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# THE DENSITY OF ZEROS OF DIRICHLET'S $L$-FUNCTIONS

D. R. HEATH-BROWN

**1. Introduction.** Let $L(s, \chi)$ be a Dirichlet $L$-function and let $N(\sigma, T, \chi)$ denote the number of zeros $\rho$ of $L(s, \chi)$, counted according to multiplicity, in the rectangle $\sigma \leqq \mathrm{Re}(\rho) \leqq 1$, $|\mathrm{Im}(\rho)| \leqq T$, $(T \geqq 1)$. In this paper we shall prove several new estimates for the sum

$$\sum(Q) = \sum_{q \leqq Q} \sum_{\chi (\mathrm{mod}\ q)}^{*} N(\sigma, T, \chi)$$

where $\sum^*$ denotes summation over primitive characters only. These estimates will all be of the type

$$(1) \qquad \sum(Q) \ll (Q^2 T^a)^{A(\sigma)(1-\sigma)+\epsilon},$$

where $\epsilon$ denotes any fixed positive quantity.

Extending the well-known density hypothesis for the Riemann Zeta-function, which is given by the case $Q = 1$, it is generally conjectured that (1) holds with $a = 1$ and $A(\sigma) = 2$ for the interval $1/2 \leqq \sigma \leqq 1$. At present the widest interval on which the conjecture is known to hold is $21/26 \leqq \sigma \leqq 1$, due to Jutila [**6**]. In this paper we shall extend the range of admissible values to $11/14 \leqq \sigma \leqq 1$. Note that $21/26 = 0.807\ldots$, whereas $11/14 = 0.785\ldots$.

THEOREM 1. *The estimate* (1) *holds with* $a = 1$, $A(\sigma) = 2$ *and* $11/14 \leqq \sigma \leqq 1$, *uniformly in* $\sigma$, $Q$ *and* $T$.

In the case $Q = 1$, (that is, for the Riemann Zeta-function) the theorem gives

$$N(\sigma, T) \ll T^{2-2\sigma+\epsilon}, \quad (11/14 \leqq \sigma \leqq 1),$$

in the usual notation; this is in fact the same as Jutila's estimate [**6**], which is the best result to date in connection with the ordinary density hypothesis. Jutila's work improved upon several earlier theorems, in particular Montgomery [**7**] and Huxley [**1**] obtained the first unconditional results in this context, with $\sigma \geqq 9/10$ and $\sigma \geqq 5/6$ respectively, and it has long been known, from the classical work of Ingham [**4**], that if the Lindelöf hypothesis

$$\zeta(1/2 + it) \ll t^{\epsilon},$$

is true, then so also is the density hypothesis. It is easy to verify that a result of the latter kind holds more generally for arbitrary $Q$.

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Huxley [**3**] has shown that the bounds for $A(\sigma)$ in (1) may be improved if $a$ is allowed to take the value 2. This gives estimates that are sharper with respect to $Q$ but weaker with respect to $T$. Huxley showed, in particular, that (1) holds with $a = 2$, $A(\sigma) = 20/9$ for the range $1/2 \leqq \sigma \leqq 1$. We shall prove the following sharpening of Huxley's results.

THEOREM 2. *The estimate* (1) *holds uniformly in* $\sigma$, $Q$ *and* $T$ *for* $1/2 \leqq \sigma \leqq 1$ *and*

(2) $\quad a = 6/5, A(\sigma) = 5/(3 - \sigma),$ *or*

(3) $\quad a = 6/5, A(\sigma) = 20/9.$

The estimates (2) and (3) should be compared with the corresponding results of Huxley [**3**], which have the same value of $A(\sigma)$, but have $a = 2$. We have thus reduced the exponent of $T$, while leaving the exponent of $Q$ unchanged.

Huxley [**3**] also showed that (1) is valid with $a = 2$, $A(\sigma) = 2$ for the range $11/14 \leqq \sigma \leqq 1$. This is now superseded by Theorem 1. However Jutila [**6**] showed that one may take $a = 2$, $A(\sigma) = 2$ in (1), for the wider range $7/9 \leqq \sigma \leqq 1$. We shall improve this result further, to $129/167 \leqq \sigma \leqq 1$. Note that $7/9 = 0.7777\ldots$ and $129/167 = 0.7724\ldots$.

THEOREM 3. *The estimate* (1) *holds, with* $a = 2$, $A(\sigma) = 2$ *for* $129/167 \leqq \sigma \leqq 1$, *uniformly in* $Q$, $T$ *and* $\sigma$.

Our proofs are based on the use of Dirichlet polynomials as in Montgomery [**7**], with the developments of Huxley [**1**], [**2**], [**3**] and Jutila [**5**], [**6**].

We adopt the following convention in the use of $\epsilon$ to denote a small positive quantity, namely that at certain points, which we shall not specify, we shall change $\epsilon$ by a constant factor. Thus, for example, we may write $x^\epsilon \log x \ll x^\epsilon$, for $x \geqq 2$.

**2. The Estimates of Huxley and Jutila.** We shall not repeat the details of the zero detection method, which are given in Huxley [**3**]. We summarize the results as follows:

Define $l = \log (QT)$ and

$$Y = (QT)^{1/2+4\epsilon/(2\sigma-1)}$$

For convenience in writing we shall let

$$\alpha = 1/2 + 4\epsilon/(2\sigma - 1).$$

There exists an integer $n$ such that

$$(QT)^\epsilon \leqq 2^n Y \leqq l^2 Y$$

and

$$\sum(Q) \ll (1 + R_n)l^4.$$

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Here $R_n = R$ is the number of zeros $\rho$, counted by $\sum(Q)$, for which

$$(4)\quad \left|\sum_{2^{n-1}Y<m\leqq 2^nY} b(m)\chi(m)m^{-\rho}\right| \geqq 1/(12\,l).$$

The coefficients $b(m)$ depend on $Q$, $T$, $\epsilon$ and $\sigma$ but not on $\chi$ or $\rho$; they satisfy

$$(5)\quad |b(m)| \leqq d(m).$$

We now take $U \leqq (QT)^4$ and suppose that

$$(l^2Y)^a \leqq U < (l^2Y)^{a+1},$$

for some positive integer $a$. We then raise the sum on the left of (4) to the power $b$, where $b$ is the integer for which

$$(2^nY)^b \leqq U < (2^nY)^{b+1}.$$

Thus

$$\left|\sum_{KW<m\leqq W} c(m)\chi(m)m^{-\rho}\right| \geqq V,$$

where $W = (2^nY)^b$, $K = 2^{-b}$, $V = (12l)^{-b}$ and, by (5), $|c(m)| \leqq (d(m))^{2b}$. Since $2^nY \leqq l^2Y$ we have $b \leqq a$. Also, since $2^nY \geqq (QT)^\epsilon$, we have $b \ll \epsilon^{-1} \ll 1$. Thus $K \gg 1$, $V \gg (QT)^{-\epsilon}$ and $c(m) \ll (QT)^\epsilon$. Moreover

$$(6)\quad U^{a/(a+1)} \leqq U^{b/(b+1)} < (2^nY)^b = W \leqq U.$$

To bound $R$ we may use either a mean value estimate, or some form of the Halász method. The mean value estimate, Theorem 7.5 of Montgomery [**7**], with $\delta = 1$ yields

$$(7)\quad R \ll GV^{-2}(W + Q^2T)(QT)^\epsilon,$$

where

$$(8)\quad G = \sum_{KW<m\leqq W} |c(m)|^2 m^{-2\sigma} \ll W^{1-2\sigma}(QT)^\epsilon.$$

This suffices for the proof of (2).

The Halász method, in the form due to Huxley [**2**], Theorem 1, yields

$$(9)\quad R \ll (GWV^{-2} + Q^2TG^3WV^{-6})(QT)^\epsilon,$$

under the condition

$$(10)\quad V \gg G^{1/2}W^{1/4}l^2.$$

We use this to prove (3).

For the proofs of Theorems 1 and 3 we use the method of Jutila [**6**]. We may divide the range $-T \leqq \mathrm{Re}(\rho) \leqq T$ into $1 + [2T/T_0]$ subintervals of length at most $T_0$ and apply (1.3) of Jutila [**6**] to each. This shows that, for a given positive integer $k$, each subinterval contains

$$\ll (GWV^{-2} + Q^2T_0(G^4W^2V^{-8})^k + (Q^2T_0G^2V^{-4})^k)(QT)^\epsilon$$

points. Hence, under the condition $1 \leqq T_0 \leqq T$, we have

(11) $R \ll (T/T_0)(GWV^{-2} + Q^2T_0(G^4W^2V^{-8})^k + (Q^2T_0G^2V^{-4})^k)(QT)^\epsilon.$

This estimate will be used to prove Theorem 1.

Finally, the proof of Theorem 3 requires, in addition to the ideas which lead to (11), a variation in Jutila's method. We postpone the description of this to § 4.

**3. The proofs of Theorems 1 and 2.** In this section we specify $U$ in each of the cases corresponding to Theorem 1, (2) and (3). We then verify that the above mentioned estimates do indeed follow from (7), (9) and (11). We shall write $D = Q^2T$ for brevity.

For the proof of Theorem 1 we choose $U = D^{2\alpha}l^6$. We distinguish two cases, $T \leqq Q$ and $T > Q$. If $T \leqq Q$ we may take $a = 3$, since

$$(l^2Y)^3 = l^6(QT)^{3\alpha} \leqq l^6(Q^2T)^{2\alpha} = U < l^8(QT)^{4\alpha} = (l^2Y)^4.$$

Otherwise $T > Q$ and we have $a = 2$ similarly.

For the case $T \leqq Q$, $a = 3$ we have by (6)

$$D^{3/4} \leqq W \ll D^{1+\epsilon}.$$

We use (11) with $T_0 = T$, $k = 3$, whence, by (8),

$$R \ll (W^{2-2\sigma} + DW^{18-24\sigma} + D^3W^{6-12\sigma})D^\epsilon.$$

Using the bounds for $W$, we have, assuming $11/14 \leqq \sigma \leqq 1$,

$$R \ll (D^{2-2\sigma} + D^{(29-36\sigma)/2} + D^{(15-18\sigma)/2})D^\epsilon \ll D^{2-2\sigma+\epsilon}$$

This proves Theorem 1 in the case $T \leqq Q$.

When $T > Q$ and $a = 2$ we set $T_0 = T(W/D)^{2-2\sigma}$ unless $W \geqq D$, when $T_0 = T$. Then, for $3/4 \leqq \sigma \leqq 1$, (6) yields

$$T(W/D)^{2-2\sigma} \geqq TD^{-(2-2\sigma)/3} \geqq TD^{-1/6} \geqq T^{5/6}Q^{-1/3} \geqq 1.$$

Hence $T_0 \geqq 1$; and clearly $T_0 \leqq T$ also. Thus we may apply (11) with $k = 3$, whence

$$R \ll (D^{2-2\sigma} + DW^{18-24\sigma} + D^{4\sigma-1}W^{10-16\sigma})D^\epsilon.$$

From (6) we have

$$D^{2/3} \leqq W \ll D^{1+\epsilon}.$$

Thus, for $11/14 \leqq \sigma \leqq 1$, we have

$$R \ll (D^{2-2\sigma} + D^{13-16\sigma} + D^{(17-20\sigma)/3})D^\epsilon \ll D^{2-2\sigma+\epsilon}.$$

This completes the proof of Theorem 1.

We turn now to the proof of Theorem 2. We shall prove (2) for $\sigma_0 \leqq \sigma \leqq 1$ where $\sigma_0$ is any number in the interval $1/2 < \sigma_0 \leqq 1$. Hence we have, on recalling our convention in the use of $\epsilon$,

$$Y = (QT)^{1/2+4\epsilon/(2\sigma-1)} \ll (QT)^{1/2+\epsilon}.$$

This avoids difficulties that arise when $\sigma$ is close to 1/2. We may then take $\sigma_0$ arbitrarily close to 1/2, and use the trivial estimate

$$\sum(Q) \ll D^{1+\epsilon}$$

for the remaining range $1/2 \leqq \sigma \leqq \sigma_0$. In this way we obtain an estimate valid uniformly for $1/2 \leqq \sigma \leqq 1$.

For the proof of (2) we choose

$$U = (Q^5T^3)^{1/(3-\sigma)}(QT)^{20\epsilon/(2\sigma-1)}l^{10}.$$

The estimate (7), together with (6) and (8) yields

$$R \ll (U^{2-2\sigma} + DU^{(1-2\sigma)a/(a+1)})D^{\epsilon}$$

under the condition

$$(l^2Y)^a \leqq U < (l^2Y)^{a+1}.$$

The case $a = 1$ is clearly impossible. If $a = 2$ then $U < (l^2Y)^3$, whence

$$(Q^5T^3)^{1/(3-\sigma)} \leqq (QT)^{3/2},$$

which simplifies to yield

$$(12) \quad Q^{(3\sigma+1)/(3-3\sigma)} \leqq T.$$

Then

$$DU^{(1-2\sigma)2/3} \leqq Q^2T(Q^5T^3)^{(2-4\sigma)/(9-3\sigma)} \leqq (Q^5T^3)^{(2-2\sigma)/(3-\sigma)}$$

using the inequality (12). This proves (2) in case $a = 2$. For $a = 3$ we have $U < (l^2Y)^4$, whence

$$Q^{(2\sigma-1)/(3-2\sigma)} \leqq T.$$

As before we have

$$DU^{(1-2\sigma)3/4} \leqq Q^2T(Q^5T^3)^{(3-6\sigma)/(12-4\sigma)} \leqq (Q^5T^3)^{(2-2\sigma)/(3-\sigma)},$$

and (2) follows. Finally, if $a \geqq 4$,

$$DU^{(1-2\sigma)4/5} \leqq Q^2T(Q^5T^3)^{(4-8\sigma)/(15-5\sigma)} \leqq (Q^5T^3)^{(2-2\sigma)/(3-\sigma)},$$

for all $Q, T \geqq 1$. This completes the proof of (2).

The proof of (3) is very similar. (3) follows from Theorem 1 for $11/14 \leqq \sigma \leqq 1$, and from (2) for $1/2 \leqq \sigma \leqq 3/4 + 2\epsilon$. For the remaining range we use the estimate (9), which yields by (6) and (8),

$$R \ll (U^{2-2\sigma} + DU^{(4-6\sigma)a/(a+1)})D^{\epsilon}.$$

By (8) the condition (10) becomes

$$W^{\sigma-3/4} \gg D^{\epsilon}.$$

We shall take

$$U = (Q^2T^{6/5})^{10/9}(QT)^{20\epsilon/(2\sigma-1)}l^{10},$$

whence $W^{\sigma-3/4} \geqq W^{\epsilon} \geqq U^{\epsilon/2}$, so that the required condition always holds.

It remains to show that

$$DU^{(4-6\sigma)a/(a+1)} \leqq (Q^2T^{6/5})^{20(1-\sigma)/9}$$

always. Since

$$(l^2Y)^a \leqq U < (l^2Y)^{a+1}$$

the case $a = 1$ is impossible. If $a = 2$ then

$$(Q^2T^{6/5})^{10/9} \leqq (QT)^{3/2},$$

whence

$$Q^{13/3} \leqq T.$$

Hence

$$DU^{(4-6\sigma)2/3} \leqq Q^2T(Q^2T^{6/5})^{20(4-6\sigma)/27} \leqq (Q^2T^{6/5})^{20(1-\sigma)/9}$$

as required. If $a = 3$ then

$$(Q^2T^{6/5})^{10/9} \leqq (QT)^2,$$

whence $Q^{1/3} \leqq T$. In this case

$$DU^{(4-6\sigma)3/4} \leqq Q^2T(Q^2T^{6/5})^{15(2-3\sigma)/9} \leqq (Q^2T^{6/5})^{20(1-\sigma)/9}$$

also.

Finally, if $a \geqq 4$ we have

$$DU^{(4-6\sigma)4/5} \leqq Q^2T(Q^2T^{6/5})^{8(4-6\sigma)/9} \leqq (Q^2T^{6/5})^{20(1-\sigma)/9},$$

for all $Q, T \geqq 1$. This completes the proof of (3).

**4. The Proof of Theorem 3.** In this section we develop the method of Jutila [**6**]. We denote the zeros counted by $R$, $\rho_r$, $(1 \leqq r \leqq R)$, their imaginary parts $\gamma_r$, and their associated characters $\chi_r$. By a further subdivision of the zeros in § 2, we may suppose that $|\gamma_r - \gamma_s| \geqq l^4$, if $\chi_r = \chi_s$. We define $h = l^2$,

$$e_n = e^{-(n/W)^h} - e^{-(n/KW)^h}, \; B = KW$$

and

$$H(s, \chi) = \sum_{n=1}^{\infty} e_n\chi(n)n^{-s}.$$

In Jutila's work, the constant $K$ is replaced by 1/2, but clearly this is not essential.

We begin by applying Lemma 1.7 of Montgomery [7], from which it follows that

$$(13)\quad R^2V^2 \ll GRW + G \sum_{r,s \leqq R, r\neq s} |H(\rho_r + \bar{\rho}_s - 2\sigma_s \chi_r \bar{\chi}_s)|.$$

We now apply Lemma 1 of Jutila [**6**]. For ease of reference we quote the lemma here.

LEMMA 1. *Let $\chi$ be a character* (mod $q$), $q \leqq Q$, *and let* $0 \leqq \sigma \leqq 1$, $|t| \leqq T$. *If $\chi$ is principal let $|t| \geqq h^2$ also. Then, for $B \leqq qT$ and* $q(|t| + h^3)(\pi B)^{-1} \leqq M \leqq (qT)^2$,

$$H(s, \chi) \ll 1 + B^{1/2}Q^\epsilon \int_{-h^2}^{h^2} \left| \sum_1^M \bar{\chi}(n) n^{-1/2+i(t+\tau)} \right| d\tau.$$

It is clear from the proof of the lemma that the conditions $B \leqq qT$ and $M \leqq (qT)^2$ may be dropped.

Lemma 1 yields

$$H(\rho_r + \bar{\rho}_s - 2\sigma, \chi_r\bar{\chi}_s) \ll \left( W^{1/2} \int_{-h^2}^{h^2} \left| \sum_1^M \bar{\chi}_r\chi_s(n) n^{-1/2+i(\gamma_r - \gamma_s + \tau)} \right| d\tau + 1 \right) D^\epsilon,$$

where $M = h^3D/(KW)$. We now write

$$\sum_{n=1}^M = \sum_{1 \leqq q \leqq 2 \log M} \sum_{M/2^q < n \leqq M/2^{q-1}},$$

$N = M/2^q$, and

$$\sum(\tau) = \sum_{r,s=1}^R \left| \sum_{N<n\leqq 2N} \bar{\chi}_r\chi_s(n) n^{-1/2+i(\gamma_r-\gamma_s+\tau)} \right|.$$

Thus (13) shows that, for some integer $q$,

$$(14)\quad R^2 \ll \left( GRW + GR^2 + GW^{1/2} \int_{-h^2}^{h^2} \sum(\tau) d\tau \right) D^\epsilon.$$

By Hölder's inequality we have, for any integer $k$,

$$(15)\quad \sum(\tau) \leqq R^{2-1/k} \left( \sum_{r,s=1}^R \left| \left( \sum_{N<n\leqq 2N} \right)^k \right|^2 \right)^{1/2k}.$$

We shall apply Lemma 2 of Jutila [**6**] to the right hand side of the above inequality. We quote the lemma here.

LEMMA 2. *Let $a_n$ be complex numbers such that $|a_n| \leqq A$. Then*

$$\sum_{r,s=1}^R \left| \sum_{n=1}^N a_n \bar{\chi}_r\chi_s(n) n^{-1/2+i(t_r-t_s)} \right|^2 \leqq A^2 \sum_{r,s=1}^R \left| \sum_{n=1}^N \bar{\chi}_r\chi_s(n) n^{-1/2+i(t_r-t_s)} \right|^2.$$

This yields

(16) $$\sum_{r,s=1}^{R}\left|\left(\sum_{N<n\leqq 2N}\right)^k\right|^2 \ll D^\epsilon \sum_{r,s=1}^{R}\left|\sum_{N^k<n\leqq (2N)^k} \bar{\chi}_r\chi_s(n)n^{-1/2+i(\gamma_r-\gamma_s)}\right|^2.$$

Alternatively, writing

$$f_n = e^{-(n/(2N)^k)^h} - e^{-(n/N^k)^h}$$

and

$$J(s,\chi) = \sum_{n=1}^{\infty} f_n\chi(n)n^{-s},$$

we have, also by Lemma 2,

(17) $$\sum_{r,s=1}^{R}\left|\left(\sum_{N<n\leqq 2N}\right)^k\right|^2 \ll D^\epsilon N^{-k}\sum_{r,s=1}^{R}|J(i(\gamma_r-\gamma_s),\ \chi_r\bar{\chi}_s)|^2.$$

We now apply Lemma 3 of Jutila [**6**], which we also quote here.

LEMMA 3. *For each $r$, $(1 \leqq r \leqq R)$, let $\chi_r$ be a primitive character of conductor at most $Q$ and let $t_r$ be a real number satisfying $|t_r| \leqq T$. Suppose that $|t_r - t_s| \geqq 1$ whenever $\chi_r = \chi_s$. Then*

$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N}\chi_r(n)\chi(n)n^{-1/2+it_r}\right|^2 \ll (N+(RD)^{1/2})D^\epsilon,$$

*where $\chi$ is any character of modulus at most $Q$.*

Lemma 3, in conjunction with (15) and (16) yields

$$\sum(\tau) \ll R^{2-1/k}(R(N^k+(RD)^{1/2}))^{1/2k}D^\epsilon$$
$$\ll (R^{2-1/(2k)}N^{1/2} + R^{2-1/(4k)}D^{1/4k})D^\epsilon.$$

We now have, by (8) and (14),

$$R^2 \ll (RW^{2-2\sigma} + R^2W^{1-2\sigma} + R^{2-1/(2k)}N^{1/2}W^{3/2-2\sigma} + R^{2-1/(4k)}D^{1/(4k)}W^{3/2-2\sigma})D^\epsilon,$$

whence

(18) $$R \ll (W^{2-2\sigma} + (NW^{3-4\sigma})^k + DW^{(6-8\sigma)k})D^\epsilon.$$

Alternatively we may estimate the expression on the right hand side of (17) by repeating the procedure of the preceding paragraphs. Lemma 1 yields

$$|J(i(\gamma_r-\gamma_s),\ \chi_r\bar{\chi}_s)|^2 \ll \left(1 + N^k\int_{-h^2}^{h^2}\left|\sum_{n=1}^{P}\bar{\chi}_r\chi_s(n)n^{-1/2+i(\gamma_r-\gamma_s+\tau)}\right|^2 d\tau\right)D^\epsilon,$$

for $B = N^k$. Here $r \neq s$, and $P = Dh^3/N^k$. Hence, on writing

$$S(\tau) = \sum_{r,s=1}^{R}\left|\sum_{n=1}^{P}\bar{\chi}_r\chi_s(n)n^{-1/2+i(\gamma_r-\gamma_s+\tau)}\right|^2,$$

we have

$$(19)\quad \sum_{r,s=1}^{R} |J(i(\gamma_r - \gamma_s), \chi_r\bar{\chi}_s)|^2 \ll \left(RN^{2k} + R^2 + N^k \int_{-h^2}^{h^2} S(\tau)d\tau\right)D^\epsilon.$$

By Hölder's inequality we have, for any integer $j$,

$$S(\tau) \leqq R^{2-2/j}\left(\sum_{r,s}\left|\left(\sum_{n\leqq P}\right)^j\right|^2\right)^{1/j}.$$

Moreover, by Lemma 2,

$$\sum_{r,s}\left|\left(\sum_{n\leqq P}\right)^j\right|^2 \ll D^\epsilon \sum_{r,s}\left|\sum_{n\leqq P^j} \bar{\chi}_r\chi_s(n)n^{-1/2+i(\gamma_r-\gamma_s)}\right|^2.$$

We apply Lemma 3 to the right hand side, whence

$$\sum_{r,s}\left|\left(\sum_{n\leqq P}\right)^j\right|^2 \ll R(P^j + (RD)^{1/2})D^\epsilon.$$

This yields

$$S(\tau) \ll (R^{2-1/j}P + R^{2-1/(2j)}D^{1/(2j)})D^\epsilon,$$

whence, by (15), (17) and (19)

$$\sum(\tau) \ll (R^{2-1/(2k)}N^{1/2} + R^2N^{-1/2} + R^{2-1/(2kj)}P^{1/(2k)} + R^{2-1/(4kj)}D^{1/(4kj)})D^\epsilon.$$

Thus, by (8) and (14)

$$R^2 \ll (RW^{2-2\sigma} + R^2W^{1-2\sigma} + R^{2-1/(2k)}W^{3/2-2\sigma}N^{1/2} + R^2W^{3/2-2\sigma}N^{-1/2} + R^{2-1/(2kj)}W^{3/2-2\sigma}P^{1/(2k)} + R^{2-1/(4kj)}W^{3/2-2\sigma}D^{1/(4kj)})D^\epsilon,$$

which reduces to

$$(20)\quad R \ll (W^{2-2\sigma} + (NW^{3-4\sigma})^k + (DN^{-k}W^{(3-4\sigma)k})^j + DW^{(6-8\sigma)kj})D^\epsilon.$$

We now choose $U = (l^2Y)^4$, whence $a = 4$ and

$$(QT)^{8/5} \leqq W \ll (QT)^{2+\epsilon}.$$

We distinguish two cases, according as $N \leqq D^{(1-\sigma)/2}W^{4\sigma-3}$, or not. In the first case we use (18) with $k = 4$. This yields

$$W^{2-2\sigma} \ll (QT)^{4-4\sigma+\epsilon},$$

$$(NW^{3-4\sigma})^k \ll (D^{(1-\sigma)/2})^4 \ll (QT)^{4-4\sigma+\epsilon},$$

and, for $129/167 \leqq \sigma \leqq 1$,

$$DW^{(6-8\sigma)k} \ll Q^2T^2(QT)^{32(6-8\sigma)/5} \ll (QT)^{4-4\sigma},$$

since $3/4 < 91/118 < 129/167$. This deals with the first case.

We now suppose that $N > D^{(1-\sigma)/2}W^{4\sigma-3}$. We use the estimate (20) with $k = 3$ and $j = 2$. For the first term of (20)

$$W^{2-2\sigma} \ll (QT)^{4-4\sigma+\epsilon}.$$

For the second term of (20) we note that $N \leqq M \ll D^{1+\epsilon}W^{-1}$. Hence

$$(NW^{3-4\sigma})^k \ll (DW^{2-4\sigma})^3 D^\epsilon \ll (Q^2T^2)^3(QT)^{24(2-4\sigma)/5}D^\epsilon \ll (QT)^{4-4\sigma+\epsilon},$$

where, in the final estimate, we have used the inequalities $3/4 < 29/38 < 129/167 \leqq \sigma$. For the third term of (20) we have, using the fact that $N > D^{(1-\sigma)/2}W^{4\sigma-3}$,

$$D^j N^{-kj} W^{(3-4\sigma)kj} \leqq D^2 D^{3\sigma-3} W^{-6(4\sigma-3)} W^{6(3-4\sigma)}.$$

Since $\sigma \geqq 129/167$, this expression is

$$\ll (Q^2T^2)^{3\sigma-1}(QT)^{-96(4\sigma-3)/5} \ll (QT)^{4-4\sigma}.$$

Finally, the fourth term of (20) is

$$DW^{6(6-8\sigma)} \ll Q^2T^2(QT)^{48(6-8\sigma)/5} \ll (QT)^{4-4\sigma},$$

since $139/182 < 129/167 \leqq \sigma$. This completes the proof of Theorem 3.

## References


1. M. N. Huxley, *On the difference between consecutive primes*, Invent. Math., *15* (1972), 164–170.
2. ——— *Large values of Dirichlet polynomials*, III, Acta Arith., *26* (1974), 435–444.
3. ——— *An imperfect hybrid zero-density theorem*, J. London Math. Soc., (2) *13* (1976), 53–56.
4. A. E. Ingham, *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser., *8* (1937), 255–266.
5. M. Jutila, *On a density theorem of H. L. Montgomery for L-functions*, Ann. Acad. Sci. Fenn. Ser. A I (1972), No. 520.
6. ——— *Zero-density estimates for L-functions*, Acta. Arith. *32* (1977), 55–62.
7. H. L. Montgomery, *Topics in multiplicative number theory* (Springer, Berlin, 1971).


*Trinity College,*
*Cambridge, England*