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APÉRY LIMITS FOR ELLIPTIC *L*-VALUES

CHRISTOPH KOUTSCHAN and WADIM ZUDILIN

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Abstract

For an (irreducible) recurrence equation with coefficients from $\mathbb{Z}[n]$ and its two linearly independent rational solutions u_n, v_n, the limit of u_n/v_n as $n \to \infty$, when it exists, is called the Apéry limit. We give a construction that realises certain quotients of L-values of elliptic curves as Apéry limits.

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Apéry's famous proof [10] of the irrationality of $\zeta(3)$ displayed a particular phenomenon (which could certainly have been dismissed if discussed only in the arithmetic context). One considers the recurrence equation

$$(n+1)^3 v_{n+1} - (2n+1)(17n^2+17n+5)v_n + n^3 v_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots \tag{1}$$

and its two *rational* solutions u_n and v_n, where $n \geq 0$, originating from the initial data $u_0 = 0$, $u_1 = 6$ and $v_0 = 1$, $v_1 = 5$. Then v_n are, in fact, integral for any $n \geq 0$ and the denominators of u_n have a moderate growth as $n \to \infty$, and are certainly not like $n!^3$, as suggested by the recursion, but are $O(C^n)$ for some $C > 1$. In fact, $D_n^3 u_n \in \mathbb{Z}$ for all $n \geq 1$, where D_n denotes the least common multiple of $1, 2, \ldots, n$; the limit $D_n^{1/n} \to e$ as $n \to \infty$ is a consequence of the prime number theorem. An important additional property is that the quotient $u_n/v_n \to \zeta(3)$ as $n \to \infty$ (and also $u_n/v_n \neq \zeta(3)$ for *all* n). Even sharper, $v_n\zeta(3) - u_n \to 0$ as $n \to \infty$, and, at the highest level of sharpness, $D_n^3(v_n\zeta(3) - u_n) \to 0$ as $n \to \infty$. It is the latter sharpest form that leads to the conclusion $\zeta(3) \notin \mathbb{Q}$. But already the arithmetic properties of u_n, v_n coupled with the 'irrational' limit relation $u_n/v_n \to \zeta(3)$ as $n \to \infty$ are phenomenal.

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One way of proving all the above claims in one go is to recast the sequence $I_n = v_n\zeta(3) - u_n$ as the Beukers triple integral [4]

$$I_n = \frac{1}{2}\int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}}\,dx\,dy\,dz \quad \text{for } n = 1, 2, \ldots.$$

A routine use of creative telescoping machinery, based on the Almkvist–Zeilberger algorithm [2] (in fact, its multivariable version [3]), then shows that I_n indeed satisfies (1), while the evaluations $I_0 = \zeta(3)$ and $I_1 = 5\zeta(3) - 6$ are straightforward. The arithmetic and analytic properties follow from the analysis of the integrals I_n performed in [4]; more *practically*, they can be predicted and checked numerically based on the recurrence equation (1).

A common belief is that we have a better understanding of the phenomenon these days. Namely, we possess some (highly nonsystematic!) recipes and strategies (see, for example, [1, 6, 7, 13, 15, 16]) for getting other meaningful constants c as *Apéry limits*. In other words, there are (irreducible) recurrence equations with coefficients from $\mathbb{Z}[n]$ such that, for two *rational* solutions u_n, v_n, we have $u_n/v_n \to c$ as $n \to \infty$ and the denominators of u_n, v_n are growing at most exponentially in n. (We may also consider *weak* Apéry limits when the latter condition on the growth of denominators is dropped.) Although one would definitely like to draw some conclusions about the irrationality of those constants c, this constraint for the arithmetic to be in the sharpest form would severely shorten the existing list of known Apéry limits; for example, it would throw out Catalan's constant from the list. A very basic question is then as follows.

QUESTION 1. What real numbers can be realised as Apéry limits?

Without going into this in any detail, we present here a ('weak') construction of Apéry limits which are related to the L-values of elliptic curves (or of weight two modular forms). The construction emanates from identities, most of which remain conjectural, between the L-values and Mahler measures.

Consider the family of double integrals

$$\begin{aligned} J_n(z) &= \int_0^1\int_0^1 \frac{x^{n-1/2}(1-x)^{n-1/2}y^{n-1/2}(1-y)^n}{(1-zxy)^{n+1/2}}\,dx\,dy \\ &= \frac{\Gamma(n+\frac{1}{2})^3\Gamma(n+1)}{\Gamma(2n+1)\Gamma(2n+\frac{3}{2})}\cdot {}_3F_2\left(\begin{matrix} n+\frac{1}{2},\ n+\frac{1}{2},\ n+\frac{1}{2} \\ 2n+1,\ 2n+\frac{3}{2} \end{matrix}\,\middle|\, z\right). \end{aligned}$$

Thanks to the nice hypergeometric representation, a recurrence equation satisfied by the double integral can be computed using Zeilberger's fast summation algorithm [3, 14], which is based on the method of creative telescoping. It leads to the third-order recurrence equation:

$$\begin{aligned} &4z^4(2n+1)^2(n+1)^2(16(27z-32)n^4 - 16(69z-86)n^3 \\ &\quad + 8(108z-143)n^2 - 4(55z-76)n + 3(7z-10))J_{n+1} \end{aligned}$$

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$$
\begin{aligned}
&+ z^2(256(3z+8)(27z-32)n^8 - 256(3z+8)(15z-22)n^7 \\
&\quad - 64(651z^2+661z-1744)n^6 + 192(59z^2-186)n^5 \\
&\quad + 16(1503z^2+697z-3610)n^4 - 16(79z^2-290z+116)n^3 \\
&\quad - 4(569z^2-381z-580)n^2 + 4(11z^2-44z+18)n + 3(4z+3)(7z-10))J_n \\
&+ 4n(64(3z^2-20z+16)(27z-32)n^7 - 384(3z^2-20z+16)(7z-9)n^6 \\
&\quad - 16(411z^3-2698z^2+3988z-1696)n^5 + 64(183z^3-1372z^2+2339z-1134)n^4 \\
&\quad + 4(531z^3-1400z^2-424z+1240)n^3 - 8(571z^3-4001z^2+6532z-3060)n^2 \\
&\quad + (151z^3-4742z^2+11596z-6888)n + 12(14z^2-29z-30)(z-1))J_{n-1} \\
&+ 4n(n-1)(2n-3)^2(z-1)(16(27z-32)n^4 + 48(13z-14)n^3 \\
&\quad + 8(18z-11)n^2 - 4(19z-24)n - (7z+6))J_{n-2} = 0.
\end{aligned}
$$

Furthermore, if we take

$$
\lambda(z) = J_0(z) = 2\pi\, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2}\end{matrix} \,\middle|\, z\right) = \int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{x(1-x)y(1-zxy)}},
$$

$$
\rho_1(z) = \pi\, {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix} \,\middle|\, z\right) = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-zx)}},
$$

$$
\rho_2(z) = \pi\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix} \,\middle|\, z\right) = \int_0^1 \frac{\sqrt{1-zx}}{\sqrt{x(1-x)}}\,dx,
$$

then $J_0(z) = \lambda(z)$,

$$
J_1(z) = -\frac{3+4z}{4z^2}\lambda - \frac{5(1-z)}{z^2}\rho_1 + \frac{13}{2z^2}\rho_2,
$$

$$
J_2(z) = \frac{105+480z+64z^2}{64z^4}\lambda + \frac{3151-2167z-984z^2}{144z^4}\rho_1 - \frac{7247+3452z}{288z^4}\rho_2.
$$

In other words, each $J_n(z)$ is a $\mathbb{Q}(z)$-linear combination of $\lambda(z), \rho_1(z), \rho_2(z)$. For $z^{-1} \in \mathbb{Z} \setminus \{\pm 1\}$, we find experimentally that the coefficients a_n, b_n, c_n (depending, of course, on this z^{-1}) in the representation

$$
J_n(z) = a_n\lambda(z) + b_n\rho_1(z) + c_n\rho_2(z)
$$

satisfy

$$
z^n 2^{4n} a_n,\ z^n 2^{4n} D_{2n}^2 b_n,\ z^n 2^{4n} D_{2n}^2 c_n \in \mathbb{Z} \quad \text{for } n = 0, 1, 2, \ldots.
$$

Now observe that

$$
\det\begin{pmatrix} J_n & J_{n+1} \\ c_n & c_{n+1}\end{pmatrix} = \det\begin{pmatrix} a_n & a_{n+1} \\ c_n & c_{n+1}\end{pmatrix} \cdot \lambda(z) + \det\begin{pmatrix} b_n & b_{n+1} \\ c_n & c_{n+1}\end{pmatrix} \cdot \rho_1(z)
$$

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for $n = 0, 1, 2, \ldots$. The sequences

$$A_n = \det\begin{pmatrix} a_n & a_{n+1} \\ c_n & c_{n+1} \end{pmatrix} \quad \text{and} \quad B_n = -\det\begin{pmatrix} b_n & b_{n+1} \\ c_n & c_{n+1} \end{pmatrix}$$

satisfy the following third-order (again!) recurrence equation which is the exterior square of the recurrence for J_n:

$$\begin{aligned}
&4(n+1)(n+2)^2(2n+1)^2(2n+3)^2z^8p_0(n)p_0(n-1)A_{n+1} \\
&\quad - 4(n+1)^2(2n+1)^2z^4p_0(n-1)(64(3z^2-20z+16)(27z-32)n^7 \\
&\qquad + 64(3z^2-20z+16)(147z-170)n^6 + 16(3369z^3-26678z^2+44012z-20576)n^5 \\
&\qquad + 16(2457z^3-20918z^2+34376z-15896)n^4 \\
&\qquad + 4(843z^3-16808z^2+29432z-13736)n^3 - 4(1445z^3-6794z^2+9600z-4144)n^2 \\
&\qquad - (741z^3-6922z^2+10772z-4728)n + z^2(131z-66))A_n \\
&\quad - n(2n-1)^2(1-z)z^2p_0(n+1)(256(3z+8)(27z-32)n^8 \\
&\qquad - 256(3z+8)(15z-22)n^7 - 64(651z^2+661z-1744)n^6 + 192(59z^2-186)n^5 \\
&\qquad + 16(1503z^2+697z-3610)n^4 - 16(79z^2-290z+116)n^3 \\
&\qquad - 4(569z^2-381z-580)n^2 + 4(11z^2-44z+18)n + 3(4z+3)(7z-10))A_{n-1} \\
&\quad - 4(n-1)n^2(2n-3)^2(2n-1)^2(1-z)^2p_0(n)p_0(n+1)A_{n-2} = 0,
\end{aligned}$$

where

$$p_0(n) = 16(27z-32)n^4 + 48(13z-14)n^3 + 8(18z-11)n^2 - 4(19z-24)n - (7z+6),$$

and

$$A_0 = \frac{13}{2z^2}, \quad A_1 = \frac{395z^2-1051z+591}{72z^6},$$

$$A_2 = \frac{15196z^4-201551z^3+548091z^2-543600z+183120}{3600z^{10}}$$

and

$$B_0 = 0, \quad B_1 = \frac{1117z^2-2299z+1182}{72z^6},$$

$$B_2 = \frac{6867z^4-65547z^3+156430z^2-143530z+45780}{450z^{10}}.$$

Furthermore, by construction,

$$\lim_{n\to\infty} \frac{B_n}{A_n} = \frac{\lambda}{\rho_1}$$

and, still only experimentally and for $z^{-1} \in \mathbb{Z} \setminus \{\pm 1\}$,

$$z^{2n+2}2^{2n}D_{2n}(n+1)(2n+1)^2A_n, \; z^{2n+2}2^{2n}D_{2n}^2(n+1)(2n+1)^2B_n \in \mathbb{Z}$$

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for $n = 0, 1, 2, \ldots$. In other words, the number λ/ρ_1 (but also the quotients λ/ρ_2 and ρ_1/ρ_2) are (weak) Apéry limits for the values of z under consideration.

For real $k > 0$ with $k^2 \in \mathbb{Z} \setminus \{0, 16\}$, the Mahler measure

$$\mu(k) = \mathrm{m}(X + X^{-1} + Y + Y^{-1} + k) = \frac{1}{(2\pi i)^2} \iint\limits_{|X|=|Y|=1} \log|X + X^{-1} + Y + Y^{-1} + k| \frac{dX}{X} \frac{dY}{Y}$$

is expected to be rationally proportional to the L-value

$$L'(E, 0) = \frac{N}{(2\pi)^2} L(E, 2)$$

of the elliptic curve $E = E_k : X + X^{-1} + Y + Y^{-1} + k = 0$ of conductor $N = N_k = N(E_k)$. This is actually proven [5] when $k = 1, \sqrt{2}, 2, 2\sqrt{2}$ and 3 for the corresponding elliptic curves `15a8`, `56a1`, `24a4`, `32a1` and `21a4` labelled in accordance with the database [9]; the first number in the label indicates the conductor.

For the range $0 < k < 4$,

$$\mu(k) = \frac{k}{4} \cdot {}_3F_2\biggl(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \biggm| \frac{k^2}{16}\biggr),$$

which thus links $\mu(k)$ to $z^{-1/2}\lambda(z)/\pi$ at $z = k^2/16$. Furthermore, the quantity $z^{-1/2}\rho_1(z)$ in this case is rationally proportional to the imaginary part of the nonreal period of the same curve, while $z^{-1/2}\rho_2(z)$ is a $\mathbb{Q}$-linear combination of the imaginary parts of the nonreal period and the corresponding quasiperiod. This means that, in many cases, we can record $z^{-1/2}\rho_1(z)$ as a rational multiple of the central L-value of a quadratic twist of the curve E. For example, when $k = 2\sqrt{2}$ (and hence $z = 1/2$) the quadratic twist of the elliptic curve of conductor 32 coincides with itself and

$$\lambda\Bigl(\frac{1}{2}\Bigr) = 2\sqrt{2}\,\pi L'(E, 0) = 16\sqrt{2}\,\frac{L(E, 2)}{\pi} \quad \text{and} \quad \rho_1\Bigl(\frac{1}{2}\Bigr) = 4\sqrt{2}\,L(E, 1).$$

From this, we see that the last recursion above with the choice $z = 1/2$ realises the quotient $L(E, 2)/(\pi L(E, 1))$ as an Apéry limit for an elliptic curve of conductor 32. When $k = 1$,

$$\lambda\Bigl(\frac{1}{16}\Bigr) = 8\pi L'(E, 0) = 30\,\frac{L(E, 2)}{\pi} \quad \text{and} \quad \rho_1\Bigl(\frac{1}{16}\Bigr) = \frac{1}{2} L(E, \chi_{-4}, 1)$$

for the twist of the elliptic curve by the quadratic character $\chi_{-4} = (\frac{-4}{\cdot})$; this means that the quotient $L(E, 2)/(\pi L(E, \chi_{-4}, 1))$ for an elliptic curve of conductor 15 is realised as an Apéry limit.

Clearly, the range $0 < k < 4$ has a limited supply of elliptic L-values. When $k > 4$, one can write

$$\mu(k) = \frac{1}{2\pi} f\Bigl(\frac{16}{k^2}\Bigr),$$

where

$$f(z) = -\pi\left(\log\frac{z}{16} + \frac{z}{4}\,{}_4F_3\left(\begin{matrix}\frac{3}{2},\ \frac{3}{2},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,z\right)\right)$$

$$= -\int_0^1 x^{-1/2}(1-x)^{-1/2}\log\frac{1-\sqrt{1-zx}}{1+\sqrt{1-zx}}\,dx$$

$$= \int_0^1\int_0^1 \frac{x^{-1/2}(1-x)^{-1/2}(1-zx)^{1/2}y^{-1/2}}{1-(1-zx)y}\,dx\,dy$$

$$= Z\int_0^1\int_0^1 \frac{x^{-1/2}(1-x)^{-1/2}(1-x/Z)^{1/2}(1-y)^{-1/2}}{x(1-y)+yZ}\,dx\,dy,$$

with $Z = z^{-1} > 1$. At this point, we see that the integrals resemble the integrals

$$Z^{-l-m}\int_0^1\int_0^1 \frac{x^j(1-x)^h y^k(1-y)^l}{(x(1-y)+yZ)^{j+k-m+1}}\,dx\,dy,$$

where h, j, k, l, m are nonnegative integers, appearing in the linear independence results for the dilogarithm [11, 12]. This similarity suggests looking at the family

$$L_n(Z) = \int_0^1\int_0^1 \frac{x^{n-1/2}(1-x)^{2n-1/2}(1-x/Z)^{1/2}y^n(1-y)^{n-1/2}}{(x(1-y)+yZ)^{n+1}}\,dx\,dy,$$

where $Z = z^{-1}$ is a large (positive) integer. We tackle this double integral by iterated applications of creative telescoping: while the first integration (regardless of whether one starts with x or with y) can be done with the Almkvist–Zeilberger algorithm, the second one requires more general holonomic methods, since the integrand is no longer hyperexponential. Using the `Mathematica` package `HolonomicFunctions` [8], where these algorithms are implemented, we find that the integral $L_n(Z)$ satisfies a lengthy fourth-order recurrence equation. Moreover, it turns out that $L_n(Z)$ is a $\mathbb{Q}(Z)$-linear combination of $\rho_1 = \rho_1(1/Z)$, $\rho_2 = \rho_2(1/Z)$, $\sigma_1 = L_0(Z)$ and

$$\sigma_2 = \sigma_2(Z) = \int_0^1\int_0^1 \frac{x^{-1/2}(1-x)^{1/2}(1-x/Z)^{1/2}(1-y)^{1/2}}{x(1-y)+yZ}\,dx\,dy.$$

One can produce a recurrence equation out of the one for $L_n(Z)$ to cast, for example, σ_1/ρ_1 as an Apéry limit. Because this finding does not meet any reasonable aesthetic requirements and does not imply anything (to be claimed) irrational, we do not include it in this article.

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CHRISTOPH KOUTSCHAN,
Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences, Altenberger Straße 69, A-4040 Linz, Austria
e-mail: christoph.koutschan@ricam.oeaw.ac.at

WADIM ZUDILIN, Department of Mathematics, IMAPP,
Radboud University PO Box 9010, 6500 GL, Nijmegen, Netherlands
e-mail: w.zudilin@math.ru.nl