ASYMPTOTIC FORMULÆ FOR THE NUMBER OF PARTITIONS OF A MULTI-PARTITE NUMBER

by M. M. ROBERTSON
(Received 20th January 1960)

1. Introduction

A multi-partite number of order j is a j dimensional vector, the components of which are non-negative rational integers. A partition of $(n_1, n_2, \ldots, n_j)$ is a solution of the vector equation

$$\sum_k (n_{1k}, n_{2k}, \ldots, n_{jk}) = (n_1, n_2, \ldots, n_j) \qquad (1)$$

in multi-partite numbers other than $(0, 0, \ldots, 0)$. Two partitions, which differ only in the order of the multi-partite numbers on the left-hand side of (1), are regarded as identical. We denote by $p_1(n_1, \ldots, n_j)$ the number of different partitions of $(n_1, \ldots, n_j)$ and by $p_2(n_1, \ldots, n_j)$ the number of those partitions in which no part has a zero component. Also, we write $p_3(n_1, \ldots, n_j)$ for the number of partitions of $(n_1, \ldots, n_j)$ into different parts and $p_4(n_1, \ldots, n_j)$ for the number of partitions into different parts none of which has a zero component.

By adaptations to $j>1$ of the celebrated Hardy-Ramanujan method (1) for the $j = 1$ case, several authors have recently obtained asymptotic expressions for $p_r(n_1, \ldots, n_j)$, which are valid under certain restrictions upon the relative rates at which the different n_l tend to infinity. Auluck (3) obtained a formula for $p_1(n_1, n_2)$, where n_1 and n_2 are large but of the same order of magnitude, i.e. the ratio n_1/n_2 is bounded above and below, and, under the same conditions, Wright (7) found asymptotic expressions for $p_r(n_1, n_2)$, where $r = 1, 2, 3$ and 4. In his article, Wright also gave without proof the first few terms of an asymptotic formula for $\log p_2(n_1, \ldots, n_j)$, where every n_l is of the same order of magnitude. Meinardus (4) had just previously published a paper in which he had found the first term of this formula for multi-partites. Later, Wright (8) obtained asymptotic expressions for $p_r(n_1, n_2)$ which hold for $n_1^{\frac{1}{2}+\varepsilon_1}<n_2<n_1^{2-\varepsilon_2}$, where $r = 1, 2, 3$ and 4 and ε_1 and ε_2 are any fixed positive numbers. This is a substantial relaxation of the restrictions imposed upon n_1 and n_2 in both (3) and (7). In his article, Auluck also obtained a formula for $p_1(n_1, n_2)$ when n_2 is fixed and n_1 is large, and Nanda (5) has shown that this formula remains valid when n_2 is large, provided that $n_2 = o(n_1^{\frac{1}{4}})$. In an article in preparation, I extend Wright's method to derive formulæ for $p_r(n_1, \ldots, n_j)$ for $r = 1, 2, 3$ and 4 and $n_1 \ldots n_j < \bar{n}^{j+1-\varepsilon_3}$, where $\bar{n} = \min n_l$ and ε_3 is any fixed positive number. In this article, I evaluate $p_r(n_1, \ldots, n_j)$ for $r = 1$ and 3 when one particular n_l tends to infinity more rapidly than the fourth power of every other n_l by means of an extension of Nanda's method † and I also

† This problem was suggested to me by Professor E. M. Wright to whom I am also grateful for much valuable advice in the course of the investigation.

https://doi.org/10.1017/S0013091500025025 Published online by Cambridge University Press

obtain an asymptotic formula for $p_r(n_1, \ldots, n_j)$ for $r = 2$ and 4 when one particular n_l tends to infinity more slowly than the cube root of every other n_l.

The letters $h, k, l, m, n, N, q, r, R, R'$ and v represent non-negative integers which may be fixed or variable according to the context and j is used for a fixed integer greater than unity. C is a positive number, not necessarily the same at each occurrence, which may depend upon j but not upon any n_l. When there is no statement to the contrary, the symbols $O(\ )$, $o(\ )$ and $\sim$ refer to the passage of the n_l to infinity.

2. Asymptotic Formulæ for $p_r(n_1, \ldots, n_j)$

It is easily seen that $p_r(n_1, \ldots, n_j)$ is a symmetric function of $n_1, \ldots, n_j$ and so, without any loss of generality, we may suppose that $n_1 \geqq n_2 \geqq \ldots \geqq n_j$. Nanda (5) has shown that the asymptotic formula

$$p_1(n_1, n_2) \sim \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}n_2} \{4\sqrt{3}n_1(n_2!)\}^{-1} \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}$$

as $n_1 \to \infty$ holds for $n_2 = o(n_1^{\frac{1}{3}})$. If we write $R_j = \sum_{l=2}^{j} n_l$, the above formula is seen to be a particular case of the following more general theorem.

Theorem 1. *If $n_l = o(n_1^{\frac{1}{3}})$ for $2 \leqq l \leqq j$, then*

$$p_1(n_1, \ldots, n_j) \sim \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}R_j} \left(4\sqrt{3}n_1 \prod_{l=2}^{j} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}$$

as $n_1 \to \infty$.

Asymptotic formulæ can also be obtained for $p_r(n_1, \ldots, n_j)$ when $r = 2$, 3 and 4, and indeed the following theorems will be proved.

Theorem 2. *If $n_j = o(n_l^{\frac{1}{3}})$ for $1 \leqq l \leqq j-1$, then*

$$p_2(n_1, \ldots, n_j) \sim (n_1 \ldots n_{j-1})^{n_j - 1} \{(n_j - 1)!\}^{1-j} (n_j!)^{-1}$$

as $n_l \to \infty$ for $1 \leqq l \leqq j-1$.

Theorem 3. *If $n_l = o(n_1^{\frac{1}{3}})$ for $2 \leqq l \leqq j$, then*

$$p_3(n_1, \ldots, n_j) \sim \left(\frac{12n_1}{\pi^2}\right)^{\frac{1}{2}R_j} \left(4.3^{\frac{1}{4}} n_1^{\frac{3}{4}} \prod_{l=2}^{j} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{n_1}{3}\right)}\right\}$$

as $n_1 \to \infty$.

Theorem 4. *If $n_j = o(n_l^{\frac{1}{3}})$ for $1 \leqq l \leqq j-1$, then*

$$p_4(n_1, \ldots, n_j) \sim (n_1 \ldots n_{j-1})^{n_j - 1} \{(n_j - 1)!\}^{1-j} (n_j!)^{-1}$$

as $n_l \to \infty$ for $1 \leqq l \leqq j-1$.

3. Two Lemmas

We put

$$\alpha_1(h_1, \ldots, h_j) = \alpha_2(h_1, \ldots, h_j) = (1 - x_1^{h_1} \ldots x_j^{h_j})^{-1}$$

and

$$\alpha_3(h_1, \ldots, h_j) = \alpha_4(h_1, \ldots, h_j) = 1 + x_1^{h_1} \ldots x_j^{h_j},$$

where $|x_l| < 1$ for $1 \leqq l \leqq j$. Then we write

$$f_r(x_1, \ldots, x_j) = \prod_{h_1, \ldots, h_j} \alpha_r(h_1, \ldots, h_j),$$

where, for $r = 2$ and 4, $h_1, \ldots, h_j$ each take all positive integral values, while, for $r = 1$ and 3, $h_1, \ldots, h_j$ each take all non-negative integral values except $h_1 = \ldots = h_j = 0$. If we put $p_r(0, 0, \ldots, 0) = 1$, we can easily verify from the definitions of $p_r(n_1, \ldots, n_j)$ that

$$f_r(x_1, \ldots, x_j) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_j=0}^{\infty} p_r(n_1, \ldots, n_j) x_1^{n_1} \ldots x_j^{n_j}$$

for $r = 1, 2, 3$ and 4.

Before proceeding with the proof of Theorem 1, we require the following lemma.

Lemma 1. *If, when $2 \leqq k < C$ and $n_l = o(n_1^{\frac{1}{4}})$ for $2 \leqq l \leqq k$,*

$$p_1(n_1, \ldots, n_k) = \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}R_k} \left(4\sqrt{3}n_1 \prod_{l=2}^{k} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1 + \sum_{r=1}^{R_k+1} O(rn_1^{-\frac{1}{2}})\right\}$$

as $n_1 \to \infty$, and if

$$f_1(x_1, \ldots, x_k) \prod_{h=1}^{q} \prod_{l=1}^{k} (1 - x_l^{N_h})^{-1} = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} \varpi_{N_1 \ldots N_q}(n_1, \ldots, n_k) x_1^{n_1} \ldots x_k^{n_k},$$

where $N_h \geqq 1$ for $1 \leqq h \leqq q$, then, provided that

$$R' = \sum_{h=1}^{q} N_h = o(n_1^{\frac{1}{4}}),$$

$$\varpi_{N_1 \ldots N_q}(n_1, \ldots, n_k)$$

$$= \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}(R_k+q)} \left(4\sqrt{3}n_1 \prod_{h=1}^{q} N_h \prod_{l=2}^{k} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}\left\{1 + \sum_{r=1}^{R_k+R'+1} O(rn_1^{-\frac{1}{2}})\right\} \ldots (2)$$

as $n_1 \to \infty$, where the constants implicit in the "O" terms on the right-hand side of (2) *are independent of q.*

Before we prove Lemma 1, we prove

Lemma 2. *If $k \geqq 0$, $m \geqq 1$ and ρ is any fixed positive number, then*

$$\sum_{r=0}^{[\frac{1}{2}m^{-1}n]} (n - mr)^{\frac{1}{2}(k-3)} \exp\{\rho\sqrt{(n - mr)}\}$$
$$= (2/\rho m) n^{\frac{1}{2}(k-2)} \exp(\rho\sqrt{n})\{1 + O(kn^{-\frac{1}{2}}) + O(mn^{-\frac{1}{2}})\} \ldots\ldots\ldots(3)$$

as $n \to \infty$, provided that $k = o(n^{\frac{1}{4}})$ and $m = o(n^{\frac{1}{4}})$.

In order to prove Lemma 1, it is sufficient to show that Lemma 2 holds when k and m are each $o(n^{\frac{1}{4}})$, but it is evident from the following proof that Lemma 2 remains true provided that k and m are each $o(n^{\frac{1}{2}})$. If $0 \leqq t \leqq \frac{1}{2}$

and $k \geqq 2$, then

$$1-\tfrac{1}{2}kt \leqq (1-t)^{\frac{1}{2}k} \leqq 1.$$

Also, for $0 \leqq t \leqq \frac{1}{2}$,

$$1-\tfrac{1}{2}t-\tfrac{1}{2}t^2 \leqq (1-t)^{\frac{1}{2}} \leqq 1-\tfrac{1}{2}t$$

and

$$1 \leqq (1-t)^{-\frac{1}{2}} \leqq (1-t)^{-1} \leqq (1-t)^{-\frac{3}{2}} \leqq 1+4t$$

and, for all $t \geqq 0$,

$$1-t \leqq \exp(-t).$$

Hence, for all $k \geqq 0$,

$$\sum_{r=0}^{[\frac{1}{2}m^{-1}n]} \left(1-\frac{kmr}{2n}\right)\left(1-\frac{\rho m^2 r^2}{2n^{\frac{3}{2}}}\right)\exp\left(-\frac{\rho mr}{2\sqrt{n}}\right) \leqq \Sigma^* n^{-\frac{1}{2}(k-3)} \exp(-\rho\sqrt{n})$$

$$\leqq \sum_{r=0}^{[\frac{1}{2}m^{-1}n]} \left(1+\frac{4mr}{n}\right)\exp\left(-\frac{\rho mr}{2\sqrt{n}}\right),$$

where Σ^* denotes the sum on the left-hand side of (3). Therefore, since $k = o(n^{\frac{1}{4}})$ and $m = o(n^{\frac{1}{2}})$,

$$(2\sqrt{n}/\rho m)\{1+O(kn^{-\frac{1}{2}})+O(mn^{-\frac{1}{2}})\}$$

$$\leqq \Sigma^* n^{-\frac{1}{2}(k-3)} \exp(-\rho\sqrt{n}) \leqq (2\sqrt{n}/\rho m)\{1+O(mn^{-\frac{1}{2}})\}$$

and Lemma 2 follows immediately.

We now prove Lemma 1. From the definition of $\varpi_{N_1}(n_1, \ldots, n_k)$, we have

$$\varpi_{N_1}(n_1, \ldots, n_k) = \sum_{\nu_1=0}^{[N_1^{-1}n_1]} \cdots \sum_{\nu_k=0}^{[N_1^{-1}n_k]} p_1(n_1-N_1\nu_1, \ldots, n_k-N_1\nu_k).$$

From Lemma 2, we obtain

$$\sum_{\nu_1=0}^{[\frac{1}{2}N_1^{-1}n_1]} p_1(n_1-N_1\nu_1, n_2, \ldots, n_k) = \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}(R_k+1)} \left(4\sqrt{3}n_1 N_1 \prod_{l=2}^{k} n_l!\right)^{-1}$$

$$\times \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1+\sum_{r=1}^{R_k+1} O(rn_1^{-\frac{1}{2}})+O(\{R_k+3\}n_1^{-\frac{1}{2}})+O(N_1 n_1^{-\frac{1}{2}})\right\}$$

$$= \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}(R_k+1)} \left(4\sqrt{3}n_1 N_1 \prod_{l=2}^{k} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1+\sum_{r=1}^{R_k+N_1+1} O(rn_1^{-\frac{1}{2}})\right\}.$$

Clearly, when $\nu_1 > [\frac{1}{2}N_1^{-1}n_1]$,

$$p_1(n_1-N_1\nu_1, n_2, \ldots, n_k) \leqq p_1([\tfrac{1}{2}n_1], n_2, \ldots, n_k)$$

and therefore,

$$\sum_{\nu_1=[\frac{1}{2}N_1^{-1}n_1]+1}^{[N_1^{-1}n_1]} p_1(n_1-N_1\nu_1, n_2, \ldots, n_k)$$

$$< CN_1^{-1}n_1 \left(\frac{3n_1}{\pi^2}\right)^{\frac{1}{2}R_k} \left(4\sqrt{3}n_1 \prod_{l=2}^{k} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{n_1}{3}\right)}\right\}.$$

It follows that

$$\sum_{\nu_1=0}^{[N_1^{-1}n_1]} p_1(n_1-N_1\nu_1, n_2, \ldots, n_k)$$
$$= \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}(R_k+1)}\left(4\sqrt{3}n_1N_1\prod_{l=2}^{k} n_l!\right)^{-1}\exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}$$
$$\times\left\{1+\sum_{r=1}^{R_k+N_1+1} O(rn_1^{-\frac{1}{2}})\right\}.$$

Hence,

$$\bar{\omega}_{N_1}(n_1, \ldots, n_k)$$
$$=\left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}(R_k+1)}\left(4\sqrt{3}n_1N_1\prod_{l=2}^{k} n_l!\right)^{-1}\exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1+\sum_{r=1}^{R_k+N_1+1} O(rn_1^{-\frac{1}{2}})\right\}$$
$$\times\prod_{l=2}^{k}\left\{1+\frac{n_l!}{(n_l-N_1)!}\left(\frac{6n_1}{\pi^2}\right)^{-\frac{1}{2}N_1}+\frac{n_l!}{(n_l-2N_1)!}\left(\frac{6n_1}{\pi^2}\right)^{-N_1}\right.$$
$$\left.+\ldots+\frac{n_l!}{(n_l-[N_1^{-1}n_l]N_1)!}\left(\frac{6n_1}{\pi^2}\right)^{-\frac{1}{2}[N_1^{-1}n_l]N_1}\right\}$$
$$=\left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}(R_k+1)}\left(4\sqrt{3}n_1N_1\prod_{l=2}^{k} n_l!\right)^{-1}\exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1+\sum_{r=1}^{R_k+N_1+1} O(rn_1^{-\frac{1}{2}})\right\},$$

since $n_l = o(n_1^{\frac{1}{4}})$ for $2 \leqq l \leqq k$. Next, if we assume that (2) holds for any positive integers q, N_1, ..., N_q such that $\sum_{h=1}^{q} N_h = o(n_1^{\frac{1}{4}})$, an argument exactly similar to the above shows that (2) remains true when q is replaced by $q+1$ and $N_{q+1} = o(n_1^{\frac{1}{4}})$. Lemma 1 follows immediately by inductive reasoning.

4. Proof of Theorem 1

The generating function of $p_1(n_1, \ldots, n_{k+1})$ is

$$f_1(x_1, \ldots, x_{k+1}) = \prod_{h_1, \ldots, h_{k+1}} (1-x_1^{h_1}\ldots x_{k+1}^{h_{k+1}})^{-1},$$

where the product is taken over all non-negative integers h_1, ..., h_{k+1} except $h_1 = \ldots = h_{k+1} = 0$. It follows that, for $k \geqq 1$,

$$f_1(x_1, \ldots, x_{k+1}) = f_1(x_1, \ldots, x_k)\prod_{h_1, \ldots, h_{k+1}}' (1-x_1^{h_1}\ldots x_{k+1}^{h_{k+1}})^{-1}, \quad \ldots\ldots\ldots(4)$$

where the latter product is taken over all non-negative h_1, ..., h_k and all positive h_{k+1}. We write

$$\prod_{h_1, \ldots, h_{k+1}}' (1-x_1^{h_1}\ldots x_{k+1}^{h_{k+1}})^{-1} = 1+\sum_{n=1}^{\infty} A_n x_{k+1}^n, \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where

$$A_n = \sum_{(n)}\prod_{m} c_{\nu_m}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

https://doi.org/10.1017/S0013091500025025 Published online by Cambridge University Press

the sum being taken over all partitions of n of the form $n = \sum_m m\nu_m$ and the product over all the different parts m of the partition, and c_n is the coefficient of y^n in $g(y)$, where

$$g(y) = \prod_{h_1=0}^{\infty} \cdots \prod_{h_k=0}^{\infty} (1 - x_1^{h_1} \ldots x_k^{h_k} y)^{-1}$$

and $|y| < 1$. Also

$$\log g(y) = -\sum_{h_1=0}^{\infty} \cdots \sum_{h_k=0}^{\infty} \log(1 - x_1^{h_1} \ldots x_k^{h_k} y)$$

$$= \sum_{h_1=0}^{\infty} \cdots \sum_{h_k=0}^{\infty} \sum_{r=1}^{\infty} r^{-1} x_1^{rh_1} \ldots x_k^{rh_k} y^r$$

$$= \sum_{r=1}^{\infty} r^{-1} y^r \prod_{l=1}^{k} (1 - x_l^r)^{-1}$$

and so,

$$g(y) = \exp\left\{ \sum_{r=1}^{\infty} r^{-1} y^r \prod_{l=1}^{k} (1 - x_l^r)^{-1} \right\}.$$

It follows that

$$c_n = \sum_{(n)} \prod_m (\nu_m'!)^{-1} \left\{ m \prod_{l=1}^{k} (1 - x_l^m) \right\}^{-\nu_m'}, \qquad \ldots\ldots(7)$$

where the sum is taken over all partitions of n of the form $n = \sum_m m\nu_m'$ and the product over all the different parts m of the partition.

We now prove by induction that, if $n_l = o(n_1^{\frac{1}{4}})$ for $2 \leqq l \leqq j$, then

$$p_1(n_1, \ldots, n_j)$$
$$= \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}R_j} \left(4\sqrt{3}n_1 \prod_{l=2}^{j} n_l!\right)^{-1} \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1 + \sum_{r=1}^{R_j+1} O(rn_1^{-\frac{1}{2}})\right\} \ldots\ldots(8)$$

as $n_1 \to \infty$. In (**5**), Nanda has already demonstrated that (8) is true for $j = 2$. Here, we assume that (8) holds for $j = k$, where k is any fixed positive integer greater than unity. From (4) and (5), $p_1(n_1, \ldots, n_{k+1})$ is equal to the coefficient of $x_1^{n_1} \ldots x_k^{n_k}$ in $A_{n_{k+1}} f_1(x_1, \ldots, x_k)$. We see from (6) that there is a one-to-one correspondence between the terms of A_n and the partitions of n. We therefore divide the partitions of n into classes in which each partition has the same number of parts and we make a corresponding division of the terms of A_n into sets. For $0 \leqq q \leqq n-1$, the $(q+1)$th set has $p_1^{(n-q)}(n)$ terms, where $p_1^{(n-q)}(n)$ denotes the number of partitions of n into exactly $n-q$ parts. In the first set there is only one term and its contribution to $p_1(n_1, \ldots, n_{k+1})$ is equal to the coefficient of $x_1^{n_1} \ldots x_k^{n_k}$ in $c_{n_{k+1}} f_1(x_1, \ldots, x_k)$. Also, from (7) and Lemma 1, the coefficient of $x_1^{n_1}\dots x_k^{n_k}$ in $c_{n_{k+1}}f_1(x_1, \dots, x_k)$ is asymptotically equal to

$$\sum_{(n_{k+1})}\prod_m (v_m'!)^{-1} m^{-2v_m'}\left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}(R_k+\sum_m v_m')}$$
$$\times\left(4\sqrt{3n}\prod_{l=2}^{k} n_l!\right)^{-1}\exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1+\sum_{r=1}^{R_{k+1}+1} O(rn_1^{-\frac{1}{2}})\right\}$$
$$=\left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}R_{k+1}}\left(4\sqrt{3n_1}\prod_{l=2}^{k+1} n_l!\right)^{-1}\exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1+\sum_{r=1}^{R_{k+1}+1} O(rn_1^{-\frac{1}{2}})\right\}$$

provided that

$$n_{k+1}!\sum_{(n_{k+1})}\prod_m (v_m'!)^{-1} m^{-2v_m'}\left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{2}(\sum_m v_m'-n_{k+1})}=1+\sum_{r=1}^{R_{k+1}+1} O(rn_1^{-\frac{1}{2}}). \quad\text{.........(9)}$$

It is easily seen that any partition of n_{k+1} into $n_{k+1}-q$ parts, where $q<\frac{1}{2}n_{k+1}$, must contain at least $n_{k+1}-2q$ units. Therefore, for any particular partition $\sum_m mv_m'$ of n_{k+1} into $n_{k+1}-q$ parts, $\prod_m v_m'!\geqq\Lambda_{n_{k+1}-2q}$, where $\Lambda_{n_{k+1}-2q}=(n_{k+1}-2q)!$ for $q<\frac{1}{2}n_{k+1}$ and $\Lambda_{n_{k+1}-2q}=1$ for $q\geqq\frac{1}{2}n_{k+1}$. Also, $p_1^{(n_{k+1}-q)}(n_{k+1})=p_1(q)$ for $q\leqq\frac{1}{2}n_{k+1}$ and $p_1^{(n_{k+1}-q)}(n_{k+1})<p_1(q)$ for $q>\frac{1}{2}n_{k+1}$. Hence, in order to prove (9), it is sufficient to show that

$$\sum_{q=1}^{n_{k+1}-1} p_1(q)\left(\frac{6n_1}{\pi^2}\right)^{-\frac{1}{2}q}(n_{k+1}!)\Lambda_{n_{k+1}-2q}^{-1}=\sum_{r=1}^{R_{k+1}+1} O(rn_1^{-\frac{1}{2}})\text{............(10)}$$

The Hardy-Ramanujan formula (**1**) for $p_1(q)$ shows that, for all $q>0$,

$$p_1(q)<Cq^{-1}\exp\{\pi\sqrt{(2q/3)}\}.$$

Therefore,

$$\sum_{q=2}^{n_{k+1}-1} p_1(q)\left(\frac{6n_1}{\pi^2}\right)^{-\frac{1}{2}q}(n_{k+1}!)\Lambda_{n_{k+1}-2q}^{-1}$$
$$<C\sum_{q=2}^{n_{k+1}-1} q^{-1}\exp\left\{\pi\sqrt{\left(\frac{2q}{3}\right)}\right\}\left(\frac{6n_1}{\pi^2}\right)^{-\frac{1}{2}q} n_{k+1}^{2q}$$
$$=C\sum_{q=2}^{n_{k+1}-1}\exp\left\{\pi\sqrt{\left(\frac{2q}{3}\right)}-\log q-\tfrac{1}{2}q\log\left(\frac{6n_1}{\pi^2 n_{k+1}^4}\right)\right\}$$
$$<C\sum_{q=2}^{n_{k+1}-1}\exp\left\{-\tfrac{1}{4}q\log\left(\frac{6n_1}{\pi^2 n_{k+1}^4}\right)\right\}$$
$$=O(n_{k+1}^2 n_1^{-\frac{1}{2}})$$

since $n_{k+1}=o(n_1^{\frac{1}{8}})$, and (10) follows immediately.

To complete the proof of (8), we have only to show that the contributions to $p_1(n_1, \dots, n_{k+1})$ from the other terms of $A_{n_{k+1}}$ can be neglected. By repeated applications of a similar argument to that employed in determining the coefficient of $x_1^{n_1}\dots x_k^{n_k}$ in $c_{n_{k+1}}f_1(x_1, \dots, x_k)$, we can show that the coefficient of

$x_1^{n_1} \ldots x_k^{n_k}$ in $\prod_m c_{\nu_m} f_1(x_1, \ldots, x_k)$ is asymptotically equal to

$$\left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}(R_k + \sum_m \nu_m)} \left(4\sqrt{3}n_1 \prod_{l=2}^{k} n_l! \prod_m \nu_m!\right)^{-1} \times \exp\left\{\pi\sqrt{\left(\frac{2n_1}{3}\right)}\right\}\left\{1 + \sum_{r=1}^{R_k + \sum_m \nu_m + 1} O(rn_1^{-2})\right\}.$$

It therefore remains to show that

$$n_{k+1}! \sum_{(n_{k+1})} \prod_m (\nu_m!)^{-1} \left(\frac{6n_1}{\pi^2}\right)^{\frac{1}{4}(\sum_m \nu_m - n_{k+1})} = 1 + \sum_{r=1}^{R_{k+1}+1} O(rn_1^{-\frac{1}{2}});$$

and this follows in exactly the same manner as did (9). Finally, since $n_l = o(n_1^{\frac{1}{4}})$ for $2 \leqq l \leqq j$, Theorem 1 is an immediate consequence of (8).

5. Proof of Theorems 2, 3 and 4

In (**1**), Hardy and Ramanujan obtained the asymptotic formula

$$p_3(n_1) = (4.3^{\frac{1}{4}} n_1^{\frac{3}{4}})^{-1} \exp\{\pi\sqrt{(n_1/3)}\}\{1 + O(n_1^{-\frac{1}{2}})\}$$

as $n_1 \to \infty$ and we can easily deduce, by a similar method to that employed by Nanda (**5**), that

$$p_3(n_1, n_2) = \left(\frac{12n_1}{\pi^2}\right)^{\frac{1}{4}n_2} \{4.3^{\frac{1}{4}} n_1^{\frac{3}{4}}(n_2!)\}^{-1} \exp\left\{\pi\sqrt{\left(\frac{n_1}{3}\right)}\right\}\left\{1 + \sum_{r=1}^{n_2+1} O(rn_1^{-\frac{1}{2}})\right\}$$

as $n_1 \to \infty$ for $n_2 = o(n_1^{\frac{1}{4}})$. The extension to the general j-partite number can be carried out exactly as in the proof of (8) and Theorem 3 follows immediately, since $n_l = o(n_1^{\frac{1}{4}})$ for $2 \leqq l \leqq j$.

We now prove Theorems 2 and 4. We denote by $p_2^{(n_j - q)}(n_1, \ldots, n_j)$ the number of different partitions of $(n_1, \ldots, n_j)$ into exactly $n_j - q$ parts in which no part has a zero component and we write $p_4^{(n_j - q)}(n_1, \ldots, n_j)$ for the number of partitions of $(n_1, \ldots, n_j)$ into exactly $n_j - q$ unequal parts in which no part has a zero component. For any particular partition $\sum_m m\nu_m$ of n_j into exactly $n_j - q$ parts, the parts can be arranged in $(n_j - q)!/\prod_m \nu_m!$ distinguishable ways. If $\sum_{k=1}^{n_j - q} n_{lk}$ is any partition of n_l into $n_j - q$ parts for $2 \leqq l \leqq j$, then the maximum number of distinct partitions of $(n_1, \ldots, n_j)$ into $n_j - q$ parts in the set

$$\sum_{k=1}^{n_j - q} (n_{1k}, m_{2k}, \ldots, m_{jk}),$$

where, for $2 \leqq l \leqq j$, $m_{l1}, \ldots, m_{l, n_j - q}$ run through the distinguishable arrangements of $n_{l1}, \ldots, n_{l, n_j - q}$, is obtained when $\sum_{k=1}^{n_j - q} n_{1k}$ is a partition of n_1 into unequal parts. It follows that

$$p_2^{(n_j - q)}(n_1, \ldots, n_j) \leqq \{(n_j - q)!\}^{j-1} \prod_{l=1}^{j-1} p_2^{(n_j - q)}(n_l) \sum_{(n_j, q)} \prod_m (\nu_m!)^{-1},$$

https://doi.org/10.1017/S0013091500025025 Published online by Cambridge University Press

where the sum is taken over all partitions of n_j into n_j-q parts of the form $n_j=\sum_m m\nu_m$. We also have

$$p_4^{(n_j-q)}(n_1, \ldots, n_j) \geqq \{(n_j-q)!\}^{j-1} \prod_{l=1}^{j-1} p_4^{(n_j-q)}(n_l) \sum_{(n_j, q)} \prod_m (\nu_m!)^{-1}.$$

Since

$$p_4^{(n_j-q)}(n_1, \ldots, n_j) \leqq p_2^{(n_j-q)}(n_1, \ldots, n_j)$$

by definition, we obtain

$$\{(n_j-q)!\}^{j-1} \prod_{l=1}^{j-1} p_4^{(n_j-q)}(n_l) \sum_{(n_j, q)} \prod_m (\nu_m!)^{-1} \leqq p_4^{(n_j-q)}(n_1, \ldots, n_j)$$

$$\leqq p_2^{(n_j-q)}(n_1, \ldots, n_j) \leqq \{(n_j-q)!\}^{j-1} \prod_{l=1}^{j-1} p_2^{(n_j-q)}(n_l) \sum_{(n_j, q)} \prod_m (\nu_m!)^{-1} \quad \ldots\ldots\ldots(11)$$

Next, we use the formula of Erdös and Lehner (**2**),

$$p_2^{(k)}(n) \sim \frac{1}{k!}\binom{n-1}{k-1}$$

as $n\to\infty$ for $k=o(n^{\frac{1}{3}})$, in the form, more convenient for our present purposes,

$$p_2^{(k)}(n) = n^{k-1}\{k!(k-1)!\}^{-1}\{1+o(1)\}.$$

We see that

$$\begin{aligned} p_4^{(k)}(n) &= p_2^{(k)}\{n-\tfrac{1}{2}k(k-1)\} \\ &= \{n-\tfrac{1}{2}k(k-1)\}^{k-1}\{k!(k-1)!\}^{-1}\{1+o(1)\} \\ &= n^{k-1}\{k!(k-1)!\}^{-1}\{1+o(1)\} \end{aligned}$$

as $n\to\infty$, provided that $k=o(n^{\frac{1}{3}})$. Therefore, since $n_j=o(n_l^{\frac{1}{3}})$ for $1\leqq l\leqq j-1$, we obtain

$$(n_1\ldots n_{j-1})^{n_j-q-1}\{(n_j-q-1)!\}^{1-j} \sum_{(n_j, q)} \prod_m (\nu_m!)^{-1}\{1+o(1)\} \leqq p_4^{(n_j-q)}(n_1, \ldots, n_j)$$

$$\leqq p_2^{(n_j-q)}(n_1, \ldots, n_j) \leqq (n_1\ldots n_{j-1})^{n_j-q-1}\{(n_j-q-1)!\}^{1-j} \sum_{(n_j, q)} \prod_m (\nu_m!)^{-1}\{1+o(1)\} \quad \ldots(12)$$

from (11). By putting $q=0$ in (12), we obtain

$$(n_1\ldots n_{j-1})^{n_j-1}\{(n_j-1)!\}^{1-j}(n_j!)^{-1}\{1+o(1)\} \leqq p_4^{(n_j)}(n_1, \ldots, n_j)$$

$$\leqq p_2^{(n_j)}(n_1, \ldots, n_j) \leqq (n_1\ldots n_{j-1})^{n_j-1}\{(n_j-1)!\}^{1-j}(n_j!)^{-1}\{1+o(1)\}$$

and, since

$$p_2(n_1, \ldots, n_j) = \sum_{q=0}^{n_j-1} p_2^{(n_j-q)}(n_1, \ldots, n_j)$$

and

$$p_4(n_1, \ldots, n_j) = \sum_{q=0}^{n_j-1} p_4^{(n_j-q)}(n_1, \ldots, n_j),$$

https://doi.org/10.1017/S0013091500025025 Published online by Cambridge University Press

we can see from (12) that Theorems 2 and 4 are proved if we show that

$$\sum_{q=1}^{n_j-1} (n_1 \ldots n_{j-1})^{-q}\{(n_j-1)(n_j-2)\ldots(n_j-q)\}^{j-1}(n_j!) \sum_{(n_j,\, q)} \prod_m (\nu_m!)^{-1} = o(1). \quad (13)$$

Now, since any partition of n_j into $n_j - q$ parts, where $q < \frac{1}{2}n_j$, must contain at least $n_j - 2q$ units, we have $\prod_m \nu_m! \geqq \Lambda_{n_j-2q}$, where $\Lambda_{n_j-2q} = (n_j - 2q)!$ for $q < \frac{1}{2}n_j$ and $\Lambda_{n_j-2q} = 1$ for $q \geqq \frac{1}{2}n_j$. Also, $p_2^{(n_j-q)}(n_j) = p_2(q)$ for $q \leqq \frac{1}{2}n_j$ and $p_2^{(n_j-q)}(n_j) < p_2(q)$ for $q > \frac{1}{2}n_j$. Therefore, since the Hardy-Ramanujan formula (1) for $p_2(q)$ shows that, for all $q > 0$,

$$p_2(q) < Cq^{-1} \exp\{\pi\sqrt{(2q/3)}\},$$

the left-hand side of (13) is less than

$$C\sum_{q=1}^{n_j-1} q^{-1} \exp\left\{\pi\sqrt{\left(\frac{2q}{3}\right)}\right\}(n_1 \ldots n_{j-1})^{-q} n_j^{(j+1)q}$$

$$= C\sum_{q=1}^{n_j-1} \exp\left\{\pi\sqrt{\left(\frac{2q}{3}\right)} - \log q - q \log\left(\frac{n_1 \ldots n_{j-1}}{n_j^{j+1}}\right)\right\}$$

$$< C\sum_{q=1}^{n_j-1} \exp\left\{-\tfrac{1}{2}q \log\left(\frac{n_1 \ldots n_{j-1}}{n_j^{j+1}}\right)\right\} = o(1),$$

since $n_j = o(n_l^{\frac{1}{3}})$ for $1 \leqq l \leqq j-1$.

REFERENCES

(1) G. H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis, *Proc. London Math. Soc.* (2), **17** (1918), 75-115.

(2) P. Erdös and J. Lehner, The distribution of the number of summands in partitions of a positive integer, *Duke Math. J.*, **8** (1941), 335-345.

(3) F. C. Auluck, On partitions of bi-partite numbers, *Proc. Cambridge Phil. Soc.*, **49** (1953), 72-83.

(4) G. Meinardus, Zur additiven Zahlentheorie in mehreren Dimensionen Teil I, *Math. Annalen*, **132** (1956), 333-346.

(5) V. S. Nanda, Bipartite partitions, *Proc. Cambridge Phil. Soc.*, **53** (1957), 272-277.

(6) E. M. Wright, Partitions of multi-partite numbers, *Proc. American Math. Soc.*, **7** (1956), 880-890.

(7) E. M. Wright, The number of partitions of a large bi-partite number, *Proc. London Math. Soc.* (3), **7** (1957), 150-160.

(8) E. M. Wright, Partitions of large bipartites, *American J. Math.*, **80** (1958), 643-658.

Department of Mathematics
University of Aberdeen