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The Dunford-Pettis Property for Symmetric Spaces

Anna Kamińska and Mieczysław Mastyło

Abstract. A complete description of symmetric spaces on a separable measure space with the Dunford-Pettis property is given. It is shown that ℓ^1, c_0 and ℓ^∞ are the only symmetric sequence spaces with the Dunford-Pettis property, and that in the class of symmetric spaces on $(0,\alpha)$, $0 < \alpha \le \infty$, the only spaces with the Dunford-Pettis property are L^1, L^∞, $L^1 \cap L^\infty$, $L^1 + L^\infty$, $(L^\infty)^\circ$ and $(L^1 + L^\infty)^\circ$, where X° denotes the norm closure of $L^1 \cap L^\infty$ in X. It is also proved that all Banach dual spaces of $L^1 \cap L^\infty$ and $L^1 + L^\infty$ have the Dunford-Pettis property. New examples of Banach spaces showing that the Dunford-Pettis property is not a three-space property are also presented. As applications we obtain that the spaces $(L^1 + L^\infty)^\circ$ and $(L^\infty)^\circ$ have a unique symmetric structure, and we get a characterization of the Dunford-Pettis property of some Köthe-Bochner spaces.

1 Introduction

A Banach space X is said to have the *Dunford-Pettis property*, shortly (DP)-property or $X \in$ (DP), if for all weakly null sequences (x_n) in X and (f_n) in X^* (topological dual), we have $f_n(x_n) \to 0$, or equivalently, if every weakly compact operator from X into an arbitrary Banach space Y is a *Dunford-Pettis operator*. Recall that an operator $T\colon X \to Y$ between two Banach spaces is a Dunford-Pettis operator, whenever T maps weakly null sequences into norm null sequences. It is easily seen that (DP)-property is inherited by complemented subspaces and if $E, F \in$ (DP) then the direct sum $E \oplus F \in$ (DP). Clearly, every Banach space with the *Schur property* (all weakly null sequences are norm null) has the (DP)-property. Throughout the paper we will also use the obvious fact that $X^* \in$ (DP) implies $X \in$ (DP). For equivalent definitions and various characterizations of the Dunford-Pettis property we refer to [2] and [14].

It is well known that $\mathcal{L}^1$-spaces and $\mathcal{L}^\infty$-spaces (in the sense of [24]), and hence L^1 and L^∞, have the Dunford-Pettis property. In [22] (*cf.* also [28]) Kislyakov proved that the disc algebra has (DP)-property and Bourgain in [6] (*cf.* also [31]) showed that a large class of subspaces of vector-valued $C(K)$ spaces including ball algebras, polydisc-algebras and Sobolev spaces in uniform norms have (DP)-property as well. He also proved in [5] that the space H^∞ of bounded analytic function on the disc has the Dunford-Pettis property. However, all of these spaces are not isomorphic to Banach lattices; they even fail local unconditional structure [28], [31]. Note also that Lorentz spaces do not have (DP)-property [14], and the same holds for Orlicz spaces distinct from L^1 [29].

The paper is devoted to study the Dunford-Pettis property for Banach lattices. Let us outline briefly the content. Section 2 contains some introductory material, definitions, notations and some results which will be used in the sequel.

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Section 3 consists of the main results of the paper. It contains a complete characterization of symmetric spaces on a separable measure space with the Dunford-Pettis property. In particular it is shown that ℓ^1, c_0 and ℓ^∞ are the only symmetric sequence spaces (up to equivalence of norms) with (DP)-property. In the case of symmetric spaces on $(0,\infty)$ there are only (up to equivalence of norms) six symmetric spaces: $L^1, L^\infty, L^1 \cap L^\infty, L^1 + L^\infty$, $(L^\infty)^\circ$, $(L^1 + L^\infty)^\circ$ with (DP)-property, where X° denotes the norm closure of $L^1 \cap L^\infty$ in X. It is also proved that all Banach dual spaces of $L^1 \cap L^\infty$ and $L^1 + L^\infty$ have the Dunford-Pettis property. The obtained results answer the question posed to the authors by S. Ya. Novikov, whether the symmetric spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$ have (DP)-property. While working on this paper we have been kindly informed by N. J. Kalton that the result that $L^1 \cap L^\infty$ and $L^1 + L^\infty$ possess the Dunford-Pettis property has been also proved by F. L. Hernandez and N. J. Kalton [17].

Section 4 contains new examples of Banach spaces showing that the Dunford-Pettis property is not a three-space property. In fact, we present Banach spaces X which do not have the Dunford-Pettis property, while some of their subspaces Y and the corresponding quotient spaces X/Y have the hereditary Dunford-Pettis property (*i.e.*, any closed subspace of the space has (DP)-property). The first example of the Banach space with the above properties has been given in [8] (see also [9]).

In the last section we present some consequences and applications of the main theorems. In particular, it follows that a symmetric sequence Banach space has the Schur property if and only if it coincides with ℓ^1. We also show that $(L^1 + L^\infty)^\circ$ and $(L^\infty)^\circ$ on $(0,\infty)$, have unique symmetric structure and we give a characterization of some Köthe-Bochner spaces possessing the Dunford-Pettis property. We conclude the paper with some remarks on the inclusion map of $L^1 \cap L^\infty$ into E being a strictly singular operator.

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2 Definition and Notation

Our definition and terminology is standard. For unexplained notation the reader is referred to [2], [3] and [24]. However, we want to explain some frequently used terms and agree on some notations.

Let $\langle X, Y\rangle$ be a dual system of Banach spaces X, Y under the bilinear form $\langle \cdot,\cdot\rangle$. The *weak topology* $\sigma(X,Y)$ on X is the topology of pointwise convergence on X, that is, a net (x_α) in X converges to x for $\sigma(X,Y)$ if $\langle x_\alpha, y\rangle$ approaches 0 for each $y \in Y$. The *topological dual* of a normed space X is denoted by X^* and the unit ball of X by B_X.

A Banach lattice E is called an AL-*space* (respectively an AM-*space*), if for disjoint vectors $x, y \in E$, we have $\|x+y\| = \|x\| + \|y\|$ (respectively, $\|x+y\| = \max\{\|x\|, \|y\|\}$). It is well known (see [2, Thm. 12.22]) that a Banach lattice E is an AL-space (resp. an AM-space) if and only if E^* is an AM-space (resp. an AL-space). It follows by Grothendieck's result (see [2, Thm. 19.6]), that every AL-space and every AM-space has the Dunford-Pettis property.

A Banach lattice $(E, \|\cdot\|)$ is called a *semi-M-space* if it follows from $u_1 \vee u_2 \geq x_n \downarrow 0$ in E with $\|u_1\| = \|u_2\| = 1$, that $\lim_n \|x_n\| \leq 1$ (*cf.* [19], [26]).

It is well known that if E is a normed lattice, then $E^* = E_c^* \oplus E_s^*$, where E_c^* is the space of order bounded and order continuous functionals on E and E_s^* is a *singular part* of E^*, *i.e.*, $E_s^* = (E_c^*)^\perp$ is a disjoint complement of E_c^* (see [26, p. 316]).

A Banach lattice E is said to have the *Fatou property* if whenever (x_n) is a norm bounded sequence in E such that $0 \leq x_n \uparrow x = \sup x_n$, then $x \in E$ and $\lim_n \|x_n\| = \|x\|$. An element $x \in E$ is said to have an *order continuous norm* if for every sequence $x_n \downarrow 0$ in E with $x_n \leq x$, we have $\|x_n\| \to 0$. The norm in a Banach lattice E is called *order continuous* if every element in E has order continuous norm and the largest ideal consisting of all elements with order continuous norms will be denoted by E_a.

Let $(\Omega, \mathcal{B}, \mu)$ (or shortly (Ω, μ)) be a σ-finite measure space. Throughout the paper μ will be always either *nonatomic* or *purely atomic*, *i.e.*, $\Omega = \mathbb{N}$ and $\mu(\{n\}) = 1$ for each $n \in \mathbb{N}$. By $L^0 = L^0(\mu)$ denote a vector lattice of all (equivalence classes) of μ-measurable real-valued functions defined on Ω, equipped with the topology of convergence in measure on μ-finite sets. A Banach space E is said to be a *Banach lattice on* (Ω, μ) if E is a subspace in L^0 with the following two properties:

(i) $|x| \leq |y|$, $y \in E$ implies $\|x\| \leq \|y\|$,
(ii) there exists $u \in E$ such that $u > 0$ on Ω.

In what follows a Banach lattice on $\mathbb{N}$ will be called *a Banach sequence space*. The *Köthe dual* E' of a Banach lattice E is then defined as

$$E' := \left\{ x \in L^0 : \|x\|_{E'} = \sup_{\|y\|_E \leq 1} \int_\Omega |xy| \, d\mu < \infty \right\},$$

and E' is a Banach lattice under the norm $\|\cdot\|_{E'}$. The space E_c^* of order bounded and order continuous functionals on E is lattice isometric to the Köthe dual E' [21], which we denote further by $E_c^* \simeq E'$. In particular, if E has order continuous norm then the dual space E^* can be identified with E'. We will say in the sequel that two Banach lattices $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ coincide (or simply that they are equal), whenever E and F coincide as sets and the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent.

A Banach lattice on (Ω, μ) is said to be *symmetric* if whenever $x \in E$, $y \in L^0$, and x and y are *equimeasurable* then $y \in E$ and $\|x\|_E = \|y\|_E$. Recall that x and y are equimeasurable if they have identical distributions, that is, $\mu_x(\lambda) := \mu\{\omega \in \Omega : |x(\omega)| > \lambda\} = \mu_y(\lambda)$ for all $\lambda \geq 0$. Given an $x \in L^0$, by x^* we denote its nonincresing rearrangement, *i.e.*, $x^*(t) = \inf\{\lambda \geq 0 : \mu_x(\lambda) \leq t\}$, $t \geq 0$, under the convention $\inf \varnothing = 0$. Obviously x^* is a Lebesgue measurable function defined on the interval $\big(0, \mu(\Omega)\big)$, and x and x^* are equimeasurable [3] in the sense that $\mu_x(\lambda) = m_{x^*}(\lambda)$ for all $\lambda \geq 0$, where m is the Lebesgue measure on $(0, \infty)$.

Recall also that given a nonatomic measure space (Ω, μ) with $\mu(\Omega) < \infty$, we define Rademacher functions (r_n) on Ω as a sequence of independent random variables with $\mu\big(\{s \in \Omega : r_n(s) = 1\}\big) = \mu\big(\{s \in \Omega : r_n(s) = -1\}\big) = \mu(\Omega)/2$ for all $n \in \mathbb{N}$.

In what follows we agree on some notations and provide auxiliary facts from interpolation theory [3], [23]. A pair $\overline{X} = (X_0, X_1)$ of Banach spaces is called a *Banach couple* if X_0 and X_1 are both continuously embedded in a Hausdorff topological vector space $\mathcal{X}$. For a

Banach couple $\overline{X} = (X_0, X_1)$, the algebraic sum $X_0 + X_1$ and the intersection $X_0 \cap X_1$ will be denoted by $\Sigma(\overline{X})$ and $\Delta(\overline{X})$, respectively. They are both Banach spaces with the norms $\|x\|_{\Sigma(\overline{X})} = K(1, x; \overline{X})$ and $\|x\|_{\Delta(\overline{X})} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$, respectively, where

$$K(t, x; \overline{X}) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\}, \quad t > 0.$$

A Banach space X is called an *intermediate space* between X_0 and X_1 (or with respect to $\overline{X}$) if $\Delta(\overline{X}) \subset X \subset \Sigma(\overline{X})$. Given two Banach couples $\overline{X}$ and $\overline{Y}$ we denote by $L(\overline{X}, \overline{Y})$ the Banach space of all linear operators $T\colon \Sigma(\overline{X}) \to \Sigma(\overline{Y})$, and we write it shortly $T\colon \overline{X} \to \overline{Y}$, such that the restriction of T to the space X_j is a bounded operator from X_j into Y_j $(j = 0, 1)$ with the norm

$$\|T\|_{\overline{X} \to \overline{Y}} = \max\{\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}\}.$$

Intermediate spaces X and Y with respect to $\overline{X}$ and $\overline{Y}$ respectively, are called *interpolation spaces* with respect to $\overline{X}$ and $\overline{Y}$ if every operator $T\colon \overline{X} \to \overline{Y}$ maps X into Y. The closed graph theorem then implies that there exists a positive constant C such that

$$\|T\|_{X \to Y} \le C\|T\|_{\overline{X} \to \overline{Y}},$$

for any $T\colon \overline{X} \to \overline{Y}$. If $C = 1$, X and Y are called *exact* interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$. If $\overline{X} = \overline{Y}$ and $X = Y$, then X is called an (exact) interpolation space between X_0 and X_1.

The Banach lattices $L^1 \cap L^\infty$ and $L^1 + L^\infty$ over (Ω, μ) will be further denoted by $\Delta = \Delta(\mu)$ and $\Sigma = \Sigma(\mu)$, respectively.

Any symmetric space on (Ω, μ) is an intermediate space between L^1 and L^∞ and symmetric spaces with the Fatou property or with order continuous norm are exact interpolation spaces between L^1 and L^∞ [3], [23]. For an intermediate space X with respect to $\overline{X}$ we denote by X° the closure of $\Delta(\overline{X})$ in X. Further we will need the following well known equalities (*cf.* [23])

$$\Sigma^\circ = L^1 + (L^\infty)^\circ \quad \text{and} \quad \Sigma^\circ = \Sigma_a = \{x : x^*(t) \to 0 \text{ as } t \to \infty\}.$$

If (E_0, E_1) is a couple of Banach lattices on (Ω, μ) then by [25], it follows that $(E_0 + E_1)' = E_0' \cap E_1'$ and $(E_0 \cap E_1)' = E_0' + E_1'$, with equality of norms. Recall also that the spaces L^1, L^∞, Σ and Δ have the Fatou property and that the Köthe duals of L^1, L^∞, Σ, Δ, $(L^\infty)^\circ$ and Σ° are L^∞, L^1, Δ, Σ, L^1 and Δ, respectively. It is worth noticing that all of them are exact interpolation spaces between Δ and Σ [23]. Given Banach lattices E, F on (Ω, μ), the weak topology $\sigma(E, F)$ will be always considered under the bilinear form $\langle \cdot, \cdot \rangle$ defined on $E \times F$ by

$$\langle x, y \rangle := \int_\Omega xy \, d\mu$$

where $x \in E$ and $y \in F$ (in this case $F \hookrightarrow E'$).

3 Symmetric Spaces with the Dunford-Pettis Property

In this section we prove our main results. We show that given an arbitrary nonatomic measure space (Ω, μ), the spaces Σ, Δ, Σ° and $(L^\infty)^\circ$ have the Dunford-Pettis property, and if, in addition, μ is separable then the only symmetric spaces with (DP)-property are L^1, L^∞, $(L^\infty)^\circ$, Δ, Σ and Σ°. We also prove that, in the case when μ is purely atomic, the only symmetric spaces with (DP)-property are ℓ^1, c_0 and ℓ^∞.

We start with some auxiliary results concerning weak topologies and weak convergence in Δ and Σ.

Lemma 1 *Let (Ω, μ) be a separable measure space. If $E \subset \Sigma$ and F is an intermediate space between L^1 and L^∞, then the topology $\sigma(E, F)$ is metrizable on $\sigma(E, F)$-compact sets.*

Proof At first we shall show that F contains a countable subset $(y_n)_{n=1}^\infty$ which is a total set of functionals on E. We have $\Sigma' = \Delta$ isometrically. This implies that there exists a strictly positive $w \in B_\Delta$ such that $\Sigma \hookrightarrow L^1(\nu)$ with $\nu = w d\mu$. By the separability of μ, $L^1(\nu)^*$ contains a countable set of functionals $(f_n)_{n=1}^\infty$ that separates points of $L^1(\nu)$. Since $L^1(\nu)^* \simeq L^1(\nu)' \hookrightarrow \Delta$, $f_n(x) = \langle x, y_n \rangle$ for some $y_n \in \Delta \subset F$ and any $x \in \Sigma$. Then the set $(y_n)_{n=1}^\infty$ is as required. Now, the metric

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |\langle x - y, y_n \rangle|\}, \quad x, y \in K$$

generates a weaker topology than $\sigma(E, F)|_K$, and thus they coincide by compactness of K.

The well known Calderón theorem [7] (see also [3], [23]) states that if (Ω_1, μ) and (Ω_2, ν) are two measure spaces and $x \in \Sigma(\mu)$ and $y \in \Sigma(\nu)$ are such that

$$K\Big(t, y; \big(L^1(\nu), L^\infty(\nu)\big)\Big) \le K\Big(t, x; \big(L^1(\mu), L^\infty(\mu)\big)\Big)$$

for each $t > 0$, then there exists an operator $T\colon \big(L^1(\mu), L^\infty(\mu)\big) \to \big(L^1(\nu), L^\infty(\nu)\big)$ of norm at most one, such that $Tx = y$. Since for any $x \in \Sigma(\mu)$ and $t > 0$,

$$K\Big(t, x; \big(L^1(\mu), L^\infty(\mu)\big)\Big) = \int_0^t x^*(s)\, dm = K\Big(t, x^*; \big(L^1(m), L^\infty(m)\big)\Big),$$

an immediate consequence of the Calderón result is that for any $x \in \Sigma(\mu)$ there exists an operator $T\colon \big(L^1(\mu), L^\infty(\mu)\big) \longrightarrow \big(L^1(m), L^\infty(m)\big)$ of norm at most 1, such that $Tx = x^*$.

We will need also the following result.

Lemma 2 *Let X and Y be exact interpolation spaces with respect to $\overline{X} = \big(L^1(\mu), L^\infty(\mu)\big)$ and $\overline{Y} = \big(L^1(\nu), L^\infty(\nu)\big)$ defined on nonatomic measure spaces. If a set $B \subset X'$ is relatively compact for $\sigma(X', X)$, then $\{Tx : x \in B, T \in L(\overline{X}, \overline{Y}), \|T\|_{\overline{X} \to \overline{Y}} \le 1\}$ is relatively compact in Y' for $\sigma\big(Y', Y \cap \Sigma_a(\nu)\big)$.*

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The above lemma is a modification of Corollary 29 in [15]. The latter result was proved under the assumption that measure spaces are Radon measure spaces defined on locally compact Hausdorff topological spaces. By the Calderón result, the proof presented in [15] works also for arbitrary nonatomic measure spaces.

Proposition 3 *Let $x_n, x \in \Sigma$ and $x_n \to x$ for $\sigma(\Sigma, \Delta)$. Then there exists a subsequence (x_{n_k}) of (x_n) such that $(x^*_{n_k})$ is an order bounded subset in $\Sigma(m)$.*

Proof By remarks before Lemma 2 and by the lemma itself it follows that (x^*_n) is a relatively compact subset in Σ for $\sigma(\Sigma, \Delta)$, where $\Sigma = \Sigma(m)$ and $\Delta = \Delta(m)$. It is easily seen (*cf.* [15]) that the set $\mathcal{D}$ of nonnegative, nonincreasing functions is closed in Σ for $\sigma(\Sigma, \Delta)$ and $y_n \to y$ in $\mathcal{D}$ for induced topology $\sigma(\Sigma, \Delta)|_{\mathcal{D}}$ implies $y_n \to y$ a.e. Thus, by Lemma 1, passing to a subsequence if necessary, we may assume that for some $u \in \Sigma$, we have $x^*_n \to u^*$ in Σ for $\sigma(\Sigma, \Delta)$, and $x^*_n \to u^*$ a.e. It follows (*cf.* [15]) that $\int_0^\infty |x^*_n - u^*| y \, dm \to 0$ for every $0 \le y \in \Delta$, which in particular implies that $\int_0^1 |x^*_n - u^*| \, dm \to 0$. Hence there exists a subsequence $(x^*_{n_k})$ and $z \in L^1(0,1)$ such that $x^*_{n_k}\chi_{(0,1)} \le z$ a.e. (see [21, Lemma 2, p. 97]). Since $x^*_n \to u^*$ in Σ for $\sigma(\Sigma, \Delta)$, (x_n) is norm bounded, so $\sup_{n \ge 1} \|x_n\|_\Sigma = C < \infty$. Thus for every $n \in \mathbb{N}$, $x^*_n(1) \le 2\|x^*_n\chi_{(1/2,1)}\|_\Sigma \le C$. Since $x^*_n\chi_{[1,\infty)} \le x^*_n(1)\chi_{[1,\infty)}$, we conclude that for $u = z + 2C \in \Sigma$, $x^*_{n_k} \le u$ a.e. for all $k \in \mathbb{N}$. This completes the proof.

Before we prove the next theorem, recall that a Banach lattice is said to have *weakly sequentially continuous lattice operations* whenever $x_n \to 0$ weakly implies $|x_n| \to 0$ weakly. By the Riesz Representation Theorem, it follows that a sequence $(x_n) \subset C(K)$ satisfies $x_n \to 0$ weakly if and only if (x_n) is norm bounded and $x_n(t) \to 0$ holds for all $t \in K$. Therefore $x_n \to 0$ weakly in $C(K)$ implies $|x_n| \to 0$. Thus by the Kakutani-Bohnenblust and M. Krein-S. Krein representation theorem (see [2], [24]) in every AM-space the lattice operations are weakly continuous. It appears that the space Δ has a similar property.

We need also to recall that a subset A of an AL-space E is relatively weakly compact if and only if for every $\varepsilon > 0$ there exists $x \ge 0$ such that $A \subset [-x, x] + \varepsilon B_E$, where $[-x, x] = \{z \in E : -x \le z \le x\}$ is an interval in E (see [2, p. 208]).

Proposition 4 *In the space Δ the lattice operations are weakly sequentially continuous.*

Proof It is clear that Δ is order isomorphic to L^∞ or to ℓ^1 provided that μ is finite or purely atomic measure, respectively. Thus we need only to consider the case of infinite nonatomic measure space.

Let $x_n \to 0$ weakly in Δ. Then $x_n \to 0$ in Δ for $\sigma(\Delta, \Delta') = \sigma(\Delta, \Sigma)$. In particular we get that $x_n \to 0$ in Δ for $\sigma(\Delta, \Sigma_a)$. We need to show that each subsequence (y_n) of $(|x_n|)$ contains a subsequence (z_n) converging weakly to 0 in Δ. Let (y_n) be any subsequence of $(|x_n|)$ and let f_n be functionals on Σ_a defined by

$$f_n(x) = \int_\Omega x y_n \, d\mu, \quad x \in \Sigma_a.$$

Clearly, $f_n \in (\Sigma_a)^*$. Since $y_n \to 0$ weakly in Δ and $\|f_n\| = \|y_n\|_\Delta, C = \sup_n \|f_n\| < \infty$. By the continuous inclusion $L^2 \hookrightarrow \Sigma_a$ and the reflexivity of L^2, we can extract a subsequence

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(n_k) such that $\lim_k f_{n_k}(x)$ exists for each $x \in L^2$. By the density of L^2 in Σ_a and the Banach-Steinhaus theorem we conclude that there exists $y \in \Delta$ such that $y_{n_k} \to y$ in $\sigma(\Delta, \Sigma_a)$. In particular this implies that $y_{n_k} \to y$ in Δ for both topologies $\sigma(\Delta, L^1)$ and $\sigma\big(\Delta, (L^\infty)^\circ\big)$.

It is well known (*cf.* [4, Thm. 2.7.1]) that $\Delta^* = (\Delta, \|\cdot\|_{L^1})^* + (\Delta, \|\cdot\|_{L^\infty})^*$ with equality of norms. Hence a sequence in Δ is weakly convergent if and only if it is weakly convergent in both spaces L^1 and L^∞. Thus $x_n \to 0$ in Δ for both topologies $\sigma(L^1, L^\infty)$ and $\sigma\big(L^\infty, (L^\infty)^*\big)$. Now, by the characterization of weakly compact sets in AL-spaces, it follows that there exists a subsequence (z_n) of (y_{n_k}) such that for some $u \in L^1$, $z_n \to u$ weakly in L^1.

Since L^∞ has weakly sequentially continuous lattice operations, we have $|x_n| \to 0$ weakly in L^∞, and thus $z_n \to 0$ in Δ for $\sigma(\Delta, L^1)$. Hence $y = 0$, and since $z_n \to u$ in $\sigma\big(L^1, (L^\infty)^\circ\big)$ and also $z_n \to y$ in $\sigma\big(L^1, (L^\infty)^\circ\big)$, so $u = y = 0$. Thus $z_n \to 0$ weakly in both spaces L^1 and L^∞ which completes the proof of the theorem.

Proposition 5 *The Banach lattice Σ is a semi-M-space. Consequently, $\Sigma^* = \Sigma_c^* \oplus \Sigma_s^* \simeq \Delta \oplus \Sigma_s^*$, where the singular part Σ_s^* is an* AL*-space.*

Proof We need to consider only nonatomic measure space. Since $\Sigma = \Delta'$ isometrically,

$$\|x\|_\Sigma = \sup\left\{\left|\int_\Omega xy\,d\mu\right| : \|y\|_\Delta \le 1\right\}.$$

This obviously implies that $\|x\|_\Sigma \le \inf\{\big(1+\rho(kx)\big)/k : k > 0\}$, where $\rho(x) = \int_\Omega \varphi(|x|)\,d\mu$ with $\varphi(t) = 0$ for $0 < t \le 1$ and $\varphi(t) = t - 1$ for $t > 1$. On the other hand by (see [3, Prop. 3.3])

$$\|x\|_\Sigma = \int_0^1 x^*(s)\,ds = \sup\left\{\int_A |x|\,d\mu : \mu(A) \le 1\right\},$$

we have that if $\|x\|_\Sigma \le 1$, then $\mu\big(\{\omega : |x(\omega)| > 1\}\big) \le 1$ and hence $\rho(x) \le 1$. Combining this with $\|x\|_\Sigma \le 1 + \rho(x)$, it easily follows that Σ is a semi-M-space. Now, the second part follows from de Jonge's result [19] (see also [26, p. 467]) stating that given an arbitrary normed lattice E, the band E_s^* in the Banach dual E^* is an AL-space if and only if E is a semi-M-space.

Now we are ready to prove the main results of this section.

Theorem 1 *Given a σ-finite measure space (Ω, μ), Δ, Σ, $(L^\infty)^\circ$, Σ° and all their Banach dual spaces have the Dunford-Pettis property.*

Proof We need to consider only the case of infinite nonatomic measure space (Ω, μ).

We shall show at first that Δ has (DP)-property. Let $x_n \to 0$ weakly in Δ and let $f_n \to 0$ weakly in $\Delta^* = \Delta_c^* \oplus \Delta_s^*$. Since Δ_c^* is lattice isometric to the Köthe dual space $\Delta' = \Sigma$, we conclude that for the band projections $P\colon \Delta^* \to \Delta_c^*$ and $Q = \mathrm{Id} - P$, we have for some $y_n \in \Sigma$

$$Pf_n(x) = \int_\Omega xy_n\,d\mu, \quad x \in \Delta.$$

Since $P\colon \Delta^* \to \Delta_c^*$ is norm continuous, we get $y_n \to 0$ in $\sigma(\Sigma, \Sigma^*)$, and in particular $y_n \to 0$ for $\sigma(\Sigma, \Delta)$-topology. By applying Proposition 3, we may assume without loss of

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generality that $y_n^* \le u$ a.e. for all $n \in \mathbb{N}$ and some $u \in \Sigma$. This yields by an application of the well known Hardy-Littlewood inequality [3] that

$$|Pf_n(x_n)| \le \int_\Omega |x_n y_n|\, d\mu \le \int_0^\infty x_n^* y_n^*\, dm \le \int_0^\infty x_n^* u^*\, dm.$$

Now, we shall show that $x_n^* \to 0$ in $\Delta(m)$ for topology $\sigma\big(\Delta(m), \Sigma(m)\big)$. Since $x_n \to 0$ weakly in Δ, we also have $|x_n| \to 0$ weakly in Δ, by Proposition 4. In particular this implies that $\|x_n\|_{L^1} \to 0$ and so $x_n^* \to 0$ a.e.

On the other hand, by $x_n \to 0$ weakly in Δ, we get that $x_n \to 0$ in $\sigma(\Sigma', \Sigma_a)$. By Lemma 2, it follows that $(x_n^*)_{n=1}^\infty$ is a relatively compact set in $\Delta(m)$ for $\sigma\big(\Delta(m), \Sigma_a(m)\big)$. Thus, by Lemma 1 we have

$$x_{n_k}^* \longrightarrow y^* \quad \text{in} \quad \sigma\big(\Delta(m), \Sigma_a(m)\big)$$

for some subsequence $(x_{n_k}^*)$ of (x_n^*) and some $y \in \Delta$. Hence (*cf.* [15, Prop. 40]) $x_{n_k}^* \to y^*$ a.e. and thus $y^* = 0$ a.e. In view of $\|x_n^*\|_{L^1(m)} \to 0$, it is now easily seen that

$$x_n^* \longrightarrow 0 \quad \text{in} \quad \sigma\big(\Delta(m), \Sigma(m)\big).$$

It follows that $Pf_n(x_n) \to 0$. In order to finish the proof we need to show that $Qf_n(x_n) \to 0$.

Since $f_n \to 0$ weakly in Δ^*, we see by the norm continuity of the band projection $Q\colon \Delta^* \to \Delta_s^*$ that $Qf_n \to 0$ weakly in Δ_s^*. Since Δ is a semi-M-space, Δ_s^* is an AL-space by de Jonge's result [19]. Pick up $M > 0$ such that $\|x_n\|_\Delta \le M$ holds for all $n \in \mathbb{N}$. By the characterization of relatively weakly sets in AL-spaces, given $\varepsilon > 0$ we obtain that there exists a nonnegative element $g \in (\Delta_s^*)$ satisfying

$$(Qf_n)_{n=1}^\infty \subset [-g, g] + \frac{\varepsilon}{2M} B_{\Delta_s^*}.$$

By Proposition 4, Δ has weakly sequentially continuous lattice operations, so $|x_n| \to 0$ weakly in Δ. Thus there exists m such that $g(|x_n|) < \varepsilon/2$ for all $n > m$. In particular, for $n > m$ we have

$$|Qf_n(x_n)| \le |Qf_n|(|x_n|) \le g(|x_n|) + \varepsilon/2 \le \varepsilon,$$

which yields that $Qf_n(x_n) \to 0$ holds, as desired. It shows, by $Pf_n(x_n) \to 0$, that $f_n(x_n) \to 0$ and thus Δ has the Dunford-Pettis property.

In view of Proposition 5,

$$\Sigma^* = \Sigma_c^* \oplus \Sigma_s^* \simeq \Delta \oplus \Sigma_s^*,$$

where Σ_s^* is an AL-space. Thus Σ^* has (DP)-property as the direct sum of Banach spaces with (DP)-property, and so Σ has (DP)-property as well. Analogously, in view of

$$\Delta^* = \Delta_c^* \oplus \Delta_s^* \simeq \Sigma \oplus \Delta_s^*,$$

it follows that $\Delta^* \in$ (DP). In fact $\Delta_s^* \in$ (DP) by de Jonge's result since Δ is a semi-M-space. Finally, since all duals of any of the spaces Δ, Σ, $(L^\infty)^\circ$ or Σ° can be decomposed into direct sums of subspaces with (DP)-property, they also have that property.

Theorem 2 *Let E be a symmetric space on a separable measure space (Ω, μ). Then the following statements hold true.*

(i) If μ is a purely atomic measure, then a symmetric sequence space E possesses the Dunford-Pettis property if and only if it coincides with one of the spaces ℓ^1, ℓ^∞ or c_0.

(ii) If μ is nonatomic and finite then E has the Dunford-Pettis property if and only if E is either L^1 or L^∞.

(iii) If μ is nonatomic and infinite then E has the Dunford-Pettis property if and only if E coincides with one of the following spaces L^1, L^∞, Δ, Σ, $(L^\infty)^\circ$ or Σ°.

Proof In the proof we will need the following well known results.

(I) Let E be a symmetric sequence space. Then E coincides with ℓ^1 if and only if the unit vectors (e_n) do not tend weakly to zero in E.

(II) (Th. 2.c.10 in [24]) Let E be a symmetric space on a finite and nonatomic measure space (Ω, μ). Then the Rademacher functions tend weakly to zero in E if and only if E does not coincide with L^∞.

In view of Theorem 1, it is enough to prove only necessity parts of the theorem. Also by the assumption of separability of μ, we restrict our proof to the set of positive integers $\mathbb{N}$ with a counting measure or to the intervals $(0, 1)$ or $(0, \infty)$ with the Lebesgue measure, in the case when μ is purely atomic or μ is nonatomic finite or nonatomic infinite measure, respectively.

Now, assume that E has (DP)-property. If E is a symmetric sequence space then (e_n) is a basic sequence in both spaces E and its Köthe dual E'. Moreover, if $E' = \ell^1$ then E must coincide with ℓ^∞ or c_0. Therefore, assuming that E is none of the spaces ℓ^1, ℓ^∞ or c_0, E' cannot be equal to ℓ^1, and so both E and E' do not coincide with ℓ^1. Applying now (I), the sequence (e_n) is weakly null in both spaces E and E', but $\langle e_n, e_n \rangle = 1$ for every $n \in \mathbb{N}$, which contradicts (DP)-property of E.

In the case of interval $(0, 1)$, the arguments are similar. In fact, we observe that $E = L^1$ whenever $E' = L^\infty$. Assuming that E is neither L^∞ nor L^1 we obtain that both E and E' do not coincide with L^∞. It follows now by (II), that Rademacher functions r_n are weakly null in both E and E', but $\langle r_n, r_n \rangle = 1$ for every $n \in \mathbb{N}$, and this is a contradiction.

Now, consider the space E over the interval $(0, \infty)$. By E_d and E'_d denote the sets of all functions in E and E' respectively, that are constant on all intervals $(n-1, n)$, $n \in \mathbb{N}$. It is clear that they are closed sublattices of E and E', respectively. It is also clear that the functions $\chi_n = \chi_{(n-1,n)}$, $n \in \mathbb{N}$, form a symmetric basic sequence in both E_d and E'_d. Note that the spaces E_d and E'_d may be identified isometrically with symmetric sequence spaces by $\sum_{n=1}^\infty a_n \chi_n \mapsto (a_n)$, and then E'_d is a Köthe dual of E_d. We observe that (χ_n) cannot be weakly null sequence simultaneously in both spaces E and E' in view of the assumption of (DP)-property in E and the obvious fact that $\langle \chi_n, \chi_n \rangle = 1$ for every $n \in \mathbb{N}$. Therefore (χ_n) is not weakly null either in E or in E'. If it is not weakly null in E, equivalently in E_d, then by (I), (χ_n) is equivalent to the unit vector basis (e_n) in ℓ^1 and $E_d = \ell^1$. Analogously, if (χ_n) is not weakly null in E'_d, then also $E'_d = \ell^1$. Therefore $E_d = \ell^\infty$ or $E_d = c_0$.

Now, the complemented subspace $E|_{(0,1)}$ of E has (DP)-property, and since it is symmetric it must be equal either to $L^1(0, 1)$ or to $L^\infty(0, 1)$. As we have just proved, $E|_{(0,1)}$ must be one of the spaces $L^1(0, 1)$ or $L^\infty(0, 1)$, and E_d coincides with either ℓ^1 or ℓ^∞ or c_0, provided that E has (DP)-property. Combining this with the fact that $\Sigma^\circ = L^1 + (L^\infty)^\circ = \{x : x^*(t) \to 0 \text{ as } t \to \infty\}$, we obtain the six spaces listed in (iii) of the theorem.

In the case of a nonatomic finite measure space, the same proof presented above for the interval $(0, 1)$ also works, and thus by Theorem 1, we have instantly the following result.

Theorem 3 *A symmetric space E over nonatomic finite measure space* (Ω, μ) *has the Dunford-Pettis property if and only if it coincides with one of the spaces* L^1 *or* L^∞.

4 A Three-Space Property and the Dunford-Pettis Property

Recall that a property $(\mathcal{P})$ is said to be a *three-space property* if, whenever a closed subspace Y of a Banach space X and the corresponding quotient X/Y have $(\mathcal{P})$, then X also has $(\mathcal{P})$. In [8] the first example of a Banach space X without Dunford-Pettis property such that a subspace Y and the corresponding quotient X/Y have the hereditary Dunford-Pettis property has been constructed. It shows that (DP)-property is not a three-space property. A Banach space X is said to have the *hereditary Dunford-Pettis property* if any closed subspace of X has (DP)-property. It is known that c_0 possesses the hereditary Dunford-Pettis property (see [14, p. 25]). Below, we present a new example showing that the Dunford-Pettis property is not a three-space property.

Let $w = \big(w(j)\big)_{j=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $\lim_{j\to\infty} w(j) = 0$ and $\sum_{j=1}^{\infty} w(j) = \infty$. Recall that a Lorentz sequence space λ_w (*cf.* [24]) is the Banach space of all sequences of scalars $x = \big(x(j)\big)_{j=1}^{\infty}$ for which

$$\|x\|_{\lambda_w} := \sum_{j=1}^{\infty} x^*(j)w(j) < \infty.$$

It is well known [23] that the Köthe dual of λ_w coincides isometrically with the *Marcinkiewicz sequence space* m_w of all sequences of scalars $x = \big(x(j)\big)_{j=1}^{\infty}$ such that

$$\|x\|_{m_w} := \sup_{n\geq 1} \frac{\sum_{j=1}^{n} x^*(j)}{\sum_{j=1}^{n} w(j)} < \infty.$$

It is also clear that all spaces λ_w, m_w and $m_w^\circ = (m_w)_a$ are symmetric sequence spaces, and thus in view of Theorem 2, none of them has (DP)-property. Observe also that the set of unit vectors (e_n) is a basis in both spaces λ_w and m_w°.

We will consider in the sequel a Banach space $m_w^\circ \oplus_1 \ell^1$. It obviously fails (DP)-property. Defining an operator $T\colon m_w^\circ \oplus_1 \ell^1 \to c_0$ by

$$T(x, y) = x + q(y),$$

where $q\colon \ell^1 \to c_0$ is a continuous surjective operator, T is also a surjective operator, and so a quotient space $m_w^\circ \oplus_1 \ell^1/\ker T \simeq c_0$ has the hereditary Dunford-Pettis property. This and the next result yield that (DP)-property is not a three space property.

Proposition 6 *The kernel* $\ker T$ *of the operator* $T\colon m_w^\circ \oplus_1 \ell^1 \to c_0$ *defined above has the hereditary Dunford-Pettis property.*

Proof In fact, by [11] (*cf.* [8]), a Banach space Z has the hereditary Dunford-Pettis property if and only if every weakly null sequence (z_n) admits a subsequence (z_{m_k}) such that for some constant K and for all $N \in \mathbb{N}$,

$$\Big\| \sum_{k=1}^{N} z_{m_k} \Big\|_Z \leq K.$$

Let (x_n, y_n) be a weakly null sequence in $\ker T$. Since ℓ^1 has the Schur property, $y_n \to 0$ in ℓ^1. This implies by $T(x_n, y_n) = 0$ that $\|x_n\|_{c_0} \to 0$. If $\|x_n\|_{m_w^\circ} \to 0$ the proof ends. So assume that (x_n) is a non-convergent sequence in m_w°. Since $x_n \to 0$ weakly in m_w°, we may assume that (x_n) is a basic sequence. Without loss of generality, we may also assume by the Bessaga-Pełczyński selection principle that (x_n) is a normalized block of the unit vector basis (e_n) of c_0. Thus there exists an increasing sequence (p_n) of integers such that

$$x_n = \sum_{j=p_{n-1}+1}^{p_n} \alpha(j) e_j$$

and $\|x_n\|_{m_w^\circ} = 1$. Clearly we have $\lim_{n\to\infty}(p_{n+1} - p_n) = \infty$ by $\|x_n\|_{c_0} \to 0$. Let us denote $A_n = \operatorname{supp} x_n$. We construct, by induction, an increasing sequence $(n_k)_{k=1}^{\infty}$ such that for any $k \in \mathbb{N}$ we have

$$\max\{|\alpha(j)| : j \in A_{n_k}\} \leq \min\{|\alpha(j)| : j \in A_{n_{k-1}}\}$$

and

$$S(p_{n_k} - p_{n_{k-1}}) \leq S(p_{n_{k+1}} - p_{n_{k+1}-1})/2,$$

where $S(n) = \sum_{j=1}^{n} w(j)$ for $n \geq 1$. It is easily seen that such construction is possible in view of $S(n) \to \infty$, $\max\{|\alpha(j)| : j \in A_n\} \to 0$ and $p_{n+1} - p_n \to \infty$ as $n \to \infty$.

We will prove the claim if we show that for every $n \in \mathbb{N}$

$$\Big\| \sum_{k=1}^{n} x_{n_k} \Big\|_{m_w} \leq 3.$$

Since $|\alpha(j)| \leq \min\{|\alpha(j)| : j \in A_{n_{k-1}}\}$ for $j > p_{n_k-1}$, it is enough to prove that for any m and N with $p_{n_m} < N \leq p_{n_m+1}$ the following inequality holds

$$\Big\| \sum_{k=1}^{m} x_{n_k} + \sum_{j=p_{n_m+1}}^{N} \alpha^*(j) e_j \Big\|_{\ell^1} \leq 3S(N).$$

In fact, by $\|x_{n_k}\|_{m_w} = 1$ for all $k \geq 1$, we have

$$\sum_{j=1}^{p_{n_k} - p_{n_k-1}} \alpha^*(j) \leq S(p_{n_k} - p_{n_k-1})$$

and

$$\sum_{j=1}^{N-p_{n_m}} \alpha^*(j) \leq S(N - p_{n_m}).$$

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Combining the above with the inequality $2S(p_{n_k} - p_{n_{k-1}}) \le S(p_{n_{k+1}} - p_{n_{k+1}-1})$ it yields

$$\Big\|\sum_{k=1}^{m} x_{n_k} + \sum_{j=p_{n_m+1}}^{N} \alpha^*(j)e_j\Big\|_{\ell^1} \le \sum_{k=1}^{m} S(p_{n_k} - p_{n_k-1}) + S(N - p_{n_m})$$
$$\le \sum_{k=1}^{m} 2^{k-m} S(p_{n_m} - p_{n_m-1}) + S(N - p_{n_m})$$
$$< 2S(N) + S(N) = 3S(N),$$

which completes the proof.

5 Some Consequences and Remarks

In this section we give some corollaries and applications of the characterization of (DP)-property in symmetric spaces stated in Theorems 1 and 2. We start with a result which is an immediate consequence of Theorem 2(i).

Corollary 1 *A symmetric sequence space E has the Schur property if and only if $E = \ell^1$.*

Corollary 2 *Let $\mu(\Omega) = \infty$. Then the inclusion map $\Delta \hookrightarrow L^1$ is a Dunford-Pettis operator which is not weakly compact.*

Proof By the Schur property of ℓ^1 we only need to consider nonatomic measure space. Let $x_n \to 0$ weakly in Δ. Then by Proposition 3, $|x_n| \to 0$ weakly in Δ as well. Thus by the continuous inclusion $\Delta \hookrightarrow L^1$, $|x_n| \to 0$ weakly in L^1 and thus $\|x_n\|_{L^1} \to 0$. This shows that the inclusion map $\Delta \hookrightarrow L^1$ is a Dunford-Pettis operator.

In order to see that the inclusion map id: $\Delta \hookrightarrow L^1$ is not weakly compact, take any sequence of measurable sets (Ω_n) such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, $\Omega_i \cap \Omega_j = \varnothing$, $i \ne j$ and $\mu(\Omega_n) = 1$ for all $n \in \mathbb{N}$. It is clear that the restriction of id to the closure $[\chi_{\Omega_n}]$ of the linear span of (χ_{Ω_n}) in both spaces Δ and L^1 is an isometry. Since $[\chi_{\Omega_n}]$ is isometrically isomorphic to ℓ_1, id is not a weakly compact operator.

Recall that a Banach space X is said to be a *Grothendieck space* [14] whenever weak* and weak convergence of sequences in X^* coincide. In [16], Grothendieck has proved that every σ-Dedekind complete AM-space with unit, and hence L^∞, is a Grothendieck space. The following statement, useful in the sequel, is also true.

Lemma 3 *Given a separable measure space (Ω, μ), L^∞ is the only Grothendieck symmetric space with the Dunford-Pettis property.*

Proof In fact, since separable Grothendieck spaces are reflexive, in view of Theorem 2, we need only to show that $(L^\infty)^\circ$, Δ and Σ are not Grothendieck spaces on $(0, \infty)$. Clearly, $L^1(0, 1)$ (resp. c_0) is isometrically isomorphic to a complemented subspace of Σ (resp. $(L^\infty)^\circ$) and thus both Σ and $(L^\infty)^\circ$ are not Grothendieck spaces. Analogously, since $\Delta^* \simeq \Sigma \oplus \Delta_s^*$ and Σ contains a copy of c_0, Δ^* is not weakly sequentially complete, so Δ is not a Grothendieck space.

It is well known that L^1 and L^∞ have the unique Banach lattice structure [1] as well as the unique rearrangement-invariant structure [18]. We notice here that this is also a consequence of Theorem 2. As we see below, by application of Theorem 2, we obtain some other examples of symmetric spaces with the unique symmetric structure.

Corollary 3 *Let E be a symmetric space on a separable measure space* (Ω, μ)*. Then the followig statements hold true.*

(i) If E has the Fatou property, then E is isomorphic to an AM*-space if and only if* $E = L^\infty(\mu)$ *up to equivalent norms.*
(ii) E is isomorphic to an AL*-space if and only if* $E = L^1(\mu)$ *up to equivalent norms.*

Proof (i) It is well known that a dual of any AL-space is an AM-space with unit [2]. Assuming now that E is isomorphic to an AM-space F, we obtain that F^{**} is a Dedekind complete AM-space with unit. By the Grothendieck's theorem [16], F^{**} has the Grothendieck property, and so E^{**} as well. By the assumption of the Fatou property and Theorem 8 in [21, p. 297], E is one-complemented in E^{**}, and thus E has the Grothendieck property. Now, Lemma 3 yields that E coincides with $L^\infty(\mu)$.

(ii) If E is isomorphic to an AL-space, then its dual E^* coincides isometrically with its Köthe dual E', which has the Fatou property and is isomorphic to an AM-space. Now by (i), E' coincides with $L^\infty(\mu)$, and hence $E'' = L^1(\mu) = E$.

Corollary 4 *If a symmetric space E on* $(0, \infty)$ *is isomorphic to* Σ° *(resp.* $(L^\infty)^\circ$*) on* $(0, \infty)$*, then* $E = \Sigma^\circ$ *(resp.* $(L^\infty)^\circ$*) up to equivalent norms.*

Proof In view of Theorem 1, if E is isomorphic to either Σ° or $(L^\infty)^\circ$, then E possesses the Dunford-Pettis property. By Theorem 2, it is enough to show that none of the spaces Σ° or $(L^\infty)^\circ$ is isomorphic to any of the spaces L^1, L^∞, Δ, Σ, Σ°, $(L^\infty)^\circ$.

Σ° is not isomorphic to L^1 since L^1 does not contain a copy of c_0, but the sequence $(\chi_{(n-1,n)})$ in Σ° is equivalent to the unit vector basis (e_n) in c_0. Also Σ° is not isomorphic to any other spaces, since Σ° is separable.

The spaces L^∞ and $(L^\infty)^\circ$ are not isomorphic, since $(L^\infty)^\circ$ contains a complemented subspace of c_0, while L^∞ being isomorphic to ℓ^∞ does not [24].

Finally, if $(L^\infty)^\circ$ was isomorphic to either Δ or Σ, then in view of the Fatou property of both spaces Δ and Σ, and the obvious fact that $(L^\infty)^\circ$ is an AM-space, Δ or Σ would be equal to L^∞ by Corollary 3(i), which is not true.

The next consequence of the characterization obtained in Theorem 2 concerns Köthe-Bochner spaces. Recall that if E is a Banach lattice on (Ω, μ) and X is any Banach space then $E(X)$ denotes the Köthe-Bochner space of all strongly measurable functions $x \colon \Omega \to X$ such that $\|x(\cdot)\|_X \in E$, with the norm $\|x\| = \big\| \|x(\cdot)\|_X \big\|_E$.

Corollary 5 *Let E and F be two symmetric spaces on finite or purely atomic measure space* (Ω, μ)*. Then E(F) has the Dunford-Pettis property if and only if E or F is one of the spaces* L^1 *or* L^∞ *or* c_0*.*

Proof The necessity follows by Theorem 2 and by an easily verified fact that E and X embed complementably in $E(X)$.

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The sufficiency follows by [13] and [10]. In fact, in [13] it is shown that for finite measure space (Ω, μ), $L^\infty(X)$ has the Dunford-Pettis property if and only if $\ell^\infty(X)$ has it. It is also proved in [13] that if either X is any $\mathcal{L}^1$-space or any $\mathcal{L}^\infty$-space, then $L^\infty(X)$ has the Dunford-Pettis property. On the other hand in [10] it is proved that if X is any $\mathcal{L}^1$-space or any $\mathcal{L}^\infty$-space, then $L^1(X)$ has the Dunford-Pettis property. Since for any Banach lattice X, $c_0(X)$ is a closed ideal in a Banach lattice $\ell^\infty(X)$, and the Dunford-Pettis property is inherited by ideals (see [30]), the proof is finished by combining the above.

Finally, as an application of the Dunford-Pettis property of the space Δ, we obtain a generalization (to infinite interval $(0, \infty)$) of the Novikov's result [27], stating that the inclusion map $L^\infty(0,1) \hookrightarrow E(0,1)$ is a strictly singular operator, which in particular, when $E = L^p(0,1)$, is the well known Grothendieck's theorem.

Corollary 4 *Let E be a symmetric space on (Ω, μ) such that the inclusion map $\Delta \hookrightarrow E$ is weakly compact. Then $\Delta \hookrightarrow E$ is a strictly singular operator, i.e., any infinite-dimensional subspace of Δ is not a closed subspace of E.*

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Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
USA
email: kaminska@msci.memphis.edu

Faculty of Mathematics and Computer Science
A. Mickiewicz University
Matejki 48/49
60-769 Poznań
Poland
email: mastylo@math.amu.edu.pl

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