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ON THE EXISTENCE OF OPTIMAL CONTROLS IN BANACH SPACES

MOHAN JOSHI

We prove the existence of an optimal control in Banach spaces for a system characterized by Hammerstein operator equations.

1. Introduction

Let X be a Banach space with dual X^* and $Y \subset X^*$ a closed subspace. Let Z be another Banach space and $U \subset Z$ be a weakly compact set. We consider a system characterized by the operator equation

$$x + K_u Nx = w \tag{1.1}$$

where $u \in U$ is a control variable and for each $u \in U$, $K_u : X \to X^*$ is a linear operator with range in Y and $N : Y \to X$ a nonlinear operator. $w \in X^*$ is given. For a fixed u, the operator equation (1.1) is called the Hammerstein operator equation. Existence and uniqueness of solutions for such types of equations have been studied by many authors (refer Browder [2]).

Let ϕ be a lower semicontinuous functional on X^* with values in R^+ and S a closed and bounded subset of X. The problem is to find a control $u^* \in U$ such that

$$J(u) = \phi(x) \tag{1.2}$$

is minimum, subject to the constraint that $x \in S$ is the response of the system (1.1) corresponding to the control u.

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In this paper we prove existence of an optimal control under continuity assumptions on $\{K_u\}_{u\in U}$ and some mild assumptions on N. As a corollary we derive the existence of a control function u^* for an important system characterized by the equation

$$x(t) + \int_0^t f(\tau, x(\tau))u(\tau)d\tau = x_0 \tag{1.3}$$

with the associated functional $J(u)$ given by

$$J(u) = \phi(x, u) = \int_0^T g(t, x(t), u(t))dt , \tag{1.4}$$

where $T \in [0, \infty)$ is prescribed.

2. Main results

LEMMA 2.1. *Let X be a real reflexive Banach space with dual X^* and $Y \subset X^*$ a closed subspace. Let Z be another Banach space and $S \subset X^*$ be a bounded set. Let, for $u \in Z$, $K_u : X \to X$ be a bounded linear operator with range in Y and $N : Y \to X$ a continuous and bounded nonlinear operator. Further assume that the following hold:*

(a) K_u is compact for each $u \in Z$;

(b) $u_n \rightharpoonup u$ in Z implies that $K_{u_n} \to K_u$ in operator norm.

Let $\{u_n\}$ be any sequence in Z which converges weakly to u^* in Z and let $x_n \in S$ denote a solution of (1.1) *corresponding to u_n. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to x^* and x^* is a solution of* (1.1) *corresponding to u^*.*

Proof. Since $x_n \in S$ is a solution of (1.1) corresponding to u_n, we have

$$x_n + K_{u_n}Nx_n = w . \tag{2.1}$$

$\{x_n\}$ is a bounded sequence in a reflexive Banach space and hence there

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exists a subsequence $\{x_{n_k}\}$ of it which converges to x^* weakly.

Similarly boundedness of N implies that there exists a subsequence of $\{Nx_{n_k}\}$ (which we again denoted by $\{Nx_{n_k}\}$) converging to y weakly.

Since $K_{u_{n_k}} \to K_{u^*}$ in operator norm and $\{Nx_{n_k}\}$ is bounded, it follows that

(2.2) $$K_{u_{n_k}} Nx_{n_k} - K_{u^*} Nx_{n_k} \to 0 \text{ as } k \to \infty .$$

Also, since $Nx_{n_k} \rightharpoonup y$ and K_{u^*} is compact, we have

(2.3) $$K_{u^*} Nx_{n_k} \to K_{u^*} y .$$

Combining (2.2) and (2.3) we get that $K_{u_{n_k}} Nx_{n_k} \to K_{u^*} y$ as $k \to \infty$. But (2.1) gives

(2.4) $$x_{n_k} = w - K_{u_{n_k}} Nx_{n_k}$$

and hence $\{x_{n_k}\}$ is strongly convergent, that is $x_{n_k} \to x^*$. Now continuity of N implies that $Nx_{n_k} \to Nx^*$ and hence $K_{u_{n_k}} Nx_{n_k} \to K_{u^*} Nx^*$.

So (2.4) gives

$$x^* + K_{u^*} Nx^* = w .$$

That is $\{x_{n_k}\}$ converges strongly to x^* where x^* is a solution of the system (1.1) corresponding to u^*.

DEFINITION 2.1. Let the spaces X, Y, Z be as in the above lemma. Let $U \subset Z$ be a weakly compact subset. $X = \{x_u : u \in U\}$ is said to be the set of trajectories of (1.1) if

(i) $X \neq \emptyset$,

(ii) $x_u \in Y$,

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(iii) x_u satisfies the operator equation

$$x_u + K_u N x_u = w .$$

DEFINITION 2.2. $F \subset Y$ is said to be an attainable set of the system (1.1) if $F = \{y : y = x \text{ for some } x \text{ in } X\}$.

THEOREM 2.1. *Let F be an attainable set of the system* (1.1) *and* S *a closed and bounded subset of Y such that $F \cap S \neq \emptyset$. Let ϕ be a lower semicontinuous functional on X^* with values in R^+. Then there exists $u^* \in U$ where it attains a minimum to the functional ϕ on the set $F \cap S$.*

Proof. By Lemma 2.1, $F \cap S$ is a compact set. Since ϕ is a lower semicontinuous functional and $F \cap S$ is compact, it follows that ϕ attains its minimum on $F \cap S$. Hence the result.

As a corollary of the above theorem we obtain the existence of an optimal control for the system characterized by the equation

$$x(t) + \int_0^t f(\tau, x(\tau))u(\tau)d\tau = x_0 , \tag{2.5}$$

where the control function $u(t)$ lies in some weakly compact subset of $L_2^m[0, T]$, $0 \leq T < \infty$. The associated cost functional $J(u)$ to be minimized is given by

$$J(u) = \phi(x) = \int_0^T g(\tau, x(\tau), u(\tau))d\tau . \tag{2.6}$$

ASSUMPTION [A]. $f(t, x) : R \times R^n \to R^{n\times m}$ is such that

[A1] $f(t, x)$ is measurable in t for all $x \in R^n$,

[A2] $f(t, x)$ is continuous in x for almost all $t \in [0, T]$,

[A3] $\|f(t, x)\| \leq \|a(t)\| + b\|x\|$, $a(t) \in L_2^{n\times m}[0, T]$, $b > 0$

for all $(t, x) \in R \times R^n$.

Here the norm in the left hand side denotes the $R^{n\times m}$ norm and the norm in the right hand side denotes the R^n norm.

ASSUMPTION [B]. $g(t, x, u) : R \times R^n \times R^m \to R$ is such that

[B1] $g(t, x, u)$ is measurable in t for all $(x, u) \in R^n \times R^m$,

[B2] $g(t, x, u)$ is continuous in $(x, u) \in R^n \times R^m$ for almost all $t \in [0, T]$,

[B3] there exists $\Psi \in L_1[0, T]$ such that $g(t, x, u) \geq \Psi(t)$ for almost all $t \in [0, T]$ and all $(x, u) \in R^n \times R^m$,

[B4] $g(t, x, u)$ is convex in u for all t, x.

We set $X = L_2^{n\times m}[0, T]$, $Y = L_2^n[0, T]$, $Z = L_2^m[0, T]$. For each $u \in Z$, define $K_u : X \to X$ with range in Y as follows:

(2.7) $$[K_u x](t) = \int_0^t x(\tau)u(\tau)d\tau .$$

$N : Y \to X$ is defined as

(2.8) $$[Nx](t) = f(t, x(t)) .$$

With these definitions, (2.5) is equivalent to the operator equation

$$x + K_u Nx = x_0 .$$

LEMMA 2.2. *K_u is a bounded linear operator from X into itself with range Y such that*

(a) K_u is compact for each $u \in Z$,

(b) $u_n \rightharpoonup u$ in Z implies that $K_{u_n} \to K_u$.

Proof. $[K_u x](t) = \int_0^t x(\tau)u(\tau)d\tau$, where $x \in L_2^{n\times m}[0, T]$, $u \in L_2^m[0, T]$. Let

$$K(t, \tau) = \begin{cases} 0 , & t \leq \tau , \\ I , & \tau < t . \end{cases}$$

Then we get

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$$[K_u x](t) = \int_0^T [K(t, \tau)x(\tau)]u(\tau)d\tau .$$

Since $\text{ess sup} \int_0^T \|K(t, \tau)\|^2 \|u(\tau)\|^2 d\tau < \infty$ it follows by the theory of integral operators (refer Okikiolu [3]) that K_u is compact for each $u \in L_2^m[0, T]$. Similarly one can show that K_u is compact with respect to the variable u and hence by using the uniform boundedness principle we get the result.

LEMMA 2.3. *Under Assumption* [A] *the nonlinear operator* N *is a continuous and bounded operator from* $L_2^n[0, T]$ *to* $L_2^{n\times m}[0, T]$.

COROLLARY. *Let* f *and* g *satisfy Assumptions* [A] *and* [B] *respectively. Let* U *be a weakly compact subset of* $L_2^m[0, T]$. *For* $u \in U$, *let* (2.5) *possess a solution in a closed and bounded set* $S \subset L_2^n[0, T]$. *Then there exists* $u^* \in U$ *such that*

$$J(u^*) = \inf J(u)$$

where $J(u)$ *is the cost functional given by* (2.6).

Proof. We set $X = L_2^{n\times m}$, $Y = L_2^n[0, T]$, $Z = L_2^m[0, T]$ and K_u and N be as defined before. Then (2.5) is equivalent to the operator equation

$$x + K_u Nx = x_0 .$$

Let F denote the attainable set of (2.5). Then by assumption $F \cap S \neq \emptyset$. Further, by a result of Berkovitz [1], ϕ is lower semi-continuous with respect to weak convergence in u and strong convergence in x . Since all the conditions of Theorem 2.1 are satisfied, it follows that there exist $u^* \in U$ such that

$$J(u^*) = \inf_{u\in U} J(u) .$$

REMARK. Vidyasagar [4] has proved a similar result for the system (2.5). However he imposes a Lipschitz condition on $f(t, x)$ assume a simple growth condition of type [A3]. This is a significant

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improvement.

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Mathematics Group,
BITS,
Pilani 333031,
India.

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