Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1191–1197

DISTRIBUTIONAL WATSON TRANSFORMS

HSING-YUAN HSU

1. Introduction. All our notation is as defined in [**2**] with the restriction to $n = 1$. However, for our purposes, we introduce a sequence $\{||\cdot||_p\}_{p=0}^{\infty}$ of norms by

$$||\varphi||_p = \max_{0 \leqq k \leqq p} \sup_{x \in R^+} |\xi_{a,b}(x) x^{k+1} D^k \varphi(x)|$$

in $\mathscr{M}_{a,b}$. It is not difficult to see that $\mathscr{M}_{a,b}$ turns out to be a fundamental space.

It is a well-known fact that the Watson transform and the Mellin transform are connected by the fact that

$$g(x) = \int_0^\infty f(t) k(xt) dt$$

and

$$f(x) = \int_0^\infty g(t) k(xt) dt$$

if and only if $K(s)K(1 - s) = 1$, where $K(s)$ is the Mellin transform of $k(x)$. Further, the Hankel transform and Hilbert transform can be considered as special cases of Watson transforms.

In this paper we extend these transformations so that they may be applied to Schwartz distributions, and we study their interrelationship.

2. Distributional Watson transforms.

THEOREM 1. *Assume that $\mathscr{M}_{a,b}$ is constructed as above with $a + b = 1$ and $k \in \mathscr{M}_{a,b}'$, $k \neq 0$, with*

$$\{s \in K | a \leqq \text{Re } s \leqq b\} \subset \Omega_k.$$

For each $f \in \mathscr{M}_{a,b}'$, $f \neq 0$, with $\{s \in K | a \leqq \text{Re } s \leqq b\}$ in Ω_f, it makes sense to define $g = f \wedge k$ so that $g \in \mathscr{M}_{a,b}'$ with $\{s \in K | a \leqq \text{Re } s \leqq b\} \subset \Omega_g$.

Then $g = f \wedge k$ implies $f = g \wedge k$ if and only if $K(s)K(1 - s) = 1$ for $a \leqq \text{Re } s \leqq b$, where $K(s) = (\mathscr{M}k)(s)$, the distributional Mellin transform of k (see [**2**] for the definitions).

Proof. Suppose that $K(s)K(1 - s) = 1$. For $g = f \wedge k$,

$$G(s) = F(1 - s)K(s), \quad \text{where} \quad G(s) = (\mathscr{M}g)(s).\ F(s) = (\mathscr{M}f)(s).$$

Then $G(1 - s) = F(s)K(1 - s)$. Set $f_1 = g \wedge k$. Then

$$F_1(s) = G(1 - s)K(s) = F(s)K(1 - s)K(s) = F(s),$$

Received November 18, 1971.

where $F_1(s) = (\mathscr{M}f_1)(s)$. By the uniqueness theorem in [**2**], $f = f_1$. Hence $f = g \wedge k$.

Conversely, suppose that $g = f \wedge k \Rightarrow f = g \wedge k$. Then

$$G(s) = F(1-s)K(s) \quad \text{and} \quad F(s) = G(1-s)K(s).$$

Thus $G(s) = G(s)K(s)K(1-s)$. Hence $K(s)K(1-s) = 1$.

Remark. Suppose that f, k are locally integrable and $f/\xi_{a-1,b-1}$, $k/\xi_{a-1,b-1}$ are absolutely integrable on R_+.

Then g is also locally integrable and $g/\xi_{a-1,b-1}$ is absolutely integrable on R_+, and so

$$g(y) = \int_0^\infty f(x)k(xy)dx \quad \text{and} \quad f(y) = \int_0^\infty g(x)k(xy)dx.$$

In this way, we get the classical Watson transform again.

Therefore, let us call the mapping W_k defined by $W_k : f \to g = f \wedge k$ the Watson transformation; and the inverse mapping W_k^{-1} is given by $W_k^{-1} : g \to f = g \wedge k$.

Further, we call such k a distributional Watson kernel.

It is not difficult to prove the following properties:

(1) Suppose that f is a distributional Watson kernel. Then $W_k(f) = g$ is also a distributional Watson kernel.

(2) Suppose that $k_1, k_2 \in \mathscr{M}_{a,b}'$, $a + b = 1$, are two distributional Watson kernels. Then $k_1 \wedge k_2$, $k_1 \vee k_2$ (see [**2**] for the definition) are also distributional Watson kernels.

(3) Suppose that $\{k_n\}_{n=1}^\infty$ is a sequence of distributional Watson kernels in $\mathscr{M}_{a,b}'$ such that $k_n \to k$ in $\mathscr{M}_{a,b}'$ as n $\to \infty$, where $a + b = 1$. Then k is also a distributional Watson kernel.

Example. Consider $\delta(x)$. It is clear that $\delta \in \mathscr{M}_{a,b}'$ for all $a, b \in R$, $a < b$. Suppose that $a + b = 1$ and set $\delta_c(x) = \delta(x - c)$. Then $(\mathscr{M}\delta_1)(s) = (\delta(x-1), x^{s-1}) = 1^{s-1} = 1$ and $(\mathscr{M}\delta_1)(1-s) = 1$. Hence

$$(\mathscr{M}\delta_1)(s)(\mathscr{M}\delta_1)(1-s) = 1.$$

Therefore δ_1 is a distributional Watson kernel. Suppose that $f \in \mathscr{M}_{a,b}'$ with $\{s \in K | a \leqq \operatorname{Re} s \leqq b\} \subset \Omega_f$. Set $g = f \wedge \delta_1$. Then for all $\varphi \in \mathscr{M}_{a,b}$, we have

$$\begin{aligned}(g(x), \varphi(x)) &= (f(x), (\delta_1(y), x^{-1}\varphi(y/x))) \\ &= (f(x), (\delta(y-1), x^{-1}\varphi(y/x))) \\ &= (f(x), x^{-1}\varphi(1/x)) \\ &= (x^{-1}f(1/x), \varphi(x)).\end{aligned}$$

Hence $g(x) = (1/x)f(1/x)$. Similarly, $f(x) = (1/x)g(1/x)$.

https://doi.org/10.4153/CJM-1972-129-5 Published online by Cambridge University Press

3. Distributional Hankel transforms.

THEOREM 2. *Let* $V \in K$ *with* Re $V > -1$. *Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are, respectively, a strictly monotonically decreasing sequence and a strictly monotonically increasing sequence in* R *such that*

(1) $-(V + \frac{1}{2}) < a_m < b_n < V + \frac{3}{2}$ *for* $m, n = 1, 2, 3, \ldots,$

(2) $a_n + b_n = 1$ *for* $n = 1, 2, 3, \ldots,$

(3) $a_n \to -(V + \frac{1}{2}),\ b_n \to V + \frac{3}{2}$ *as* $n \to \infty$.

Set

$$\mathscr{H}_V = \bigcup_{n=1}^{\infty} \mathscr{M}_{a_n, b_n}.$$

Then $\mathscr{H}_V$ *is a fundamental space with*

$$\mathscr{H}_V' = \bigcap_{n=1}^{\infty} \mathscr{M}_{a_n, b_n}'$$

as its dual.

Proof. By the construction of $\mathscr{M}_{a_n,b_n}$ for $n = 1, 2, 3, \ldots,$ each $\mathscr{M}_{a_n,b_n}$ is a complete countably normed space.

It is easy to see that $\{\mathscr{M}_{a_n,b_n}\}_{n=1}^{\infty}$ is an increasing sequence and the topology of each $\mathscr{M}_{a_b,b_n}$ is stronger than the topology induced by $\mathscr{M}_{a_{n+1},b_{n+1}}$.

Hence $\mathscr{H}_V$ is a countable union space of the spaces $\mathscr{M}_{a_n,b_n}$. Further, $\varphi_m \to 0$ in $\mathscr{H}_V$ as $m \to \infty$ implies $\varphi_m \to 0$ in $\mathscr{M}_{a_n,b_n}$ for some n as $m \to \infty$. Hence $\varphi_m(x) \to 0$ as $m \to \infty$ on R_+. Therefore $\mathscr{H}_V$ is a fundamental space. Plainly, $\{\mathscr{M}_{a_n,b_n}'\}_{n=1}^{\infty}$ is decreasing so that the dual $\mathscr{H}_V'$ of $\mathscr{H}_V$ is equal to $\bigcap_{n=1}^{\infty} \mathscr{M}_{a_n,b_n}'$.

THEOREM 3. *Assume that* $k(x) = x^{\frac{1}{2}} J_V(x)$, *where* $J_V(x)$ *is the Bessel function of order* V.

Then the mapping $H_V : \mathscr{H}_V' \to \mathscr{H}_V'$ *by* $g = H_V(f) = f \wedge k$ *is one-one, onto, and continuous. The inverse mapping is given by* $f = \mathscr{H}_V^{-1}(g) = g \wedge k$ *which is also continuous.*

Proof. $k(x)$ is clearly locally integrable and $\dfrac{k(x)}{\xi_{a_{n-1},b_{n-1}}(x)}$ is absolutely integrable on R_+ for $n = 1, 2, 3, \ldots$ so that $k(x)$ is an $\mathscr{H}_V$ regular distribution.

Let

$$f_1(x) = \begin{cases} (1 - x^2)^{V - \frac{1}{2}}, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

and

$$g_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Gamma(a)(\cos \tfrac{1}{2}a\pi)x^{-a}, \qquad 0 < \text{Re } a < 1.$$

Then the Fourier cosine transforms $F_{1c}(x), G_{1x}(x)$, respectively, of $f_1(x), g_1(x)$ are

https://doi.org/10.4153/CJM-1972-129-5 Published online by Cambridge University Press

$$\begin{aligned} F_{1c}(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^1 (1-t^2)^{V-\frac{1}{2}} \cos xt\, dt \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \int_0^1 (1-t^2)^{V-\frac{1}{2}} t^{2n} dt \\ &= \frac{1}{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{\Gamma(V+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(V+n+1)} \\ &= \frac{1}{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Gamma(V+\tfrac{1}{2})\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}x)^{2n}}{n!(V+n+1)} \\ &= 2^{V-\frac{1}{2}}\Gamma(V+\tfrac{1}{2})x^{-V}J_V(x), \end{aligned}$$

$$\begin{aligned} G_{1c}(x) &= \frac{2}{\pi}\int_0^\infty \Gamma(a)\cos\tfrac{1}{2}a\pi \cdot t^{-a}\cos xt\,dt \\ &= \frac{2}{\pi}\Gamma(a)(\cos\tfrac{1}{2}a\pi)\Gamma(1-a)(\sin\tfrac{1}{2}a\pi)x^{a-1} \\ &= x^{a-1}. \end{aligned}$$

By Parseval's formula,

$$\int_0^\infty F_{1c}(x)G_{1c}(x)dx = \int_0^\infty f_1(x)g_1(x)dx,$$

that is,

$$2^{V-\frac{1}{2}}\Gamma(V+\tfrac{1}{2})\int_0^\infty J_V(x)x^{a-V-1}dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(a)\cos\tfrac{1}{2}a\pi\int_0^1 (1-x^2)^{V-\frac{1}{2}}x^{-a}dx.$$

Hence

$$\begin{aligned} \int_0^\infty J_V(x)x^{a-V-1}dx &= \frac{1}{\sqrt{\pi}}2^{1-V}\frac{\Gamma(a)}{\Gamma(V+\frac{1}{2})}\cos\tfrac{1}{2}a\pi\int_0^\infty (1-x^2)^{V-\frac{1}{2}}x^{-a}dx \\ &= \frac{2^{-V}}{\sqrt{\pi}}\frac{\Gamma(a)}{\Gamma(V+\frac{1}{2})}\cos\tfrac{1}{2}a\pi\int_0^1 (1-t)^{V-\frac{1}{2}}t^{-a/2-\frac{1}{2}}dt \\ &= \frac{2^{-V}}{\sqrt{\pi}}\frac{\Gamma(a)\Gamma(V+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}a)}{\Gamma(V+\frac{1}{2})\Gamma(V+1-\frac{1}{2}a)}\cos\tfrac{1}{2}a\pi \\ &= \frac{2^{-V}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}a)}{\Gamma(V+1-\frac{1}{2}a)}\cos\tfrac{1}{2}a\pi\,\frac{2^{a-1}\Gamma\left(\frac{a}{2}\right)^2\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\sqrt{\pi}} \\ &= \frac{2^{a-V-1}}{\pi}\frac{\Gamma(\frac{1}{2}a)\cos\frac{1}{2}a\pi}{\Gamma(V+1-\frac{1}{2}a)}\frac{\pi}{\sin\pi(\frac{1}{2}+\frac{1}{2}a)} \\ &= \frac{2^{a-V-1}\Gamma(\frac{1}{2}a)}{\Gamma(V+1-\frac{1}{2}a)},\ 0 < a < V+\tfrac{3}{2}. \end{aligned}$$

Put $s = a - V - \frac{1}{2}$. Then

$$\int_0^\infty x^{\frac{1}{2}} J_V(x) x^{s-1} dx = \frac{2^{(s+\frac{1}{2})-1}\Gamma(\frac{1}{2}(s+\frac{1}{2})+\frac{1}{2}V)}{\Gamma(\frac{1}{2}V - \frac{1}{2}(s+\frac{1}{2})+1)}.$$

Thus

$$K(s) = \frac{2^{s-\frac{1}{2}}\Gamma(\frac{1}{2}s + \frac{1}{4} + \frac{1}{2}V)}{\Gamma(\frac{1}{2}V - \frac{1}{2}s + \frac{3}{4})}$$

and

$$K(1-s) = \frac{2^{\frac{1}{2}-s}\Gamma(\frac{3}{4} - \frac{1}{2}s + \frac{1}{2}V)}{\Gamma(\frac{1}{2}V + \frac{1}{4} + \frac{1}{2}s)} = \frac{1}{K(s)}.$$

Hence $K(s)K(1-s) = 1$. Therefore k is a distributional Watson kernel. Thus, by Theorem 1, H_V is a one-one, onto mapping such that

$$g = f \wedge k \Leftrightarrow f = g \wedge k.$$

It is clear that $f_m \to f$ in $\mathscr{H}_V'$ as $m \to \infty$ and $g_m = f_m \wedge k$ implies $g_m \to g$ in $\mathscr{H}_V'$ as $m \to \infty$, where $g = f \wedge k$. Thus H_V is continuous. Similarly, H_V^{-1} is continuous.

Remark. If f is locally integrable and $f/\xi_{a_n, b_n}$ is absolutely integrable over R_+ for all $n \in N$, then so is g such that

$$g(y) = \int_0^\infty f(x)(xy)^{\frac{1}{2}} J_V(xy) dx,$$

$$f(y) = \int_0^\infty g(x)(xy)^{\frac{1}{2}} J_V(xy) dx$$

which coincides with the classical Hankel transforms.

Thus we call H_V the Hankel transformation and H_V^{-1} the inverse Hankel transformation and $k(x) = x^{\frac{1}{2}} J_V(x)$ the distributional Hankel kernel of Vth order.

4. Distributional Hilbert transforms. Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{k=1}^\infty$ are, respectively, a strictly monotonically decreasing sequence and a strictly monotonically increasing sequence in R such that

(1) $0 < a_m < b_n < 1$ for $m, n = 1, 2, 3, \ldots,$

(2) $a_n + b_n = 1$ for $n = 1, 2, 3, \ldots,$

(3) $a_n \to 0, b_n \to 0$ as $n \to \infty$.

As indicated in Theorem 2, $\mathscr{H}_{0,1} = \bigcup_{n=1}^\infty \mathscr{M}_{a_n, b_n}$ is a fundamental space with $\mathscr{H}_{0,1}' = \bigcap_{n=1}^\infty \mathscr{M}_{a_n, b_n}'$ as its dual.

LEMMA 4. *Suppose that*

$$k(x) = \frac{1}{1-x}.$$

https://doi.org/10.4153/CJM-1972-129-5 Published online by Cambridge University Press

Then $P_V h$ defined by

$$(P_V h, \varphi) = P_V \int_0^\infty \frac{\varphi(x)}{1-x}\, dx$$

for all $\varphi \in \mathscr{H}_{0,1}$ is an $\mathscr{H}_{0,1}$-regular distribution, where

$$P_V \int_0^\infty \frac{\varphi(x)}{1-x}\, dx$$

is the Cauchy principal value of

$$\int_0^\infty \frac{\varphi(x)}{1-x}\, dx.$$

Proof. Let $\varphi \in \mathscr{H}_{0,1}$. Then $\varphi \in \mathscr{M}_{a_n,b_n}$ for some n.

$$\begin{aligned}
\left| P_V \int_0^\infty \frac{\varphi(x)}{1-x}\, dx \right| &= \lim_{\epsilon\to 0} \left| \int_0^{1-\epsilon} \frac{\varphi(x)}{1-x}\, dx + \int_{1+\epsilon}^\infty \frac{\varphi(x)}{1-x}\, dx \right| \\
&= \lim_{\epsilon\to 0} \left| \left(\int_0^{1/e} + \int_{1/e}^{1-\epsilon} + \int_{1+\epsilon}^{e} + \int_e^\infty \right) \frac{\varphi(x)}{1-x}\, dx \right| \\
&\leqq \lim_{\epsilon\to 0} \|\varphi\|_0^{a_n,b_n} \left(\int_0^{1/e} \left| \frac{1}{x^{1-a_n}(1-x)} \right| dx \right. \\
&\quad + \int_{1/e}^{1-\epsilon} \left| \frac{1}{\mathscr{U}(x)(1-x)} \right| dx + \int_{1+\epsilon}^{e} \left| \frac{1}{\mathscr{U}(x)(1-x)} \right| dx \\
&\quad \left. + \int_e^\infty \left| \frac{1}{x^{1-b_n}(1-x)} \right| dx \right) \\
&\leqq \lim_{\epsilon\to 0} \|\varphi\|_0^{a_n,b_n} (M_1 + M \log \epsilon - M \log(1 - 1/e) \\
&\quad + M \log(e-1) - M \log \epsilon + M_2) \\
&= \|\varphi\|_0^{a_n,b_n} (M_1 + M_2 + M) = M_0, \text{ say,}
\end{aligned}$$

where $\mathscr{U}(x)$ is the part of $\xi_{a_n,b_n}(x)$ in $1/e < x < e$, and

$$M = \max |\mathscr{U}(x)|,$$

$$M_1 = \int_0^{1/e} \left| \frac{1}{x^{1-a_n}(1-x)} \right| dx,$$

$$M_2 = \int_0^\infty \left| \frac{1}{x^{1-b_n}(1-x)} \right| dx$$

which are bounded. Hence $P_V h$ is a functional on $\mathscr{H}_{0,1}$. The linearity of it as a functional on $\mathscr{H}_{0,1}$ is clear.

Suppose that $\varphi_m \to 0$ in $\mathscr{H}_{0,1}$ as $m \to \infty$. Then $\varphi_m \to 0$ in $\mathscr{M}_{a_n,b_n}$ for some n as $m \to \infty$. Thus, by an argument quite similar to that given above, we have

$$(P_V h, \varphi_m) \to 0 \text{ as } m \to \infty.$$

Hence $P_V h$ is continuous. Therefore $P_V h$ is an $\mathscr{H}_{0,1}$-regular distribution.

THEOREM 5. *The mapping* $\mathscr{H} = \mathscr{H}_{0,1}' \to \mathscr{H}_{0,1}'$ *defined by* $f \to g = f \wedge k$, *where*

$$k(x) = \frac{2}{\pi} P_V \frac{1}{1 - x^2},$$

is one-one, onto, and continuous. The inverse mapping $\mathscr{H}^{-1}$ *is given by* $\mathscr{H}^{-1} : g \to f = g \wedge k$ *which is also continuous.*

Proof. Plainly, $1/(1 - x)$ is an $\mathscr{H}_{0,1}$-regular distribution.

Now,

$$\begin{aligned} k(s) &= \frac{2}{\pi} P_V \int_0^\infty \frac{x^{s-1}}{1 - x^2} dx \\ &= \frac{2}{\pi} P_V \int_0^\infty \frac{y^{\frac{1}{2}(s-1)}}{1 - y} \frac{dy}{2y^{\frac{1}{2}}}, y = x^2, \\ &= \frac{2}{\pi} P_V \int_0^\infty \frac{y^{\frac{1}{2}s-1}}{1 - y} dy \\ &= \cot \tfrac{1}{2}s\pi. \end{aligned}$$

$$K(1 - s) = \cot \tfrac{1}{2}(1 - s)\pi = \tan \tfrac{1}{2}s\pi.$$

Thus $K(s)K(1 - s) = \cot \frac{1}{2}s\pi \tan \frac{1}{2}s\pi = 1$. Hence $k(x)$ is a distributional Watson kernel. By Theorem 1, $\mathscr{H}$ is one-one and onto. It is clear that $f_m \to f$ in $\mathscr{H}_{0,1}'$ as $m \to \infty$; and $g_m = f_m \wedge k$ implies $g_m \to g = f \wedge k$ in $\mathscr{H}_{0,1}'$ as $m \to \infty$. Hence $\mathscr{H}$ is continuous. Similarly, $\mathscr{H}^{-1}$ is given by $\mathscr{H}^{-1} : g \to f = g \wedge k$ which is also continuous.

Remark. If f is locally integrable and $f/\xi_{a_{n-1}, b_{n-1}}$ is absolutely integrable over R_+ for all $n \in N$, then $\mathscr{H}$ would coincide with the classical Hilbert transformation. Thus we call $\mathscr{H}$ the Hilbert transformation and $\mathscr{H}^{-1}$ the inverse Hilbert transformation and

$$k(x) = \frac{2}{\pi} P_V \frac{1}{1 - x^2}$$

the distributional Hilbert kernel.

REFERENCES

1. E. C. Titchmarsch, *Introduction to the theory of Fourier integrals. 2nd edition* (Oxford University Press, 1948).

2. A. Zemanian, *The distributional Laplace and Mellin transformations*, SIAM J. Appl. Math. *14* (1966), 41–59.

University of Washington,
Seattle, Washington

https://doi.org/10.4153/CJM-1972-129-5 Published online by Cambridge University Press