
[4] S. Lang, *Introduction to modular forms*. Springer-Verlag, Heidelberg, 1976.
[5] H. Laufer, *On rational singularities*. Amer. J. Math. **94**(1972), 597–608.
[6] M. H. Lee, *Mixed cusp forms and holomorphic forms on elliptic varieties*. Pacific J. Math. **132**(1988), 363–370.
[7] ———, *Periods of mixed cusp forms*. Manuscripta Math. **73**(1991), 163–177.
[8] ———, *Mixed cusp forms and Poincaré series*. Rocky Mountain J. Math. **23**(1993), 1009–1022.
[9] ———, *Mixed Siegel modular forms and Kuga fiber varieties*. Illinois J. Math. **38**(1994), 692–700.
[10] ———, *Mixed automorphic forms on semisimple Lie groups*. Illinois J. Math. **40**(1996), 464–478.
[11] ———, *Mixed automorphic vector bundles on Shimura varieties*. Pacific J. Math. **173**(1996), 105–126.

[12] T. Miyake, *Modular forms*. Springer-Verlag, Heidelberg, 1989.
[13] G. Shimura, *Introduction to the arithmetic theory automorphic functions*. Princeton Univ. Press, Princeton, 1971.
[14] V. Šokurov, *Holomorphic differential forms of higher degree on Kuga's modular varieties*. Math. USSR Sb. **30**(1976), 119–142.
[15] P. Stiller, *Special values of Dirichlet series, monodromy, and the periods of automorphic forms*. Mem. Amer. Math. Soc. **299**, 1984.

Department of Mathematics
Pohang University of Science and Technology
Pohang, 790–784
Korea
email: yjc@postech.ac.kr

Department of Mathematics
University of Northern Iowa
Cedar Falls, IA 50614
USA
email: lee@math.uni.edu