THE ASYMPTOTIC SERIES FOR A CERTAIN CLASS OF PERMUTATION PROBLEMS

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1. Introduction. This paper is concerned with problems connected with the permutations of the integers $1, 2, \ldots, n$ subject to certain special restrictions. One such class of problems, the so-called "card matching" problems, deals with conditions of the type, "the number i is in the jth position," "the number k is in the mth position," etc. The given conditions need not be compatible, i.e. a meaningful problem results from having amongst the set of conditions such conditions as "1 is second," "2 is second," "2 is third." In a permutation of $1, 2, 3, \ldots, n$ and a set of conditions S we will say that there are r "hits" if exactly r of the given conditions are fulfilled. Amongst the $n!$ permutations of the numbers $1, 2, 3, \ldots, n$, suppose there are $N(r)$ in which there are r hits. The problem of determining $N(r)$ has been treated in **(3; 6; 7)**. These results may be expressed in the language of probability by saying that $M(r) = N(r)/n!$ is the probability of exactly r hits.

A second type of problem deals with the so-called relative conditions, i.e., conditions such as "i immediately precedes j." These problems are dealt with in much the same way as the previous type in **(3)**, and for the purposes of this paper will not require a separate treatment.

For a fairly large class of problems of both types the distribution of $M(r)$ is asymptotic to a Poisson distribution. In fact, in these cases, it is possible to write $M(r)$ in the form:

$$(1) \qquad M(r) = e^{-A}\frac{A^r}{r!}\left(1 + \frac{c_1}{n} + \frac{c_2}{n(n-1)} + \frac{c_3}{n(n-1)(n-2)} + \ldots\right).$$

In general the c_i are polynomials in r of degree at most $2i$. It is the purpose of this paper to show how to compute $A, c_1, c_2, \ldots$. The determination of A, $c_1, c_2, \ldots$, does not require a knowledge of the exact expression for $N(r)$. It suffices to have a difference equation for a certain polynominal operator associated with the given set of conditions. The computation can be carried out completely from a knowledge of the coefficients which appear in the difference equation, together with the initial conditions necessary to fix the solution of the difference equation.

2. The general problem. In what follows, the discussion will be confined to the card-matching type of problem although one of the illustrations given in the end will deal with a "relative condition" problem. If p_{ij} denotes the

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condition "i is in position j" and $N(0)$ and $N(r)$ denote the number of permutations of $1, 2, \ldots, n$ in which there are 0 and r hits respectively, the method of inclusion and exclusion yields the following formulae for $N(0)$ and $N(r)$:

$$N(0) = \sum_{k=0}^{n} (-1)^k \phi_{n,k}(n-k)!$$

and

$$N(r) = \sum_{k=r}^{n} (-1)^{k+r} \phi_{n,k} \binom{k}{r} (n-k)!.$$

In these formulae ϕ_{nk} represents the number of ways in which exactly k compatible conditions may be chosen from the set of all p_{ij}.

If $M(0)$ and $M(r)$ are the probabilities of 0 and r hits respectively, the relevant formulae are:

$$M(0) = \sum_{k=0}^{n} (-1)^k \phi_{n,k} \frac{(n-k)!}{n!}$$

and

$$M(r) = \sum_{k=r}^{n} (-1)^{k+r} \phi_{n,k} \binom{k}{r} \frac{(n-k)!}{n!}.$$

If $\psi(t)$ is the generating function of the number of hits, i.e.

$$\psi(t) = \sum_{r=0}^{n} M(r)\, t^r,$$

we have:

$$\psi(t) = \sum_{r=0}^{n} \sum_{k=r}^{n} (-1)^{k+r} \phi_{n,k} \binom{k}{r} \frac{(n-k)!}{n!} t^r$$
$$= \sum_{k=0}^{n} \phi_{n,k} (t-1)^k \frac{(n-k)!}{n!}.$$

The determination of $\phi_{n,k}$ has been treated in **(3; 6; 7)**. Perhaps the most interesting representation has been given by Kaplansky and Riordan in **(6)**. In this representation, for each condition p_{ij} the cell in the ith row, jth column in an $(n \times n)$ chessboard is marked. It is easily seen that $\phi_{n,k}$ is the number of ways of putting k non-attacking rooks on the marked squares of the board. This representation makes it easy in special cases to obtain explicit formulae for $\phi_{n,k}$, and in more complicated cases it simplifies the determination of recurrence relationships.

Fréchet **(2)** gives a thorough discussion of the method of inclusion and exclusion on which the formulae of this section are based.

3. The symbolic representation. By the use of the difference operator E, defined as $Ef(n) = f(n+1)$ the formulae for $N(0)$, $N(r)$, $M(0)$ $M(r)$ may be expressed in the forms:

$$N(0) = P_n(E)\, f(0),$$
$$N(r) = P_n(E)\, g_r(0),$$
$$M(0) = P_n(E)\, f^*(0),$$
$$M(r) = P_n(E)\, g_r^*(0),$$

where

$$P_n(E) = \sum_{k=0}^{n} (-1)^k \phi_{n,k} E^k,$$
$$f(t) = (n-t)!$$
$$g_r(t) = \begin{cases} (-1)^r \binom{t}{r}(n-t)!, & t \geqslant r, \\ 0, & t < r, \end{cases}$$
$$f^*(t) = \frac{(n-t)!}{n!},$$
$$g_r^*(t) = \begin{cases} (-1)^r \binom{t}{r}\frac{(n-t)!}{n!}, & t \geqslant r, \\ 0, & t < r. \end{cases}$$

In **(3; 7)** methods of obtaining difference equations for $P_n(E)$ are given and in a number of cases these lead to explicit formulae. This paper is concerned mostly with a determination of the asymptotic series for $M(0)$ and $M(r)$ in the cases where the difference equation for $P_n(E)$ is of a special form. In a large class of problems discussed in the literature the polynomial $P_n(E)$ does indeed have a difference equation of the required form.

4. Some illustrative examples. A number of examples (mostly classical) are given here to illustrate in concrete terms the type of problem with which this discussion is concerned. These examples have also served to verify the correctness of formulae which are developed later in this paper. Such verification is necessary since the computations are quite formidable.

Example 1. *Problème des rencontres.* In this example the set of conditions are: for $i = 1, 2, \ldots, n$, "i is in position i." In this case the formulae become:

$$M(0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!},$$
$$M(r) = \frac{1}{r!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right),$$
$$P_n(E) = (1-E)^n.$$

$P_n(E)$ satisfies the recurrence formula

$$P_n(E) = (1-E)\, P_{n-1}(E).$$

Asymptotic formulae are:

$$M(0) \sim e^{-1}, \quad M(r) \sim \frac{e^{-1}}{r!}.$$

In this case the general asymptotic series of the form

$$M(r) = \frac{e^{-A}A^r}{r!}\left(1 + \frac{c_1}{n} + \frac{c_2}{n(n-1)} + \ldots\right)$$

does not exist. The reason for this is that convergence to the asymptotic value is so rapid that c_1, c_2, etc. are all 0.

Example 2. *Problème des ménages.* The conditions of this problem are: for $i = 1, 2, \ldots, n-1$, "i is in ith position" and "i is in $(i+1)$th position," together with "n is in nth position" and "n is in first position." The requisite formulae are:

$$M(0) = \sum_{i=0}^{n} (-1)^i \binom{2n}{2n-i}\binom{2n-i}{i}\frac{(n-i)!}{n!},$$

$$M(r) = \sum_{i=r}^{n} (-1)^{r+i} \binom{2n}{2n-i}\binom{2n-i}{i}\binom{i}{r}\frac{(n-i)!}{n!};$$

the recurrence formula for $P_n(E)$ is given by,

$$P_n(E) = (1-2E)\,P_{n-1}(E) - E^2P_{n-2}(E),$$

and the asymptotic formulae are:

$$M(0) = e^{-2}\left(1 - \frac{1}{(n-1)} + \frac{1}{2!(n-1)_2} + \ldots + \frac{(-1)^i}{i!(n-1)_i} + \ldots\right)$$

$$M(r) = \frac{e^{-2}2^r}{r!}\left(1 - \frac{(r-1)(r-4)}{4n} + \frac{r^4 - 14r^3 + 51r^2 - 38r - 16}{32n(n-1)}\right) + O(n^{-3}).$$

Here n_k is the Jordan factorial notation for $n(n-1)(n-2)\ldots(n-k+1)$. These results were obtained by Kaplansky and Riordan in **(5)** and check with the formula developed here.

Example 3. *Ménages non-circulaires.* This differs from Example 2 only in the omission of the condition "n is in first position." Here the formulae are:

$$M(0) = \sum_{i=0}^{n} (-1)^i \binom{2n-i}{i}\frac{(n-i)!}{n!},$$

$$M(r) = \sum_{i=r}^{n} (-1)^{r+i} \binom{2n-i}{i}\binom{i}{r}\frac{(n-i)!}{n!};$$

the recurrence formula for $P_n(E)$ is,

$$P_n(E) = (1-2E)\,P_{n-1}(E) - E^2\,P_{n-2}(E),$$

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which is identical with that of the ordinary ménages problem; the asymptotic formulae are

$$M(0) = e^{-2}\left(1 - \frac{1}{2n(n-1)}\right) + O(n^{-3}),$$

and

$$M(r) = \frac{e^{-2}2^r}{r!}\left(1 - \frac{r(r-3)}{4n} + \frac{r^4 - 10r^3 + 23r^2 + 2r - 16}{32n(n-1)}\right) + O(n^{-3}).$$

These results were also given in **(5)**.

Example 4. *Rook-king problem.* This example is a case of a relative condition problem. The conditions are "1 immediately precedes 2," "n immediately precedes $n-1$" and for $i = 2, 3, 4, \ldots, n-1$, "i immediately precedes $(i-1)$" and "i immediately precedes $(i+1)$." This problem had been treated in **(3)** and **(4)**. No convenient exact expressions for $M(0)$ and $M(r)$ have been found. The recurrence formula for $P_n(E)$ is given by

$$P_n(E) = (1 - E)\, P_{n-1}(E) - EP_{n-2}(E).$$

The formulae developed later yield the following asymptotic series:

$$M(0) = e^{-2}\left(1 - \frac{2}{n(n-1)}\right) + O(n^{-3}),$$

and

$$M(r) = \frac{e^{-2}2^r}{r!}\left(1 + \frac{r(3-r)}{2n} + \frac{r^4 - 8r^3 + 9r^2 + 22r - 16}{8n(n-1)}\right) + O(n^{-3}).$$

Example 5. The final example is one which has not appeared anywhere in the literature. The set of conditions is: "1 is 2nd," " n is $(n-1)$th" and for $i = 2, 3, 4, \ldots, n-1$, "i in $(i-1)$th" and "i is $(i+1)$th." In terms of the chessboard representations the marked squares are precisely those on the two diagonals adjacent to the main diagonal. The recurrence formula for $P_n(E)$ has been obtained by the present author by the method given in his paper **(7)**. Exact formulae for $M(0)$ and $M(r)$ are not readily obtained but $P_n(E)$ satisfies the recurrence formula

$$P_n(E) = (1 - E)\, P_{n-1}(E) + (-E + E^2)\, P_{n-2}(E) + E^3 P_{n-3}(E).$$

This together with

$$P_2(E) = 1 - 2E + E^2, \quad P_3(E) = (1 - 4E + 4E^2),$$
$$P_4(E) = 1 - 6E + 11E^2 - 6E^3 + E^4$$

are sufficient to define $P_n(E)$ for all $n \geqslant 2$. The first three terms of $P_n(E)$ are readily computed to be

$$P_n(E) = 1 - (2n - 2)\, E + (2n^2 - 7n + 7)\, E^2 + \ldots.$$

The formula to be developed yields the asymptotic expressions:

$$M(0) = e^{-2}\left(1 + \frac{1}{n} + \frac{1}{2n(n-1)}\right) + O(n^{-3})\,.$$

and

$$M(r) = \frac{e^{-2}2^r}{r!}\left(1 - \frac{(r^2 - r - 4)}{4n} + \frac{r^4 - 6r^3 + 3r^2 + 2r + 16}{32n(n-1)}\right) + O(n^{-3}),$$

A peculiarity arises here. There is a "pseudo recurrence formula"

$$P_n(E) = (1 - 2E)\,P_{n-1}(E) - E^2 P_{n-2}(E),$$

which is *not* satisfied by the $P_n(E)$ associated with this example. Nevertheless, this "pseudo recurrence formula" yields the correct asymptotic series. The reason for this is that if correct values of $P_{n-1}(E)$ and $P_{n-2}(E)$ are substituted in the "pseudo recurrence formula" the formula yields the correct polynominal $P_n(E)$ except for the term in E^n. Asymptotically, this term is of no importance. The author has constructed several examples of problems which can be associated with "pseudo recurrence formulae" which are simpler than the true recurrence formulae and which yield the proper asymptotic series. However, no general theory of this phenomenon has as yet been formulated.

5. The general theory. From this point on, only permutation problems whose polynominal operators satisfy a difference equation of the type

$$\begin{aligned}(2)\quad P_n = {} & (\alpha_1 - \beta_1 E)\,P_{n-1} + (\alpha_2 - \beta_2 E + \gamma_2 E^2)\,P_{n-2} + (\alpha_3 - \beta_3 E \\ & + \gamma_3 E^2 - \delta_3 E^3)\,P_{n-3} \\ & + \ldots + (\alpha_k - \beta_k E + \gamma_k E^2 + \ldots + (-1)^k \lambda_k E^k)\,P_{n-k}(E).\end{aligned}$$

where $P_n = P_n(E)$, k is a fixed integer and all the Greek letters are constants, will be considered. It does not seem possible to give a precise characterization of the problems whose operators satisfy such a recurrence formula but some relevant observations may be made here. The total number of conditions possible is n^2. If in a specific problem the number of conditions in the set S is of the form $an^2 + bn + c$ with $a \neq 0$, no recurrence formula of the above type is possible. It is also true that there is no asymptotic series of the type

$$M(r) = \frac{e^{-A}A^r}{r!}\left(1 + \frac{c_1}{n} + \ldots\right).$$

Examples of such problems have been given by Kaplansky and Riordan in **(6)** but they are outside our scope. If the set S had $kn + l$ conditions a recurrence formula of the given type is possible and if these conditions form a reasonably regular pattern of marked squares in the chessboard representation the existence of a suitable recurrence is likely, and probably can be obtained in a routine way, by the methods given in **(6)** or **(7)**.

Assume now that a recurrence formula for $P_n(E)$ exists and is given by equation (2). An asymptotic series of the type given in equation (1) is sought. In this connection it is possible to show that the method given by Kaplansky in **(3)** yields a result of the form

$$M(r) = \frac{e^{-A}A^r}{r!}\left(1 + \frac{c}{n}\right) + O(n^{-2})$$

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in those cases where $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ $(i = 1, 2, 3, \ldots, k)$. In what follows it is assumed that the complete asymptotic series exists and the work is confined to the computation of the c_i under this assumption. The polynomial $P_n(E)$ is given by:

$$P_n(E) = 1 - \phi_{n,1} E + \phi_{n,2} E^2 + \ldots + (-1)^n \phi_{n,n} E^n. \tag{3}$$

Using equations (2) and (3) it follows by complete induction that $\phi_{n,i}$ is a polynominal of degree i in n with coefficients which are functions of i. It is convenient to express $\phi_{n,i}$ in the form

$$\phi_{n,i} = C_0^{(i)} n_i + C_1^{(i)}(n-1)_{i-1} + C_2^{(i)}(n-2)_{i-2} + \ldots + C_i^{(i)} \tag{4}$$

The notation n_i is the Jordan factorial notation defined previously. To avoid complications of notation we define $\phi_{n,0} = 1$ and $\phi_{n,r} = 0$ if $r < 0$ or $r > n$. On substituting the expression for $P_n(E)$ as given by (3) into the recurrence (2) and arranging the result in powers of E the following relations are obtained:

$$\sum_{i=1}^{k} \alpha_i = 1,$$

(from the constant term), and

$$\phi_{n,r} = \sum_{i=1}^{k} \alpha_i \phi_{n-i,r} + \sum_{i=1}^{k} \beta_i \phi_{n-i,r-1} + \sum_{i=2}^{k} \gamma_i \phi_{n-i,r-2} + \ldots + \lambda_k \phi_{n-k,r-k}. \tag{5}$$

The expression (4) for $\phi_{n,i}$ is substituted into equation (5) to yield an expression which will be referred to as equation (6). Because of its extreme length, equation (6) is not written down here. On comparing coefficients of n^r in equation (6) and by the use of induction the following result is obtained:

$$C_0^{(r)} = \frac{A^r}{r!}, \quad A = \frac{\beta_1 + \beta_2 + \ldots + \beta_k}{\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k}. \tag{7}$$

To compute $C_1^{(r)}$ the following procedure is used. First, all the $\phi_{n,i}$ occurring in equation (6) are expressed as factorial polynomials in terms of the variable $n - k - 2$ by making use of the relationship

$$(n+t)_u = n_u + tu\, n_{u-1} + \frac{t(t-1)\, u(u-1)}{2!} n_{u-2} - \ldots .$$

Then the coefficients of $(n-k-2)_{r-2}$ in both sides of the resultant equation are equated to yield

$$C_1^{(r)} = \frac{A}{r-1} C_1^{(r-1)} - B \frac{A^{r-2}}{(r-1)!}, \tag{8}$$

where

$$B = \frac{1}{P}(KA^2 + LA + M),$$

$$K = \left\{\sum_{i=1}^{k} \alpha_i \binom{k-i+2}{2}\right\} - \binom{k+2}{2},$$

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$$L = \sum_{i=1}^{k} (k - i + 2)\, \beta_i,$$

$$M = \sum_{i=2}^{k} \gamma_i,$$

$$P = \left\{ \sum_{i=1}^{k} (k - i + 1)\, \alpha_i \right\} - (k + 1).$$

The recurrence formula for $P_n(E)$ does not determine $C_1^{(1)}$ but this can be obtained from a knowledge of $P_1(E)$. In terms of $C_1^{(1)}$, B and A it is easily seen by induction that equation (8) has as solution

$$C_1^{(r)} = \frac{A^{r-1}C_1^{(1)}}{(r-1)!} - \frac{A^{r-2}B}{(r-2)!}. \tag{9}$$

At this point the first two terms of the asymptotic series will be computed. From the relationship $M(r) = P_n(E)\, g_r^*(0)$, the result

$$M(r) = \frac{1}{n!}\left\{ \phi_{n,r}\binom{r}{r}(n-r)! - \phi_{n,r+1}\binom{r+1}{r}(n-r-1)! + \phi_{n,r+2}\binom{r+2}{r}(n-r-2)! - \ldots \right\}$$

is obtained. This reduces (on substituting for $\phi_{n,r}$ the expression (4)) to:

$$\begin{aligned} M(r) = &\left\{ C_0^{(r)} - \binom{r+1}{1} C_0^{(r+1)} + \binom{r+2}{2} C_0^{(r+2)} - \ldots \right\} \\ &- \frac{1}{n}\left\{ C_1^{(r)} - \binom{r+1}{1} C_1^{(r+1)} + \binom{r+2}{2} C_1^{(r+2)} - \ldots \right\} \\ &+ O(n^{-2}). \end{aligned}$$

Substituting for $C_0^{(i)}$ and $C_1^{(i)}$ the expression obtained in (7) and (8) yields the equation

$$\begin{aligned} M(r) = &\left\{ \frac{A^r}{r!} - \binom{r+1}{1}\frac{A^{r+1}}{(r+1)!} + \binom{r+2}{2}\frac{A^{r+2}}{(r+2)!} - \ldots \right\} \\ &+ \frac{1}{n}\Bigg[C_1^{(1)}\left\{ \frac{A^{r-1}}{(r-1)!} - \binom{r+1}{1}\frac{A^r}{r!} + \binom{r+2}{2}\frac{A^{r+1}}{(r+1)!} - \ldots \right\} \\ &- B\left\{ \frac{A^{r-2}}{(r-2)!} - \binom{r+1}{1}\frac{A^{r-1}}{(r-1)!} + \binom{r+2}{2}\frac{A^r}{r!} - \ldots \right\} \Bigg] \\ &+ O(n^{-2}). \end{aligned}$$

This reduces to

$$M(r) = \frac{e^{-A}A^r}{r!}\left[1 + \frac{1}{n}\left\{ C_1^{(1)}\left(\frac{r}{A} - 1\right) - B\left(\frac{r(r-1)}{A^2} - \frac{2r}{A} + 1\right) \right\} \right] + O(n^{-2}).$$

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In the case where $r = 0$, the result further reduces to

$$M(0) = e^{-A}\left[1 - \frac{B + C_1^{(1)}}{n}\right] + O(n^{-2}).$$

The author has computed two further terms of the asymptotic series. The results are quite complicated in form. The term in $1/n(n-1)$ has been verified by means of the examples quoted previously. It is given here for completeness but all the computations have been omitted as they are quite involved and do not utilize any new idea. The final formula contains the number $C_2^{(2)}$ which is not obtainable from the recurrence formula for $P_n(E)$. All that is required for the computation of $C_2^{(2)}$ is a knowledge of $P_2(E)$. The final result is

$$M(0) = e^{-A}\left[1 - \frac{1}{n}(B + C_1^{(1)}) + \frac{1}{n(n-1)}\left\{C_2^{(2)} - \frac{\Gamma + B^2}{A} + \frac{B^2}{2}\right\}\right] + O(n^{-3}),$$

$$M(r) = \frac{e^{-A}A^r}{r!}\left[1 + \frac{1}{n}\left\{C_1^{(1)}\left(\frac{r}{A} - 1\right) - B\left(\frac{r(r-1)}{A^2} - \frac{2r}{A} + 1\right)\right\}\right.$$
$$+ \frac{1}{n(n-1)}\left\{C_2^{(2)}\left(\frac{r(r-1)}{A^2} - \frac{2r}{A} + 1\right)\right.$$
$$+ \frac{\Gamma}{A}\left(\frac{r(r-1)(r-2)}{A^3} - \frac{3r(r-1)}{A^2} + \frac{3r}{A} - 1\right)$$
$$\left.\left. + B^2\left(\frac{r(r-1)^2(r-2)}{2A^4} - \frac{r(r-1)(2r-1)}{A^3} + \frac{3r^2}{A^2} - \frac{2r+1}{A} + \frac{1}{2}\right)\right\}\right]$$
$$+ O(n^{-3}).$$

All terms in $M(r)$ except Γ have been previously defined. The value of Γ is given by the expression:

$$\Gamma = -\frac{1}{P}\{(RA^4 + SA^3 + TA^2 + UA) + C_1^{(1)}(KA^3 + LA^2 + MA) - B(KA^2 - M)\},$$

where

$$R = \left\{\sum_{i=1}^{k} \alpha_i \binom{k-i+3}{3}\right\} - \binom{k+3}{3},$$

$$S = \sum_{i=1}^{k} \beta_i \binom{k-i+3}{2},$$

$$T = \sum_{i=2}^{k} \gamma_i (k-i+3),$$

$$U = \sum_{i=3}^{k} \delta_i.$$

The above formulae while formidable in appearance are quite simple to apply in practical cases. In none of the five examples quoted did the computations require as much as five minutes.

6. Distribution moments. The difference equation for $P_n(E)$ may be used to yield all the moments of the distribution of $M(r)$ as well as the asymptotic series. In this section formulae are established for the mean m and the variance v of the distribution.

The computation of moments is most easily carried out by the use of the notion of a factorial moment. The factorial moments are more natural to the type of problem considered in this paper than are the more usual power moments. The ith factorial moment of the distribution of the number of hits is defined as $M^{(i)}$, where

$$M^{(i)} = \sum_{r=0}^{n} r(r-1)(r-2)\ldots(r-i+1)\,M(r).$$

It has been shown in **(4)** and **(3)** that

$$M^{(i)} = \phi_{n,i}\bigg/\binom{n}{i}.$$

Actually, this result follows easily by a direct computation. In terms of these factorial moments the mean m and the variance v are given by:

$$m = M^{(1)}, \quad v = M^{(2)} + M^{(1)} - \{M^{(1)}\}^2.$$

In terms of the constants computed in this paper these formulae become:

$$m = \frac{\phi_{n,1}}{n} = C_0^{(1)} + \frac{C_1^{(1)}}{n} = A + \frac{C_1^{(1)}}{n},$$

$$\begin{aligned} v &= M^{(2)} + M^{(1)} - \{M^{(1)}\}^2 \\ &= \frac{2\phi_{n,2}}{n(n-1)} + \left(A + \frac{C_1^{(1)}}{n}\right) - \left(A + \frac{C_1^{(1)}}{n}\right)^2 \\ &= 2C_0^{(2)} + \frac{2C_1^{(2)}}{n} + \frac{2C_2^{(2)}}{n(n-1)} + A + \frac{C_1^{(1)}}{n} - \left(A + \frac{C_1^{(1)}}{n}\right)^2 \\ &= A^2 + \frac{2}{n}(AC_1^{(1)} - B) + \frac{2C_2^{(2)}}{n(n-1)} + A + \frac{C_1^{(1)}}{n} - \left(A + \frac{C_1^{(1)}}{n}\right)^2 \\ &= A + \frac{1}{n}(C_1^{(1)} - 2B) - \frac{\{C_1^{(1)}\}^2}{n^2} + \frac{2C_2^{(2)}}{n(n-1)}. \end{aligned}$$

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REFERENCES

1. T. S. Broderick, *On some symbolic formulae in probability theory*, Proc. Roy. Irish Acad., *44* (1937), 19–28.
2. M. Fréchet, *Les probabilités associées à un système d'événements compatibles et dépendents*, Actualités Scientifiques et Industrielles, nos 859 et 942 (Paris, 1940 and 1943).
3. I. Kaplansky, *Symbolic solution of certain problems in permutations*, Bull. Amer. Math. Soc., *50* (1944), 906–914.
4. ———, *The asymptotic distribution of runs of consecutive elements*, Ann. Math. Statist., *16* (1945), 200–203.
5. I. Kaplansky and J. Riordan, *Le problème des ménages*, Scripta Math., *12* (1946), 113–124.
6. ———, *The problem of the rooks and its applications*, Duke Math. J., *16* (1946), 259–268.
7. N. S. Mendelsohn, *Symbolic solution of card matching problems*, Bull. Amer. Math. Soc., *52* (1946), 918–924.
8. ———, *Applications of combinational formulae to generalizations of Wilson's theorem*, Can. J. Math., *1* (1949), 328–336.
9. J. Riordan, *Three line Latin rectangles* II, Amer. Math. Monthly, *53* (1946), 18–20.

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